Solutions to exercice sheet 5

Construction of Wiener process

1. Find a basis and a sub-basis for the canonical topology on $\mathbb{R}$.

Consider the set

$$\beta := \{]a, \infty[: a \in \mathbb{R}\} \cup \{] - \infty, a] : a \in \mathbb{R}\}.$$

Then β is a collection of open sets on $\mathbb{R}$ and intersections of elements in β yield all open intervals on $\mathbb{R}$. But this last collection is a basis for the Euclidean topology on $\mathbb{R}$, so that β is a basis for this topology.

2. Let (Ω, τ) be a topological space. Let $\mathcal{S} \subset \mathcal{P}(\tau)$ be a totally ordered set (for the partial order being the set inclusion) so that no element of $\mathcal{S}$ has a finite sub-cover for Ω. Show then that $\cup_{S \in \mathcal{S}}$ has no finite sub-cover of Ω.
For $k = 1, \ldots, n$ let $\mathcal{M}_k$ be a finite collection of open sets of some topological space (Ω, τ). For each k, Let $B_k \in \tau$ so that $\mathcal{M}_k \cup \{B_k\}$ covers Ω. Show then that $(\cup_{k=1}^n \mathcal{M}_k) \cup \{\cap_{k=1}^n B_k\}$ covers Ω.

3. Let $\{(\Omega_i, \tau_i)\}_{i \in I}$ is a family of topological spaces. Provide the cartesian product $\prod_{i \in I} \Omega_i$ with the collection τ of sets consisting of arbitrary unions of sets of the form $\prod_{i \in I} V_i$, where all but a finite number of the $V_i' s$ are equal to Ω_i.
Show that τ is a topology. For some fixed $i \in I$, let $\pi_i : \prod_{i \in I} \Omega_i \to \Omega_i$, $(x_i)_{i \in I} \mapsto x_i \in \Omega_i$ be the **canonical projection**. Show that the collection of sets $\{\pi_i^{-1}(V) : i \in I \text{ and } V \in \tau_i\}$ is a sub-basis for τ. Prove that the product topology is then the coarsest topology for which all the maps π_i are continuous.

4. The **one-point compactification** of $\mathbb{R}^N$ is defined as the set $\dot{\mathbb{R}}^N := \mathbb{R}^N \cup \{*\}$, where $* \notin \mathbb{R}^N$, provided with the topology τ consisting of the canonical open sets in $\mathbb{R}^N$ together with sets of the form $\{*\} \cup (\mathbb{R}^N \setminus K)$, with $K \subset \mathbb{R}^N$.
Prove that $\dot{\mathbb{R}}^N$ is a compact set. Prove that $f \in C(\dot{\mathbb{R}}^N)$ iff $f = \lambda + g$ with $g \in C_0(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$.
Consider $\Omega_L := \prod_{t>0} \dot{\mathbb{R}}^N$, provided with the product topology, and the set of finite functions $C_{\text{fin}}(\Omega_L)$. Prove that $C_{\text{fin}}(\Omega_L) \subset C(\Omega_L)$ and that $C_{\text{fin}}(\Omega_L)$ is uniformly dense in $C(\Omega_L)$.

5. For a fixed $x_0 \in \mathbb{R}^N$ and for some $F_{f,t_1,\dots,t_m} \in C_{\text{fin}}(\Omega_L)$, we define

$$I_{x_0}(F_{f,t_1,\dots,t_m}) := \int_{\mathbb{R}^{mN}} p^{x_0}_{t_1,\dots,t_m}(x_1,\dots,x_m) f(x_1,\dots,x_m)\mu_L(dx_1\times\dots\times dx_m),$$
$$p^{x_0}_{t_1,\dots,t_m}(x_1,\dots,x_m) = \prod_{k=1}^{m} \frac{1}{(2\pi(t_k-t_{k-1}))^{N/2}} e^{-\frac{(x_k-x_{k-1})^2}{2(t_k-t_{k-1})}}, \quad t_0 = 0.$$

Check that $I_{x_0}(1) := 1$. For $F_{f,t_1,\dots,t_m}$ and $0 < t$ with $\{t_1,\dots,t_m\}\cap\{t\} = \emptyset$, if $G_{f,t_1,\dots,t_k,t,t_{k+1},\dots,t_m}(w) := F_{f,t_1,\dots,t_m}(w)$, check that

$$I_{x_0}(G_{f,t_1,\dots,t_k,t,t_{k+1},\dots,t_m}) = I_{x_0}(F_{f,t_1,\dots,t_m}).$$

or some constant $C > 0$ and for $\epsilon, t > 0$, prove that

$$\mathbb{E}_{x_0,W}(\mathbb{1}_{\{w\in\Omega_L\,:\,|w_t-x_0|>\epsilon\}}) \leq \frac{C}{\epsilon} e^{-\frac{\epsilon^2}{2Nt}}.$$

If we set

$$\rho(\epsilon,\delta) := \text{supp}\{\mu_{x_0,W}(\{w\in\Omega_L\,:\,|w_t-x_0|>\epsilon\})\,:\,0<t\leq\delta\},$$
$$\text{then verify that} \quad \lim_{\delta\to 0^+}\frac{1}{\delta}\rho(\epsilon,\delta) = 0.$$

6. Let $\delta, \epsilon > 0$ and consider a finite set $S \subset \mathbb{R}_+$ so that $\forall t \in S$, $|t - t_{\min(S)}| \leq \delta$. For $t \in S$, let

$$C_{t,\epsilon,S} := \{w\in\Omega_L\,:\,|w_t - w_{\max(S)}| > \frac{\epsilon}{2}\},$$
$$D_{t,\epsilon,S} := \{w\in\Omega_L\,:\,|w_t - w_{\min(S)}| > \epsilon \text{ and } \forall s\in S \text{ with } s<t,\ |w_s - w_{\min(S)}| \leq \epsilon\}.$$

Shoe then that $\forall t \in S$,

$$\mu_{x_0,W}(C_{t,\epsilon,S}\cap D_{t,\epsilon,S}) \leq \rho(\frac{\epsilon}{2},\delta)\mu_{x_0,W}(D_{t,\epsilon,S}).$$

Let $f(x,w) := \mathbb{1}_{|x-w|>\frac{\epsilon}{2}}$, so that $\mathbb{1}_{C_{t,\epsilon,S}} = F_{f,t,\max(S)}$.
Similarly, let $\{t_1,\dots,t_m\} = S\cap]\min(S),t[$ and define

$$g(y,x_1,\dots,x_m,x) := \mathbb{1}_{|x-y|>\epsilon}\prod_{k=1}^{m}\mathbb{1}_{|x_k-y|\leq\epsilon}.$$

Then $\mathbb{1}_{D_{t,\epsilon,S}} = G_{g,\min(S),t_1,\dots,t_m,t}$ and

$$\mu_{x_0,W}(C_{t,\epsilon,S}\cap D_{t,\epsilon,S})$$
$$= \int_{\mathbb{R}^{N(m+3)}} p^{x_0}_{\min(S),t_1,\dots,t_m,t,\max(S)}(y,x_1,\dots,x_m,x,w)g(y,x_1,\dots,x_m,x)f(x,w)\mu_L(dy\times\dots\times dw).$$

Observe now that

$$p^{x_0}_{\min(S),t_1,\dots,t_m,t,\max(S)}(y,x_1,\dots,x_m,x,w) = p^{x_0}_{\min(S),t_1,\dots,t_m,t}(y,x_1,\dots,x_m,x)p^{x}_{t,\max(S)}(x,w).$$

We now may calculate the integral over w first and notice, that the result is bounded by $\rho(\frac{\epsilon}{2}, \delta)$. Integration over the remaining variables result in $\mu_{x_0,W}(D_{t,\epsilon,S})$ which yields the announced inequality.

7. Let $\delta, \epsilon > 0$ and consider a finite set $S \subset \mathbb{R}_+$ so that $\forall t \in S$, $|t - t_{\min(S)}| \leq \delta$. Let

$$A_{\epsilon,S} := \{w \in \Omega_L \ : \ \exists s \in S \text{ s.t. } |w_s - w_{\min(S)}| > \epsilon\}.$$

Prove then that

$$\mu_{x_0,W}(A_{\epsilon,S}) \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

Define

$$B_{\epsilon,S} := \{w \in \Omega_L . : |w_{\min(S)} - w_{\max(S)}| > \frac{\epsilon}{2}\},$$

$$\text{for } t \in S \ , \quad C_{t,\epsilon,S} := \{w \in \Omega_L \ : \ |w_t - w_{\max(S)}| > \frac{\epsilon}{2}\},$$

$$D_{t,\epsilon,S} := \{w \in \Omega_L \ : \ |w_t - w_{\min(S)}| > \epsilon \text{ and } \forall s \in S \text{ with } s < t, \ |w_s - w_{\min(S)}| \leq \epsilon\}.$$

If $w \in A_{\epsilon,S}$, then $w \in D_{t,\epsilon,S}$ for some $t \in S$.
If $w \notin B_{\epsilon,S}$ and if for some $t \in S$ $w \in D_{t,\epsilon,S}$, then $w \in C_{t,\epsilon,S}$, since w has to move a distance at least $\frac{\epsilon}{2}$ to go back from outside the ball of radius ϵ into the ball of radius $\frac{\epsilon}{2}$. Therefore

$$A_{\epsilon,S} \subset B_{\epsilon,S} \bigcup \left(\bigcup_{t \in S} C_{t,\epsilon,S} \cap D_{t,\epsilon,S} \right).$$

Thus,

$$\begin{aligned}\mu_{x_0,W}(A_{\epsilon,S}) &\leq \mu_{x_0,W}(B_{\epsilon,S}) + \sum_{t \in S} \mu_{x_0,W}(C_{t,\epsilon,S} \cap D_{t,\epsilon,S}) \\ &\leq \mu_{x_0,W}(B_{\epsilon,S}) + \rho(\frac{\epsilon}{2}, \delta) \sum_{t \in S} \mu_{x_0,W}(D_{t,\epsilon,S})\end{aligned}$$

by the previous exercice. Since $D_{t,\epsilon,S}$ and $D_{t',\epsilon,S}$ are by definition disjoint for $S \ni t \neq t' \in S$, one has

$$\mu_{x_0,W}(A_{\epsilon,S}) \leq \mu_{x_0,W}(B_{\epsilon,S}) + \rho(\frac{\epsilon}{2}, \delta) \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

8. Let $\delta, \epsilon > 0$ and consider $0 < t_0 < t_1$ with $t_1 - t_0 \leq \delta$. Define

$$E_{t_0,t_1,\epsilon} := \{w \in \Omega_L \ : \ \exists s, t \in [t_0, t_1] \text{ s.t. } |w_s - w_t| > 2\epsilon\}.$$

Prove then that

$$\mu_{x_0,W}(E_{t_0,t_1,\epsilon}) \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

Consider some finite set $S \in [t_0, t_1]$ with $t_0, t_1 \in S$ and notice, that if one defines

$$E_{\epsilon,S} := \{w \in \Omega_L \ : \ \exists t, s \in S \text{ s.t. } |w_s - w_t| > 2\epsilon\},$$

then $E_{\epsilon,S} \subset A_{\epsilon,S}$ with

$$A_{\epsilon,S} := \{w \in \Omega_L \,:\, \exists s \in S \text{ s.t. } |w_s - w_{\min(S)}| > \epsilon\},$$

since if $|w_s - w_t| > 2\epsilon$, then $|w_s - w_{t_0}|, |w_t - w_{t_0}| \leq \epsilon$ cannot both hold. Hence, by the previous exercice,

$$\mu_{x_0,W}(E_{\epsilon,S}) \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

We now are going to make use of the regularity of the measure $\mu_{x_0,W}$ and note that the aforementioned sets $E_{\epsilon,S}$ are open in the product topology for any finite set $S \subset [t_0, t_1]$ with $t_0, t_1 \in S$.
If we consider the collection of open sets

$$\Gamma_{\epsilon,t_0,t_1} := \{E_{\epsilon,S} \,:\, S \subset [t_0, t_1] \text{ is a finite set with } t_0, t_1 \in S\},$$

then

$$E_{t_0,t_1,\epsilon} = \bigcup_{V \in \Gamma_{\epsilon,t_0,t_1}} V$$

and by the regularity of the measure $\mu_{x_0,W}$,

$$\mu_{x_0,W}(E_{t_0,t_1,\epsilon}) = \sup\{\mu_{x_0,W}(E_{\epsilon,S})\} \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

□

10. Let