Solutions to exercice sheet 1

Schwartz space

1. Prove the Riesz & Fréchet theorem, namely, that given a Hilbert space H, the anti-linear map

$$J : H \ni x \mapsto \langle x, \ \rangle =: J(x) \in H'$$

is an isometric isomorphism.
(Hint: if $V \subset H$ is a closed subspace, then $H = V \oplus V^{\perp}$.)

Let $x \in H$ and we shall show, that $J(x) \in H'$. Indeed, for any $y \in H$, $|J(x)(y)| = |\langle x, y\rangle| \leq \|x\|_H \|y\|_H$. This shows that $J(x) \in H'$.
For $x \neq 0$, one may take $y = \frac{x}{\|x\|_H}$, and one has $J(x)(y) = \|x\|$. This shows, that $\|J(x)\|_{H'} = \|x\|_H$ and J is hence isometric and injective.
Let $\xi \in H'$ and let us find an $x \in H$, so that $\xi = J(x)$. If $\xi = 0_{H'}$, then one may chose $x = 0_H$. Let's therefore assume that $\xi \neq 0_{H'}$. The continuity of ξ implies, that $\ker(\xi) = \xi^{-1}\{0\} = V$ is a closed subspace of H. One therefore has $H = V \oplus V^{\perp}$ and since $\xi \neq 0_{H'}$, one obviously has $V^{\perp} \neq \{0_H\}$.
Moreover, if $V^{\perp} \ni x, y \neq 0_H$, then $\xi\left(\xi(y)x - \xi(x)y\right) = 0$, so that $\xi(y)x - \xi(x)y \in V \cap V^{\perp} = \{0_H\}$. Consequently, $V^{\perp}$ is a one-dimensional closed subspace of H.
Chose now $0_H \neq y \in V^{\perp}$ and set $x := \frac{y}{\|y\|_H^2}\overline{\xi(y)}$. For $v = \lambda y$, one has therefore

$$J(x)(v) = \langle x, \lambda y\rangle = \lambda\xi(y) = \xi(\lambda y) = \xi(v).$$

Thus, $J(x)\Big|_{V^{\perp}} = \xi\Big|_{V^{\perp}}$, $J(x)\Big|_{V} = \xi\Big|_{V}$, and since $H = V \oplus V^{\perp}$, one may conclude.

2. Let H be a Hilbert space and consider a Banach space B. Prove that any continuous and linear map $\xi : D \to B$, defined on a dense set $D \subset H$ has a unique and isometric extension $\overline{\xi} : H \to B$.

Let $x \in H$. Since $D \subset H$ is dense, there is a Cauchy sequence $(d_k)_{k\in\mathbb{N}} \subset D$ whose limit in H is x.
Consider then the sequence $(\xi(d_k))_{k\in\mathbb{N}} \subset B$. The continuity of the map $\xi : H \to B$ is equivalent to ξ being bounded, so that $(\xi(d_k))_{k\in\mathbb{N}}$ is Cauchy in B. This latter space being complete, there is a limit $B \ni y = \lim_k \xi(d_k)$.
If $(d_k')_{k\in\mathbb{N}} \subset D$ is another Cauchy sequence whose limit in H is x, then $(d_k' - d_k)_{k\in\mathbb{N}} \subset D$ is a sequence converging to 0_H. The continuity of ξ then implies, that $\lim_k \xi(d_k' - d_k) = 0_B$. Therefore, $\lim_k \xi(d_k') = \lim_k \xi(d_k) = y$ and thus y depends only on x and not on the particular sequence $(d_k)_{k\in\mathbb{N}} \subset D$ chosen. We set $\overline{\xi}(x) := y$.
It remains to show that this extension of the map ξ is isometric. For a given $\epsilon > 0$, there is by construction some $d \in D$, so that $\|\overline{\xi}(x) - \xi(d)\|_B < \epsilon$. Consequently, $\|\overline{\xi}(x)\|_B \leq \|\xi(d)|_B + \epsilon \leq \|\xi\|_{\mathcal{L}(D,B)} + \epsilon$. This being true for any $\epsilon > 0$, one must have $\|\overline{\xi}(x)\|_B \leq \|\xi\|_{\mathcal{L}(D,B)}$. This being true for any $x \in H$, one concludes, that $\|\overline{\xi}\|_{\mathcal{L}(D,B)} \leq \|\xi\|_{\mathcal{L}(D,B)}$.
The inverse inequality follows from the fact that $\overline{\xi}\Big|_D = \xi$.

3. Let V be a $\mathbb{K}-$vector space and let $\| \ \|_{1,2} : V \to \mathbb{R}_+$ be two norms:

(a) Show that if $\forall x \in V$, $\|x\|_1 \leq \|x\|_2$, then $\{x \in V : \|x\|_2 < 1\} \subset \{x \in V : \|x\|_1 < 1\}$.

(b) Show that the topology τ_2 defined by $\| \ \|_2$ is finer than the topology τ_1 defined by $\| \ \|_1$, i.e. $\tau_1 \subset \tau_2$. Prove as a consequence, that any sequence $(x_n)_{n\in\mathbb{N}} \subset V$ converges for τ_1 if it does so for τ_2.

(c) Show that if W is a $\mathbb{K}-$vector space with a topology τ_W and if $f : V \to W$ is a continuous map with respect to τ_1, then f is also continuous with respect to τ_2.

(d) Show that if in addition, there is a positive constant C so that $\forall x \in V$, $\|x\|_2 \leq C\|x\|_1$, then $\tau_1 = \tau_2$.

(a) For a given $x \in V$, if $\|x\|_2 < 1$, then $\|x\|_1 \leq \|x|_2 < 1$. Consequently, $\{x \in V : \|x\|_2 < 1\} \subset \{x \in V : \|x\|_1 < 1\}$.

(b) Let $U \in \tau_1$ be an open set for the topology induced by the norm $\| \ |_1$. By definition, this means, that $\forall x \in U$, there is an $\epsilon > 0$, so that U contains the ball $\{y \in V : \|x - y\|_1 < \epsilon\}$. By the previous point, this implies, that U contains the open ball $\{y \in V : \|x - y\|_2 < \epsilon\}$ as well, so that by definition, $U \in \tau_2$.
A sequence $(x_n)_{n\in\mathbb{N}} \subset V$ converges for τ_2 iff there is a $x \in V$, so that for any open set $x \in U \in \tau_2$, there is an $N \in \mathbb{N}$, so that $n \geq N$ implies $x_n \in U$. By the previous point, $\tau_1 \subset \tau_2$, so that if the statement holds for any $x \in U \in \tau_2$, it must hold for any $x \in U \in \tau_1$. As a consequence, the sequence $(x_n)_{n\in\mathbb{N}} \subset V$ converges for τ_1 if it does so for τ_2.

(c) If $f : V \to W$ is a continuous map with respect to τ_1, then by definition, this means that for any $U \in \tau_W$, $f^{-1}\{U\} \in \tau_1$. But since $\tau_1 \subset \tau_2$, this then means, that $U \in \tau_W$, $f^{-1}\{U\} \in \tau_2$. Hence, f is continuous with respect to τ_2 as well.

(d) Suppose that in addition, there is a positive constant C so that $\forall x \in V$, $\|x\|_2 \leq C\|x\|_1$. Let $U \in \tau_2$. For any $x \in U$, there is hence an $\epsilon > 0$, so that $\{y \in V : \|x - y\|_2 < \epsilon\} \subset U$. But then, the set $\{y \in V : \|x - y\|_1 < \epsilon/C\} \subset U$ as well and $\tau_1 = \tau_2$.

4. (a) For $f, g \in C^n(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq n$, show that

$$\partial^\alpha(fg)(x) = \sum_{\substack{\beta,\gamma\in\mathbb{N}^N, \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^\gamma g(x).$$

(b) Show that the cardinality of the set $\mathbb{N}^N_{\leq n} := \{\alpha \in \mathbb{N}^N : |\alpha| \leq n\}$ is $\binom{n+N}{n}$.
(Hint: define $\mathbb{N}^N_{=k} := \{\alpha \in \mathbb{N}^N : |\alpha| = k\}$ and observe, that $\mathbb{N}^N_{\leq n} = \cup_{k=0}^n M_{=k}$.)

(a) For fixed $N, n \in \mathbb{N}^*$ and $|\alpha| = 1$, this is juste the well-known Leibnitz rule. Suppose then that the result is true for $|\alpha| = k < n$. Set $\alpha' = \alpha + \delta$ with

$|\delta| = 1$. One then has

$$\begin{aligned}
\partial^{\alpha'}(fg) = \partial^{\delta}(\partial^{\alpha}(fg)) &= \partial^{\delta}\left(\sum_{\substack{\beta,\gamma\in\mathbb{N}^N\\ \beta+\gamma=\alpha}}\binom{\alpha}{\beta}\partial^{\beta}f\partial^{\gamma}g\right)\\
&= \sum_{\substack{\beta,\gamma\in\mathbb{N}^N,\\ \beta+\gamma=\alpha}}\binom{\alpha}{\beta}\partial^{\beta+\delta}f\partial^{\gamma}g + \sum_{\substack{\beta,\gamma\in\mathbb{N}^N,\\ \beta+\gamma=\alpha}}\binom{\alpha}{\beta}\partial^{\beta}f\partial^{\gamma+\delta}g\\
&= \sum_{\substack{\beta',\gamma'\in\mathbb{N}^N,\\ \beta'+\gamma'=\alpha',\ \beta'\geq\delta}}\binom{\alpha'-\delta}{\beta'-\delta}\partial^{\beta'}f\partial^{\gamma'}g + \sum_{\substack{\beta',\gamma'\in\mathbb{N}^N,\\ \beta'+\gamma'=\alpha',\ \gamma'\geq\delta}}\binom{\alpha'-\delta}{\beta'}\partial^{\beta'}f\partial^{\gamma'}g.
\end{aligned}$$

Note that in the first sum, if $\beta' + \gamma' = \alpha'$ and $\gamma' \geq \delta$ is false, then $\beta'\delta = \alpha'\delta$. Similarly, if $\beta' + \gamma' = \alpha'$ and $\beta' \geq \delta$ is false, then $\gamma'\delta = \alpha'\delta$. We thus split the sums accordingly and obtain

$$\begin{aligned}
\partial^{\alpha'}(fg) &= \sum_{\substack{\beta',\gamma'\in\mathbb{N}^N,\\ \beta'+\gamma'=\alpha',\ \beta'\geq\delta}}\binom{\alpha'-\delta}{\beta'-\delta}\partial^{\beta'}f\partial^{\gamma'}g + \sum_{\substack{\beta',\gamma'\in\mathbb{N}^N,\\ \beta'+\gamma'=\alpha',\ \gamma'\geq\delta}}\binom{\alpha'-\delta}{\beta'}\partial^{\beta'}f\partial^{\gamma'}g\\
&= \sum_{\substack{\beta',\gamma'\in\mathbb{N}^N,\\ \beta'+\gamma'=\alpha',\ \beta'\delta=\alpha'\delta}}\binom{\alpha'-\delta}{\beta'-\delta}\partial^{\beta'}f\partial^{\gamma'}g + \sum_{\substack{\beta',\gamma'\in\mathbb{N}^N,\\ \beta'+\gamma'=\alpha',\ \gamma\delta=\alpha'\delta}}\binom{\alpha'-\delta}{\beta'}\partial^{\beta'}f\partial^{\gamma'}g\\
&\quad + \sum_{\substack{\beta',\gamma'\in\mathbb{N}^N,\\ \beta'+\gamma'=\alpha',\ \beta',\gamma'\geq\delta}}\left(\binom{\alpha'-\delta}{\beta'-\delta}+\binom{\alpha'-\delta}{\beta'}\right)\partial^{\beta'}f\partial^{\gamma'}g.
\end{aligned}$$

In the first sum, note that if $\beta' + \gamma' = \alpha'$ and $\beta'\delta = \alpha'\delta$, then $\binom{\alpha'-\delta}{\beta'-\delta} = \binom{\alpha'}{\beta'}$. A similar argument for the second sum shows, that $\binom{\alpha'-\delta}{\beta'} = \binom{\alpha'-\delta}{\alpha'-\delta-\beta'} = \binom{\alpha'-\delta}{\gamma'-\delta} = \binom{\alpha'}{\gamma'} = \binom{\alpha'}{\beta'}$.
In the last sum, observe that $\binom{\alpha'-\delta}{\beta'-\delta} + \binom{\alpha'-\delta}{\beta'} = \binom{\alpha'}{\beta'}$. Adding therefore these three sums gives the desired result.

(b) $\mathbb{N}^N_{=k}$ may be viewed as the number of sampling with replacement of k indistinguishable elements among N distinguishable ones. Hence, the cardinality of $\mathbb{N}^N_{=k}$ is $\binom{N+k-1}{k}$.
This may be shown as follows: For a given $N \geq 1$, $k = 0$ and $k = 1$, one obviously has $\mathbb{N}^N_{=0} = 1$ and $\mathbb{N}^N_{=1} = N$. Obviously, one also has $\mathbb{N}^1_{=k} = 1$.
One then proceeds by induction on N and k: the possible choices for $\mathbb{N}^N_{=k+1}$ are then to chose the first element as the $N+1^{\text{th}}$ and the other k elements among all $N+1$ choices, or to chose all $k+1$ elements among the first N elements. Therefore:

$$\mathbb{N}^{N+1}_{=k+1} = \mathbb{N}^{N+1}_{=k} + \mathbb{N}^{N}_{=k+1} = \binom{N+1+k-1}{k} + \binom{N+k+1-1}{k+1} = \binom{N+k+1}{k+1}.$$

Again by induction on k, one then has

$$\mathbb{N}^{N+1}_{\leq k+1} = \mathbb{N}^{N+1}_{=k+1} + \mathbb{N}^{N+1}_{\leq k} = \binom{N+1+k}{k+1} + \binom{N+k+1}{k} = \binom{N+k+2}{k+1}.$$

5. Let X be a $\mathbb{K}-$vector space endowed with a family $\{\| \ \|_j\}_{j\in I}$ of norms. For $\epsilon > 0$, $x \in X$ and $\{j_1, \ldots, j_n\} \subset I$, one defines

$$U_{x,\epsilon,j_1,\ldots,j_n} := \{y \in X \ : \ \forall k = 1, \ldots, n, \ \|y - x\|_{j_k} < \epsilon\}.$$

Show that the collection of all subsets $U \subset X$, so that

$$\forall x \in U, \ \exists \epsilon > 0, \ \exists \{j_1, \ldots, j_n\} \subset I \text{ s.t. } U_{x,\epsilon,j_1,\ldots,j_n} \subset U$$

is a topology τ_X on X, i.e.:

- $\forall \mathcal{F} \subset \tau_X$, $|\mathcal{F}| \in \mathbb{N}$ implies $\cap_{U\in\mathcal{F}} U \in \tau_X$,
- $\forall \mathcal{F} \subset \tau_X$, $\cup_{U\in\mathcal{F}} U \in \tau_X$.

- Let $\mathcal{F} \subset \tau_X$, $|\mathcal{F}| \in \mathbb{N}$. Without loss of generality, we may suppose $\mathcal{F} = \{U_1, \ldots, U_n\}$. Let $x = \cap_{U\in\mathcal{F}} U$. By definition, there are positive numbers $\epsilon_1, \ldots, \epsilon_n > 0$ and finite sets of indices $I_1, \ldots, I_n$, so that for each $k = 1, \ldots$,

  $$U_{x,\epsilon_k,j\in I_k} \subset U_k.$$

  Set now $\epsilon := \min\{\epsilon_1, \ldots, \epsilon_n$ and $I = \cup_{k=1}^n I_k$. It is then clear, that

  $$U_{x,\epsilon,j\in I} \subset \cap_{U\in\mathcal{F}} U,$$

  which shows that $\cap_{U\in\mathcal{F}} U \in \tau_X$.
- Let $\mathcal{F} \subset \tau_X$ and $x \in \cup_{U\in\mathcal{F}} U$. There is hence some $U \in \mathcal{F}$ so that $x \in U$. By definition, there is then an $\epsilon > 0$ and finite indices $j_1, \ldots, j_n$, so that

  $$U_{x,\epsilon,j_1,\ldots,j_n} \subset U.$$

  But it is the clear, that

  $$U_{x,\epsilon,j_1,\ldots,j_n} \subset \cup_{U\in\mathcal{F}} U$$

  as well, showing that $\cup_{U\in\mathcal{F}} U \in \tau_X$.

We end by the remark, that by convention, if $\tau_X \supset \mathcal{F} = \emptyset$, then $\cap_{U\in\mathcal{F}} U = X$ and $\cup_{U\in\mathcal{F}} U = \emptyset$, so that as a consequence, $\emptyset, X \in \tau_X$ as a consequence of the two previously checked rules.

6. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ be a sequence which is Cauchy for all the norms $|\!|\!| \ |\!|\!|_n$. Show, that this sequence converges to some $f \in \mathcal{S}(\mathbb{R}^N)$ for τ_S.
(Hint: you might wanna use the Stone-Weierstrass theorem and the uniform continuity of the Riemann integral.)

Since $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ is Cauchy for all the norms $|\!|\!| \ |\!|\!|_n$, then for any $\alpha \in \mathbb{N}^N$ and any $n \in \mathbb{N}$, we have that $(\partial^\alpha f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ and $((1 + x \cdot x)^n \partial^\alpha f_k(x))_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ are Cauchy for the norm $\| \ \|_\infty$.
By the Stone Weierstrass theorem, all sequences $(\partial^\alpha f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^N)$ converge uniformly on $\mathbb{R}^N$ to some continuous functions f_α, which are all of rapid decrease.

(Strictly speaking, one has to apply the Stone Weierstrass theorem to the one-point compactification $(\mathbb{R}^N)^+$. This space is defined as the set $\mathbb{R}^N \cup \{\star\}$, endowed with the topology consisting of all open sets in $\mathbb{R}^N$ and the sets $(\mathbb{R}^N \setminus K) \cup \{\star\}$, where $K \subset \mathbb{R}^N$ are compact sets. A sequence $(f_k)_{k\in\mathbb{N}}$ of continuous functions on $\mathbb{R}^N$ which converge to 0 as $x \cdot x \to \infty$ can then be extended to a sequence $(F_k)_{k\in\mathbb{N}}$ of continuous functions on $(\mathbb{R}^N)^+$, defined by $F_k(\star) = 0$ and $F\Big|_{\mathbb{R}^N} = f_k$. If $(f_k)_{k\in\mathbb{N}}$ is Cauchy for $\| \ \|_\infty$ on $\mathbb{R}^N$, then $(F_k)_{k\in\mathbb{N}}$ is Cauchy for $\| \ \|_\infty$ on $(\mathbb{R}^N)^+$, so that The Stone-Weierstrass theorem can be applied to this sequence, which converges uniformly to a continuous fonction F on $(\mathbb{R}^N)^+$. Obviously, $F(\star) = 0$ and $(f_k)_{k\in\mathbb{N}}$ converges uniformly on $\mathbb{R}^N$ to the continuous fonction $f = F\Big|_{\mathbb{R}^N}$.)

It remains to be shown, that $f_\alpha = \partial^\alpha f$. In order to do so we proceed by induction on $\alpha \in \mathbb{N}^N$. For $\alpha = \overline{0}$ this is just stating $f = f_{\overline{0}} = \partial^{\overline{0}} f = f$.
Suppose then that $f_\alpha = \partial^\alpha f$ and let $\delta \in \mathbb{N}^N_{\leq 1}$. We then have for a given $x \in \mathbb{R}^N$

$$f_\alpha(x) = \lim_{k\to\infty} \partial^\alpha f_k(x) = \lim_{k\to\infty} \int_{-\infty}^{x\cdot\delta} \partial^{\alpha+\delta} f_k(x(\overline{1}-\delta) + y\delta) d(\delta \cdot y).$$

Since $(\partial^{\alpha+\delta} f_k)_{k\in\mathbb{N}}$ converges uniformly on $\mathbb{R}^N$ to $f_{\alpha+\delta}$, one may exchange the limit and the Riemann integration to get

$$f_\alpha(x) = \int_{-\infty}^{x\cdot\delta} f_{\alpha+\delta}(x(\overline{1}-\delta) + y\delta) d(\delta \cdot y).$$

$f_{\alpha+\delta}$ is a continuous function, so that by the fundamental theorem of calculus, this last equality yields

$$\partial^\delta f_\alpha(x) = f_{\alpha+\delta}(x).$$

7. Let $g \in L^2(\mathbb{R}^N, \mu_L)$. For $x \in \mathbb{R}^N$, set $E_x := \{y \in \mathbb{R}^N : y = \delta x \text{ s.t. } \delta \in [0,1]^N\}$. Show that the function

$$\mathbb{R}^N \ni x \mapsto G(x) := \operatorname{sgn}(x) \int_{E_x} g(y)\mu_L(dy)$$

is well-defined, continuous and polynomially bounded. If $f \in \mathcal{S}(\mathbb{R}^N)$, show that

$$\int_{\mathbb{R}^N} G(x)\partial^{\overline{1}} f(x)\mu_L(dx) = (-1)^N \int_{\mathbb{R}^N} g(x)f(x)\mu_L(dx).$$

(Hint: for the second part, show it first when $g(x) = \prod_{k=1}^N g_k(x)$ and all $g_k(t)$ are continuous and compactly supported on $\mathbb{R}$. Use then a density argument.)

The set E_x is obviously compact and consequently, $1_{E_x} \in L^2(\mathbb{R}^N, \mu_L)$. Since $\int_{E_x} g(y)\mu_L(dy) = \int_{\mathbb{R}^N} 1_{E_x}(y)g(y)\mu_L(dy) = \langle 1_{E_x}, g\rangle_{L^2}$, it is well-defined and if $x \to x'$, then manifestly $1_{E_x}(y) \to 1_{E_{x'}}(y)$ for all $y \in \mathbb{R}^N$ and by the dominated convergence theorem, $\lim_{x\to x'} G(x) = G(x)$.
Using the Cauchy-Schwarz inequality, one gets that $|G(x)| \leq \|g\|_{L^2}\mathrm{Vol}(E_x)$, which is obviously bounded by $(x \cdot x)^{N/2}$.
Consider first the case where $g = \prod_{k=1}^N g_k(x_k)$, where all functions $g_k(t)$ are continuous and compactly supported. Clearly, $g(x) \in L^2(\mathbb{R}^N, \mu_L(dx))$ and all functions

$g_k(t)$ have continuous primitive functions $G_k(t)$, which are constant outside a compact support. For a given $x \in \mathbb{R}^N$ and by integrating all N dimensions in successive order, one gets $G(x) = \prod_{k=1}^N (G_k(x_k) - G_k(0))$.
Now, $G(x)\partial^{\bar{1}} f(x)$ is continuous and square summable, so that one may replace the Lebesgue integral by Riemann integration to get

$$\int_{\mathbb{R}^N} G(x)\partial^{\bar{1}} f(x)\mu_L(dx)$$

$$= \int_{\mathbb{R}} dx_N \ldots \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_1 \prod_{k=1}^N (G_k(x_k) - G_k(0)) \frac{\partial}{\partial x_1}\left(\frac{\partial^{N-1}}{\partial x_2 \ldots \partial x_N} f(x_1, x_2 \ldots, x_N)\right)$$

$$= \int_{\mathbb{R}} dx_N \ldots \int_{\mathbb{R}} dx_2 \prod_{k=2}^N (G_k(x_k) - G_k(0))$$

$$\times \int_{\mathbb{R}} dx_1 (G_1(x_1) - G_1(0)) \frac{\partial}{\partial x_1}\left(\frac{\partial^{N-1}}{\partial x_2 \ldots \partial x_N} f(x_1, x_2 \ldots, x_N)\right)$$

$$= \int_{\mathbb{R}} dx_N \ldots \int_{\mathbb{R}} dx_2 \prod_{k=2}^N (G_k(x_k) - G_k(0))$$

$$\times \Big[\int_{\mathbb{R}} dx_1 (-1) g_1(x_1) \left(\frac{\partial^{N-1}}{\partial x_2 \ldots \partial x_N} f(x_1, x_2 \ldots, x_N)\right)$$

$$+ (G(x_1) - G(0)) \frac{\partial^{N-1}}{\partial x_2 \ldots \partial x_N} f(x_1, x_2 \ldots, x_N)\Big|_{-\infty}^{\infty} \Big]$$

$$= -\int_{\mathbb{R}} dx_N \ldots \int_{\mathbb{R}} dx_2 \prod_{k=2}^N (G_k(x_k) - G_k(0)) \int_{\mathbb{R}} dx_1 g_1(x_1) \left(\frac{\partial^{N-1}}{\partial x_2 \ldots \partial x_N} f(x_1, x_2 \ldots, x_N)\right).$$

By iteration, one finally gets

$$\int_{\mathbb{R}^N} G(x)\partial^{\bar{1}} f(x)\mu_L(dx) = (-1)^N \int_{\mathbb{R}^N} g(x) f(x)\mu_L(dx).$$

Linearity of the integral implies that this last relation remains valid for $g(x)$ being a linear combination of the type $g(x) = \sum_{l=1}^M \prod_{k=1}^N g_{l,k}(x_k)$ with all $g_{l,k}(t)$ being continuous and of compact support.
We know use the density of these latter functions in $L^2(\mathbb{R}^N, \mu(dx))$. Let $g \in L^2(\mathbb{R}^N, \mu(dx))$ and consider a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2(\mathbb{R}^N, \mu(dx))$, so that $\lim_k g_k = g$ in $L^2(\mathbb{R}^N, \mu(dx))$. Suppose that for any $k \in \mathbb{N}$, the relation $\int_{\mathbb{R}^N} G_k(x)\partial^{\bar{1}} f(x)\mu_L(dx) = (-1)^N \int_{\mathbb{R}^N} g_k(x) f(x)\mu_L(dx)$ holds. We then have

$$G(x) = \mathrm{sgn}(x) \int_{E_x} g(y)\mu_L(dy) = \mathrm{sgn}(x)\langle I_{E_x}, g\rangle_{L^2}$$

$$= \mathrm{sgn}(x)\langle I_{E_x}, \lim_k g_k\rangle_{L^2} = \lim_k \mathrm{sgn}(x)\langle I_{E_x}, g_k\rangle_{L^2} = \lim_k G_k(x),$$

and since $|G(x) - G_k(x)| = |\langle I_{E_x}, g - g_k\rangle_{L^2}| \leq \|I_{E_x}\|_{L^2}\|g - g_k\|_{L_2} = \mathrm{Vol}(E_x)^{1/2}\|g - g_k\|_{L_2}$, which is polynomially bounded in x, we have that $G_k(x)\partial^{\bar{1}} f(x)$ converges uniformly to $G(x)\partial^{\bar{1}} f(x)$. Hence,

$$\int_{\mathbb{R}^N} G(x)\partial^{\bar{1}} f(x)\mu_L(dx) = \lim_k \int_{\mathbb{R}^N} G_k(x)\partial^{\bar{1}} f(x)\mu_L(dx)$$
$$= (-1)^N \lim_k \int_{\mathbb{R}^N} g_k(x)f(x)\mu_L(dx) = (-1)^N \int_{\mathbb{R}^N} g(x)f(x)\mu_L(dx),$$

where the last limit is in $L^2(\mathbb{R}^N, \mu_L)$.

8. Find an $n \in \mathbb{N}$ and continuous and polynomially bounded functions $(g_\alpha(x))_{\alpha \in \mathbb{N}^N_{\leq n}}$ on $\mathbb{R}$, so that

$$\varphi(f) = \sum_{\alpha \in \mathbb{N}^N_{\leq n}} \int_{\mathbb{R}} g_\alpha(x)\partial^\alpha f(x)\mu_L(dx)$$

for

(a) $\varphi = \delta(x)$,

(b) $\varphi = \text{p.v.}(\frac{1}{x})$.

Can you find more than one such representations?

(a) A simple integration, that for $\varphi = \delta(x)$, one has

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \varphi(f) = -\int_{\mathbb{R}^+} f'(x)dx.$$

Hence, integration by parts yields

$$\varphi(f) = \int_{\mathbb{R}} (0 \vee x) f''(x)dx.$$

Therefore, one may chose $n = 2$, $g_0(x) = g_1(x)$ and $g_2(x) = (0 \vee x)$. This last function is certainly continuous and bounded by x^2.
One may also chose $g_2(x) = (0 \vee x) + cx + d$ for any constant c, d.

(b) Two integration by parts show, that for $\varphi = \text{p.v.}(\frac{1}{x})$, one has

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \varphi(f) = \int_{\mathbb{R}^+} x(\ln(x) - 1)(f''(x) - f''(-x))dx.$$

Hence,

$$\varphi(f) = \int_0^\infty x(\ln(x) - 1)f''(x)dx + \int_0^\infty (-x)(\ln(x) - 1)f''(-x)dx$$
$$= \int_0^\infty x(\ln(x) - 1)f''(x)dx + \int_0^\infty (-x)(\ln(x) - 1)f''(-x)dx$$
$$= \int_{-\infty}^\infty x(\ln(|x|) - 1)f''(x)dx.$$

Therefore, one may chose $n = 2$, $g_0(x) = g_1(x)$ and $g_2(x) = x(\ln(|x|) - 1)$. This last function is certainly continuous (since $\lim_{x\to 0} x\ln(|x|) = 0$) and bounded by x^2.
One may also chose $g_2(x) = x(\ln(|x|) - 1) + cx + d$ for any constant c, d.

9. Prove that the sequence $(h_k)_{k\in\mathbb{N}^*} \subset \mathcal{S}'(\mathbb{R})$ converges in the weak* topology to $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ for

(a) $\varphi = \delta(x)$ and $h_k(x) = 1_{[-1,1]}\frac{n}{2}e^{-n|x|}$,

(b) $\varphi = \text{p.v.}(\frac{1}{x})$ and $h(k) = \frac{x}{x^2+k^{-2}}$.

(a) For a given $k \in \mathbb{N}$ and a fixed $f \in \mathcal{S}(\mathbb{R})$, one has

$$\begin{aligned}\langle \overline{h_k}, f\rangle_{L^2} &= \int_{\mathbb{R}} 1_{[-1,1]}\frac{k}{2}e^{-k|x|}f(x)\mu_L(dx)\\ &= \int_{\mathbb{R}} 1_{[-k,k]}\frac{1}{2}e^{-|y|}f(yk^{-1})\mu_L(dy).\end{aligned}$$

The integrand is bounded by $\frac{1}{2}e^{-|y|}\|f|_\infty$, which is certainly in $L^1(\mathbb{R},\mu_L)$. By the dominated convergence theorem, we therefore may conclude, that

$$\lim_{k\to\infty}\langle \overline{h_k}, f\rangle_{L^2} = \int_{\mathbb{R}}\frac{1}{2}e^{-|y|}f(0)\mu_L(dy) = \delta(x)(f).$$

(b) For a given $k \in \mathbb{N}$ and a fixed $f \in \mathcal{S}(\mathbb{R})$, one has

$$\begin{aligned}\langle \overline{h_k}, f\rangle_{L^2} &= \int_{\mathbb{R}}\frac{xf(x)}{x^2+k^{-2}}\mu_L(dx)\\ &= \frac{1}{2}\int_{\mathbb{R}} f(x)\left(\frac{1}{x+ik^{-1}}+\frac{1}{x-ik^{-1}}\right)\mu_L(dx)\\ &= -\frac{1}{2}\int_{\mathbb{R}} f'(x)\left(\ln(x+ik^{-1})+\ln(x-ik^{-1})\right)\mu_L(dx)\\ &= \frac{1}{2}\int_{\mathbb{R}} f''(x)\left((x+ik^{-1})\ln(x+ik^{-1})-x+(x-ik^{-1})\ln(x-ik^{-1})-x\right)\mu_L(dx)\\ &= \int_{\mathbb{R}} f''(x)\left(\frac{x}{2}\ln(x^2+k^{-2})-x+\frac{i}{2k}\ln(\frac{x+ik^{-1}}{x-ik^{-1}})\right)\mu_L(dx).\end{aligned}$$

Because $f''(x)$ is rapidly decreasing, the integrand is bounded by $|f''(x)\,(x^2+|x|+|x+1|^2)$, which is in $L^1(\mathbb{R},\mu_L)$. By the dominated convergence theorem, we therefore may conclude, that

$$\lim_{k\to\infty}\langle \overline{h_k}, f\rangle_{L^2} = \int_{\mathbb{R}} x(\ln(|x|)-1)f''(x)\mu_L(dx) = \text{p.v.}(\frac{1}{x})(f).$$

10. Prove that the weak* topology on $\mathcal{S}'(\mathbb{R}^N)$ is a topology (see exercice 5). Prove that $\mathcal{S}'(\mathbb{R}^N)$ is complete when endowed with this topology.

Let $\mathcal{F} \subset \tau(\mathcal{S}'(\mathbb{R}^N),\mathcal{S}(\mathbb{R}^N))$ and $|\mathcal{F}| \in \mathbb{N}$. Let $\varphi \in \cap_{U\in\mathcal{F}}U$. Then $\forall U \in \mathcal{F}$, $x \in U$, and there are for every $U \in \mathcal{F}$ a finite number of Schwartz functions $f_1^U,\dots,f_{n_U}^U$, so that

$$\{\eta \in \mathcal{S}'(\mathbb{R}^N)\,:\,\forall k = 1,\dots,n_U,\,|\eta(f_k^U)-\varphi(f_k^U)|<1\} \subset U.$$

But then, $\cup_{U\in\mathcal{F}}\{f_1^U,\dots,f_{n_U}^U\}$ is also a finite set of Schwartz functions and

$$\{\eta \in \mathcal{S}'(\mathbb{R}^N)\,:\,\forall U\in\mathcal{F},\forall k = 1,\dots,n_U,\,|\eta(f_k^U)-\varphi(f_k^U)|<1\} \subset \cap_{U\in\mathcal{F}}U$$

and $\cap_{U\in\mathcal{F}}U \in \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$.
Let $\mathcal{F} \subset \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ and suppose $\varphi \in \cup_{U\in\mathcal{F}}U$. Then there is an open set $U \in \mathcal{F}$, so that $\varphi \in U$ and there is a finite number of Schwartz functions $f_1^U, \ldots, f_{n_U}^U$, so that

$$\{\eta \in \mathcal{S}'(\mathbb{R}^N) \,:\, \forall k = 1, \ldots, n_U,\, |\eta(f_k^U) - \varphi(f_k^U)| < 1\} \subset U.$$

But then obviously

$$\{\eta \in \mathcal{S}'(\mathbb{R}^N) \,:\, \forall k = 1, \ldots, n_U,\, |\eta(f_k^U) - \varphi(f_k^U)| < 1\} \subset \cup_{U\in\mathcal{F}}U$$

and $\cup_{U\in\mathcal{F}}U \in \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$.
Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence for $\tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$. Then this means, that for any $U \in \tau(\mathcal{S}'(\mathbb{R}^N), \mathcal{S}(\mathbb{R}^N))$ with $\varphi_0 \in U$, there is an $n_U \in \mathbb{N}$, so that $k, l \geq n_U$ implies $\varphi_k - \varphi_l \in U$.
In particular, this means, that for any fixed $f \in \mathcal{S}(\mathbb{R}^N)$ and any $\epsilon > 0$, there is an $N_{f,\epsilon}$, so that for $k, l \geq N_{f,\epsilon}$, $\varphi_k - \varphi_l \in \{\eta \in \mathcal{S}'(\mathbb{R}^N) \,:\, |\eta(\frac{f}{\epsilon})| < 1\}$.
Thus, for any fixed $f \in \mathcal{S}(\mathbb{R}^N)$ and any $\epsilon > 0$, there is an $N_{f,\epsilon}$, so that $k, l \geq N_{f,\epsilon}$ implies $|\varphi_k(f) - \varphi_l(f)| < \epsilon$. Consequently, for any fixed $f \in \mathcal{S}(\mathbb{R}^N)$, $(\varphi_k(f))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and the limit $\lim_k \varphi_k(f)$ exists in $\mathbb{C}$.
By linearity of the limits and all tempered distributions φ_k, this implies, that the map

$$\mathcal{S}(\mathbb{R}^N) \ni f \mapsto \lim_k \varphi_k(f) =: \varphi(f)$$

is a linear functional on $\mathcal{S}(\mathbb{R}^N)$. It remains to be shown, that φ is continuous.
The family of tempered distributions $\{\varphi_k\}_{k\in\mathbb{N}}$ is simply bounded for any $f \in \mathcal{S}(\mathbb{R}^N)$, since $(\varphi_k(f))_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$. By the uniform boundedness principle, this family is therefore equicontinuous, meaning there is an open set $0 \in U \subset \mathcal{S}(\mathbb{R}^N)$, so that $\forall f \in U$, $|\varphi_k(f)| < \frac{1}{2}$ for any $k \in \mathbb{N}$. By simple convergence, $\forall f \in U$, $|\varphi(f)| \leq \frac{1}{2} < 1$, and $\varphi \in \mathcal{S}'(\mathbb{R}^N)$.