Solutions to exercice sheet 4

Brownian motions & probability

1. For $t \neq 0$, the **heat kernel** is given by

$$u(t,x) = \frac{1}{(2\pi Dt)^{3/2}} \exp(-\frac{1}{2Dt}(x\cdot x)).$$

Verify that $\partial_t u(t,x) = \frac{D}{2}\Delta u(t,x)$ with $\lim_{t\to 0^+} \varphi_{u(t,x)} = \delta_0^{(3)}$. Conclude, that if $f : \mathbb{R}^3 \to \mathbb{C}$ is continuous and polynomially bounded, $(u * f)(t)$ is also a solution to the heat equation with $(u*f)(0,x) = f(x)$. Compute the solution with $u(0,x) = 0 \vee x$ in the case $(t,x) \in \mathbb{R}^2$. Substitute t by it and D by $(m\hbar)^{-1}$ in the heat kernel. Verify that now we have a solution $v(t,x)$ for the Schrödinger equation. Prove that if $f \in \mathcal{S}(\mathbb{R}^3)$, $(v*f)(t)$ is a solution with initial value $f(x)$. What happens when $f(x)$ is of compact support?

By direct computation, one gets

$$\partial_t u(t,x) = \partial_t \frac{1}{(2\pi Dt)^{3/2}} \exp(-\frac{1}{2Dt}(x\cdot x))$$

$$= -\frac{3\pi D}{(2\pi Dt)^{5/2}} \exp(-\frac{1}{2Dt}(x\cdot x)) + \frac{x\cdot x}{2Dt^2} u(t,x) = (\frac{x\cdot x}{2Dt^2} - \frac{3}{2t})u(t,x),$$

$$\Delta u(t,x) = \nabla\cdot(\nabla u(t,x)) = \frac{1}{(2\pi Dt)^{3/2}} \nabla\cdot\left(-\frac{x}{Dt}\exp(-\frac{1}{2Dt}(x\cdot x))\right)$$

$$= \frac{1}{(2\pi Dt)^{3/2}}\left(-\frac{3}{Dt}\exp(-\frac{1}{2Dt}(x\cdot x)) + \frac{x\cdot x}{(Dt)^2}\exp(-\frac{1}{2Dt}(x\cdot x))\right)$$

$$= \frac{2}{D}(\frac{x\cdot x}{2Dt^2} - \frac{3}{2t})u(t,x),$$

which yields the heat equation. For a test function $f \in \mathcal{S}(\mathbb{R}^3)$, one has

$$\lim_{t\to 0^+} \varphi_{u(t,x)}(f(x)) = \lim_{t\to 0^+} \int_{\mathbb{R}^3} u(t,x) f(x)\mu_L(dx)$$

$$= \lim_{t\to 0^+} \frac{1}{(2\pi Dt)^{3/2}} \int_{\mathbb{R}^3} \exp(-\frac{1}{2Dt}(x\cdot x)) f(x)\mu_L(dx)$$

$$= \lim_{t\to 0^+} \frac{1}{(2\pi Dt)^{3/2}} \int_{\mathbb{R}^3} \exp(-\frac{1}{2}(y\cdot y)) f(\sqrt{Dt}y)(Dt)^{3/2}\mu_L(dy).$$

The integrand being dominated by $(2\pi)^{-3/2}\exp(-\frac{1}{2}(y\cdot y))|||f|||_0$, which is Lebesgue-summable, one may use the dominated convergence theorem and get

$$\lim_{t\to 0^+} \varphi_{u(t,x)}(f(x)) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \lim_{t\to 0^+}\left(\exp(-\frac{1}{2}(y\cdot y)) f(\sqrt{Dt}y)\right)\mu_L(dy)$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp(-\frac{1}{2}(y\cdot y)) f(0)\mu_L(dy) = f(0) = \delta_0^{(3)}(f).$$

therefore, $\lim_{t\to 0^+} \varphi_{u(t,x)} = \delta_0^{(3)}$ in the weak*-topology.

If $f : \mathbb{R} \to \mathbb{C}$ is continuous and polynomially bounded, then

$$\partial_t (u * f)(t,x) = \partial_t \int_{\mathbb{R}^3} u(t, x-y) f(y) \mu_L(dy)$$
$$= \partial_t \int_{\mathbb{R}^3} u(t,v) f(x-v) \mu_L(dv) = \int_{\mathbb{R}^3} (\partial_t u(t,v))\, f(x-v) \mu_L(dv),$$

where we used the Leibnitz integral rule, which is here legitimised by the fact that for each fixed $x \in \mathbb{R}^3$, the partiel derivative $(\partial_t u(t,v))\, f(x-v)$ is Lebesgue-summable, thanks to the exponential decay in y of $\partial_t u(t,y)$. Applying the heat equation to the heat kernel inside the integral one gets

$$\begin{aligned}
\partial_t (u * f)(t,x) &= \frac{D}{2} \int_{\mathbb{R}^3} (\Delta_v u(t,v))\, f(x-v) \mu_L(dv) \\
&= \frac{D}{2} \int_{\mathbb{R}^3} (\Delta u)(t, x-y) f(y) \mu_L(dy) \\
&= \frac{D}{2} \Delta_x \int_{\mathbb{R}^3} u(t, x-y) f(y) \mu_L(dy),
\end{aligned}$$

where the interchange of the partiel derivatives with respect to x is again justified by the Leibnitz integral rule. Furthermore,

$$\lim_{t\to 0^+} (u * f)(t,x) = \lim_{t\to 0^+} \frac{1}{(2\pi D t)^{3/2}} \int_{\mathbb{R}^3} \exp(-\frac{1}{2Dt}(u\cdot u)) f(x-u) \mu_L(du)$$
$$\underset{v=\frac{u}{\sqrt{t}}}{=} \lim_{t\to 0^+} \frac{|t|^{3/2}}{(2\pi D t)^{3/2}} \int_{\mathbb{R}^3} \exp(-\frac{1}{2D}(v\cdot v)) f(x-\sqrt{t}v) \mu_L(dv).$$

For a fixed value of x and $t \in]0,1[$, the integrand is bounded by $\exp(-\frac{1}{2D}(v\cdot v)) M(v)$, with $M(v) = \max\{|f(s)| : |s-x| \leq |v|\}$, which is certainly polynomially bounded since f is. Therefore, we may interchange limit and integration by dominated convergence and obtain

$$\lim_{t\to 0^+} (u * f)(t,x) = \frac{|t|^{3/2}}{(2\pi D t)^{3/2}} \int_{\mathbb{R}^3} \lim_{t\to 0^+} \exp(-\frac{1}{2D}(v\cdot v)) f(x-\sqrt{t}v) \mu_L(dv)$$
$$= \frac{1}{(2\pi D)^{3/2}} \int_{\mathbb{R}^3} \exp(-\frac{1}{2D}(v\cdot v)) f(x) \mu_L(dv) = f(x).$$

In the special case where $f(x) = 0 \vee x$, we obtain

$$\begin{aligned}
(u * f)(t,x) &= \frac{1}{\sqrt{2\pi D}} \int_{\mathbb{R}} \exp(-\frac{1}{2D} v^2)(0 \vee (x - \sqrt{t}v)) \mu_L(dv) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{Dt}}}^{\infty} \exp(-\frac{1}{2} v^2)(x - \sqrt{Dt}v) \mu_L(dv) \\
&= x\mathcal{N}(\frac{x}{\sqrt{Dt}}) - \frac{\sqrt{Dt}}{\sqrt{2\pi}} \exp(-\frac{x^2}{2Dt}),
\end{aligned}$$

where $\mathcal{N}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{(}2)} dt$.

Substituting now t for it amd D for $\frac{1}{\hbar m}$ in the heat kernel yields

$$v(t,x) = \frac{(m\hbar)^{3/2}}{(2\pi i t)^{3/2}} \exp(\frac{i\hbar m}{2t}(x \cdot x)),$$

where one takes $i^{3/2} = \exp(i\frac{3\pi}{4})$. By direct computation one gets

$$i\hbar\partial_t v(t,x) = i\hbar\partial_t \frac{(m\hbar)^{3/2}}{(2\pi i t)^{3/2}} \exp(\frac{i\hbar m}{2t}(x \cdot x))$$

$$= \frac{3\pi(m\hbar)^{3/2}\hbar}{(2\pi i t)^{5/2}} \exp(\frac{i\hbar m}{2t}(x \cdot x)) + \hbar^2 m \frac{x \cdot x}{2t^2} u(t,x) = \hbar^2 m(\frac{x \cdot x}{2t^2} - \frac{3i}{2m\hbar t})u(t,x),$$

$$\Delta v(t,x) = \nabla \cdot (\nabla v(t,x)) = \frac{(m\hbar)^{3/2}}{(2\pi i t)^{3/2}} \nabla \cdot \left(\frac{i\hbar m x}{t} \exp(\frac{i\hbar m}{2t}(x \cdot x)) \right)$$

$$= \frac{(m\hbar)^{3/2}}{(2\pi i t)^{3/2}} \left(\frac{3i\hbar m}{t} \exp(\frac{i\hbar m}{2t}(x \cdot x)) - \frac{x \cdot x \hbar^2 m^2}{t^2} \exp(\frac{i\hbar m}{2t}(x \cdot x)) \right)$$

$$= -2\hbar^2 m^2 (\frac{x \cdot x}{2t^2} - \frac{3i}{2m\hbar t})v(t,x),$$

from which the Schrödinger equation follows.

If $f \in \mathcal{S}(\mathbb{R})$, then

$$i\hbar\partial_t (v * f)(t,x) = i\hbar\partial_t \int_{\mathbb{R}^3} v(t, x-y) f(y) \mu_L(dy)$$

$$= i\hbar\partial_t \int_{\mathbb{R}^3} v(t,w) f(x-w)\mu_L(dw) = i\hbar \int_{\mathbb{R}^3} (\partial_t u(t,w)) f(x-w) \mu_L(dv),$$

where we used the Leibnitz integral rule, which this time is legitimised by the fact that for each fixed $x \in \mathbb{R}^3$, the partiel derivative $(\partial_t v(t,w)) f(x-w)$ is Lebesgue-summable, thanks to the rapid decay in w of $f(x-w)$. Applying the Schrödinger equation to $v(t,w)$ inside the integral one gets

$$i\hbar\partial_t (v * f)(t,x) = -\frac{1}{2m} \int_{\mathbb{R}^3} (\Delta_w v(t,w)) f(x-w)\mu_L(dw)$$

$$= -\frac{1}{2m} \int_{\mathbb{R}^3} (\Delta v)(t, x-y) f(y) \mu_L(dy)$$

$$= -\frac{1}{2m} \Delta_x \int_{\mathbb{R}^3} v(t, x-y) f(y) \mu_L(dy),$$

where the interchange of the partiel derivatives with respect to x is again justified by the Leibnitz integral rule.

For a fixed $x \in \mathbb{R}^N$ and $t > 0$, one has

$$(v * f)(t,x) = \int_{\mathbb{R}^3} v(t,w) f(x-w) \mu_L(dw) = \varphi_{v(t,w)}(g),$$

where $g(w) := f(x-w)$ is a Schwartz function. By definition, $\varphi_{v(t,w)}(g) = \mathcal{F}(\varphi_{v(t,w)})(\check{g})$. The Fourier transform is weak*-continuous and for this topology, dominated convergence implies that $\lim_{\epsilon\to 0^+} \varphi_{v(t,w)e^{-\frac{\epsilon\hbar m}{2t}w\cdot w}} = \varphi_{v(t,w)}$. Note that for any $\epsilon > 0$, $v(t,w)e^{-\frac{\epsilon\hbar m}{2t}w\cdot w} \in \mathcal{S}(\mathbb{R}^3)$ and a direct computation yields

$$\mathcal{F}(\varphi_{v(t,w)e^{-\frac{\epsilon\hbar m}{2t}w\cdot w}})(p) = \varphi_{\mathcal{F}(v(t,w)e^{-\frac{\epsilon\hbar m}{2t}w\cdot w})},$$

$$\mathcal{F}(v(t,w)e^{-\frac{\epsilon\hbar m}{2t}w\cdot w}) = \frac{1}{\sqrt{2\pi}^3}\frac{(m\hbar)^{3/2}}{(2\pi it)^{3/2}}\int_{\mathbb{R}^3}\exp(-w\cdot w\frac{\hbar m}{2t}(\epsilon - i))\exp(-iw\cdot p)\mu_L(dw)$$

$$= \frac{(m\hbar)^{3/2}}{(2\pi)^3(it)^{3/2}}\int_{\mathbb{R}^3}\exp(-\frac{\hbar m(\epsilon-i)}{2t}(w\cdot w + w\cdot p\frac{i2t}{\hbar m(\epsilon-i)}))\mu_L(dw)$$

$$= \frac{(m\hbar)^{3/2}}{(2\pi)^3(it)^{3/2}}\int_{\mathbb{R}^3} e^{-\frac{\hbar m(\epsilon-i)}{2t}(w+p\frac{it}{\hbar m(\epsilon-i)})\cdot(w+p\frac{it}{\hbar m(\epsilon-i)})}e^{-\frac{tp\cdot p}{2\hbar m(\epsilon-i)}}\mu_L(dw).$$

This complex gaussian integral is now performed using a triple contour integral. For each of the three variables x_1, x_2, x_3, chose a path γ along the line $z = s\left(\frac{\hbar m(\epsilon-i)}{t}\right)^{1/2} + ip\left(\frac{t}{\hbar m(\epsilon-i)}\right)^{1/2}$, $s \in \mathbb{R}$. Note that along this path, $dz = \left(\frac{\hbar m(\epsilon-i)}{t}\right)^{1/2} ds$ and hence,

$$\mathcal{F}(v(t,w)e^{-\frac{\epsilon\hbar m}{2t}w\cdot w}) = \frac{1}{(i\epsilon+1)^{3/2}(2\pi)^3}e^{-\frac{tp\cdot p}{2\hbar m(\epsilon-i)}}\int_{\gamma^3} e^{-\frac{z^2}{2}}dz.$$

Each of the contours γ may now be closed along the real line. The boundary terms of this closed path tend to 0 thanks to the exponential decline of the exponent and we finally get

$$\mathcal{F}(v(t,w)e^{-\frac{\epsilon\hbar m}{2t}w\cdot w}) = \frac{1}{(i\epsilon+1)^{3/2}(2\pi)^{3/2}}e^{-\frac{tp\cdot p}{2\hbar m(\epsilon-i)}}.$$

Taking now the weak*-limit for $\epsilon \to 0^+$ first, followed by the weak*-limit for $t \to 0^+$, we get

$$\lim_{t\to 0^+}\mathcal{F}(\varphi_{v(t,w)}) = \varphi_{\frac{1}{(2\pi)^{3/2}}} \quad\Rightarrow\quad \lim_{t\to 0^+}\varphi_{v(t,w)} = \delta^{(0)}(w).$$

Therefore,

$$\lim_{t\to 0^+}(v*f)(t,x) = f(x).$$

If $f(x)$ has compact support, then so does $f(x-w)$ for a fixed $x \in \mathbb{R}^3$ and

$$(v*f)(t,x) = \int_{\mathbb{R}^3} v(t,x-w)f(w)\mu_L(dw)$$

$$= \frac{(m\hbar)^{3/2}}{(2\pi it)^{3/2}}\int_{B(0,r)}\exp(\frac{i\hbar m}{2t}(x-w)\cdot(x-w))f(w)\mu_L(dw)$$

for some fixed $r > 0$. Substituting $x+iy$ for x the integrand is still bounded on compact sets for w with bounded and summable derivative w.r.t. x or y. One may therefore use the Leibnitz integral rule to conclude, that $(v*f)(t,x+iy)$ is holomorphic for any $x+iy \in \mathbb{C}^3$ and hence entire for any $t > 0$. Such functions cannot have compact supports.

2. Let $X : \Omega \to \mathbb{R}$ be a random variable on a probability space (Ω, Σ, μ) together with its cumulative probability distribution $F_X(x) := \mu(X^{-1}]-\infty, x])$.
Prove that $F_X(x)$ is **càdlàg**, i.e.

- $\forall x, y \in \mathbb{R}$, $x \leq y$ implies $0 \leq F_X(x) \leq F_X(y) \leq 1$,
- $\forall x \in \mathbb{R}$, $\lim_{y\to x^+} F_X(y) = F_X(x)$,
- $\forall x \in \mathbb{R}$, $\lim_{y\to x^-} F_X(y) \leq F_X(x)$.

We begin first by the observation, that if $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ is a set of nested measurable sets of the measure space (Ω, Σ, μ), then $\lim_n \mu(E_n) = \mu(\cup_{n\in\mathbb{N}} E_n)$. Indeed, define iteratively $F_0 := E_0$ and $F_n := E_n \setminus E_{n-1}$. Observe, that all $F_n's$ are disjoint measurable sets and that $E_n = \cup_{k=0}^n F_k$. Hence, by σ-additivity of μ

$$\mu(\cup_{n\in\mathbb{N}} E_n) = \mu(\cup_{n\in\mathbb{N}} F_n) = \sum_{n\geq 0} F_n = \lim_n \mu(E_n).$$

- Let $x, y \in \mathbb{R}$ with $x \leq y$. One has $X^{-1}]-\infty, x] \subset X^{-1}]-\infty, y]$ and thus

  $$F_X(x) = \mu(X^{-1}]-\infty, x]) \leq \mu(X^{-1}]-\infty, y]) = F_X(y).$$

  Moreover, $\forall x \in \mathbb{R}$, since μ is a probability measure, $0 \leq \mu(X^{-1}]-\infty, x]) = F_X(x) \leq 1$.
- For any decreasing sequence $(y_n)_{n\in\mathbb{N}}$ with $\{y_n\}_{n\in\mathbb{N}} \subset]x, \infty[$ and $\lim_n y_n = x$, one has $w \leq x \iff \forall n \geq 0,\ w \leq y_n$. Therefore, $\cap_{n\geq 0}]-\infty, y_n] =]-\infty, x]$, which is equivalent to $\cup_{n\geq 0}]y_n, \infty[=]x, \infty[$. By σ-additivity of μ one finds

  $$\begin{aligned}\lim_n F_X(y_n) &= \lim_n \mu(X^{-1}]-\infty, y_n]) = \lim_n \left(\mu(\Omega \setminus X^{-1}]y_n, \infty[)\right) \\ &= \mu(\Omega) - \lim_n \left(\mu(X^{-1}]y_n, \infty[)\right) = \mu(\Omega) - \left(\mu(\cup_{n\geq 0} X^{-1}]y_n, \infty[)\right) \\ &= \mu(\Omega) - \mu(X^{-1}]x, \infty[) = \mu(X^{-1}]-\infty, x]) = F_X(x).\end{aligned}$$

- For any increasing sequence $(y_n)_{n\in\mathbb{N}}$ with $\{y_n\}_{n\in\mathbb{N}} \subset]-\infty, x[$ and $\lim_n y_n = x$, one has $w \geq x \iff \forall n \geq 0,\ w > y_n$. Therefore, $\cap_{n\geq 0}]y_n, \infty[= [x, \infty[$, which is equivalent to $\cup_{n\geq 0}]-\infty, y_n] =]-\infty, x[$. By σ-additivity of μ one finds

  $$\begin{aligned}\lim_n F_X(y_n) &= \lim_n \mu(X^{-1}]-\infty, y_n]) = \mu(\cup_{n\geq 0} X^{-1}]-\infty, y_n]) \\ &= \mu(X^{-1}]-\infty, x[) \leq \mu(X^{-1}]-\infty, x]) = F_X(x).\end{aligned}$$

3. Prove that any real number $x \in]0,1]$ may be uniquely expressed as $x = \sum_{n\geq 1} \frac{x_n}{3^n}$ where $x_n \in \{0,1,2\}$ and $|\{n \geq 1 : x_n \neq 0\}| = |\mathbb{N}^*|$.
The **Cantor set** $\mathcal{C}$ is then defined as the set of $x \in]0,1]$ so that $x_n \in \{0,2\}$. Prove that $\mu_L(\mathcal{C}) = 0$ and that there is a bijection from $\mathcal{C}$ to $]0,1]$.
For $x \in]0,1] \setminus \mathcal{C}$, denote by N_x as the smallest $n \geq 1$, so that $x_n = 1$. The **Cantor function** is defined as

$$c(x) := \begin{cases} \frac{1}{2}\sum_{n\geq 1} \frac{x_n}{2^n} & \text{if } x \in \mathcal{C}, \\ \frac{1}{2^{N_x}} + \frac{1}{2}\sum_{n=1}^{N_x-1} \frac{x_n}{2^n} & \text{otherwise} \end{cases}.$$

Prove that $c(x)$ is a continuous and increasing function on $]0,1]$. Prove $c(x)$ is locally constant μ_L-a.e. and conclude from these properties, that there is no Lebesgue summable function $d(x)$, so that $c(x) = \int_{[0,x]} d(s)\mu_L(ds)$.

Let $x \in]0,1]$. There are then three mutually exclusive possibilities: $x \in]0,\frac{1}{3}]$ in which case we set $x_1 := 0$, $x \in]\frac{1}{3},\frac{2}{3}]$ in which case we set $x_1 := 1$ or $x \in]\frac{2}{3},1]$ in which case we set $x_1 := 2$. In all cases we will have $0 < x - \frac{x_1}{3} \leq \frac{1}{3}$.

Suppose we are given a finite set $\{x_k\}_{k=1,\dots,n} \subset \{0,1,2\}^n$, so that $0 < x - \sum_{k=1}^n \frac{x_k}{3^k} \leq \frac{1}{3^n}$. We have again three mutually exclusive possibilities:

$x \in]\sum_{k=1}^n \frac{x_k}{3^k}, \sum_{k=1}^n \frac{x_k}{3^k} + \frac{1}{3^{n+1}}]$ in which case we set $x_{n+1} := 0$,
$x \in]\sum_{k=1}^n \frac{x_k}{3^k} + \frac{1}{3^{n+1}}, \sum_{k=1}^n \frac{x_k}{3^k} + \frac{2}{3^{n+1}}]$ in which case we set $x_{n+1} := 1$ or
$x \in]\sum_{k=1}^n \frac{x_k}{3^k} + \frac{2}{3^{n+1}}, \sum_{k=1}^n \frac{x_k}{3^k} + \frac{1}{3^n}]$ in which case we set $x_{n+1} := 2$.
In all cases we will have $0 < x - \sum_{k=1}^n \frac{x_k}{3^k} + \frac{x_{n+1}}{3^{n+1}} \leq \frac{1}{3^{n+1}}$.

We may inductively define in this way a sequence $(x_n)_{n\in\mathbb{N}^*} \in \{0,1,2\}^{\mathbb{N}^*}$, so that $\forall n \geq 1$, $0 < x - \sum_{k=1}^n \frac{x_k}{3^k} \leq \frac{1}{3^n}$. Obviously one has $x = \lim_n \sum_{k=1}^n \frac{x_k}{3^k} = \sum_{k\geq 1} \frac{x_k}{3^k}$. More precisely,

$$\forall n \in \mathbb{N}^*, \quad x_n = \lfloor 3^n x \rfloor, \quad d(x) := (x_n)_{n\in\mathbb{N}^*}.$$

Suppose now that this map $d :]0,1] \to \{0,1,2\}^{\mathbb{N}^*}$ has an image $(x_n)_{n\geq 1}$ with $x_n = 0$ for $n > N$. Set $a = \sum_{k=1}^N \frac{x_k}{3^k}$. By definition of the map d, this means, that the pre-image x to this sequence $(x_n)_{n\geq 1}$ is a number in $]a, a + \frac{1}{3^n}]$ for each $n > N$. But then $x \in \cap_{n>N}]a, a + \frac{1}{3^n}] = \emptyset$. Such an x does not exist and every sequence $(x_n)_{n\geq 1}$ thus obtained verifies $|\{n \in \mathbb{N}^* : x_n \neq 0\}| = |\mathbb{N}|$.

Consider now two such sequences $(x_n)_{n\geq 1} \neq (y_n)_{n\geq 1}$. There is hence some index $n \geq 1$, so that $y_m = x_m$ if $m < n$ and $y_n \neq x_n$. Without loss of generality, one may suppose $x_n < y_n$. By posing $a = \sum_{k=1}^n \frac{y_k}{3^k}$, one notices that by construction of the sequences, $x \leq a < y$. Hence, $x \neq y$ and the map $d :]0,1] \to \{0,1,2\}^{\mathbb{N}^*}$ in question is injective and manifestly surjective when the co-domain is restricted to sequences $\{(x_n)_{n\geq 1} \in \{0,1,2\}^{\mathbb{N}^*} : |\{n \geq 1 : x_n \neq 0\}| = |\mathbb{N}^*|\}$.

The cantor set is then

$$\mathcal{C} := d^{-1}\{(x_n)_{n\geq 1} \in \{0,2\}^{\mathbb{N}^*} : |\{n \geq 1 : x_n \neq 0\}| = |\mathbb{N}^*|\}.$$

$x \notin \mathcal{C}$ is equivalent to $\exists n \geq 1$ so that $x_n = 1$ for $(x_n)_{n\geq 1} = d(x)$. We may identify those sequences iteratively:

- $x_1 = 1$. Then $x \in E_1 :=]\frac{1}{3}, \frac{2}{3}]$.
- $x_1 \neq 1$ and $x_2 = 1$. Then $x \in E_2 :=]\frac{1}{9}, \frac{2}{9}] \cup]\frac{7}{9}, \frac{8}{9}]$.
- $x_1, x_2 \neq 1$ and $x_3 = 1$, Then $x \in E_3 :=]\frac{1}{27}, \frac{2}{27}] \cup]\frac{7}{27}, \frac{8}{27}] \cup]\frac{19}{27}, \frac{20}{27}] \cup]\frac{25}{27}, \frac{26}{27}]$.

- $x_1, \ldots, x_{n-1} \neq 1$ and $x_n = 1$. Then

$$x \in E_n := \cup_{(x_k)\in\{0,2\}^{n-1}}]\frac{1}{3^n} + \sum_{k=1}^{n-1} \frac{x_k}{3^k}, \frac{2}{3^n} + \sum_{k=1}^{n-1} \frac{x_k}{3^k}].$$

By definition, $E_n \cap E_m = \emptyset$ if $n \neq m$ and each E_n is the disjoint union of 2^{n-1} intervals of length $\frac{1}{3^n}$. Therefore, $\mu_L(E_n) = 2^{n-1}\frac{1}{3^n}$ and

$$\mu_L(]0,1] \setminus \mathcal{C}) = \sum_{n\geq 1} \mu_L(E_n) = \frac{1}{3} \sum_{n\geq 0} \frac{2^n}{3^n} = \frac{1}{3} \frac{1}{1 - \frac{2}{3}} = 1.$$

Consequently, $\mu_L(\mathcal{C}) = 0$.
On the other hand, the map $\mathcal{C} \ni x \mapsto y \in]0,1]$ defined by

$$x = \sum_{n\geq 1} \frac{x_n}{3^n} \mapsto y = b(x) := \sum_{n\geq 1} \frac{x_n/2}{2^n}$$

defines a bijection form $\mathcal{C}$ to $]0,1]$, since any y in the image of this map is the binary expression of some number in $]0,1]$.

The function

$$c(x) := \begin{cases} \frac{1}{2} \sum_{n\geq 1} \frac{x_n}{2^n} & \text{if } x \in \mathcal{C}, \\ \frac{1}{2^{N_x}} + \frac{1}{2} \sum_{n=1}^{N_x - 1} \frac{x_n}{2^n} & \text{otherwise} \end{cases}$$

may be rewritten in the following way: if $x \in]0,1]$, set $\bar{x} := \sup]0,x] \cap \mathcal{C}$.
If $x \in \mathcal{C}$, then obviously $x = \bar{x}$.
If $x \notin \mathcal{C}$, then $\bar{x} = \sum_{k=1}^{N_x - 1} \frac{x_k}{3^n} + \sum_{n\geq N_x+1} \frac{2}{3^n}$.
In all cases $c(x) = c(\bar{x})$ and has the following properties:

- c is a surjection from $]0,1]$ to $]0,1]$. Indeed, by the previous surjection $b(x)$, $c\Big|_{\mathcal{C}} =]0,1]$.
- c is increasing. Indeed, if $x, y \in]0,1]$ and $x \leq y$, then $\bar{x} \leq \bar{y}$, so that $\forall n \in \mathbb{N}$ $\bar{x}_n \leq \bar{y}_n$. Thus,

$$c(x) = c(\bar{x}) = \sum_{n\geq 1} \frac{\bar{x}_n}{2^n} \leq \sum_{n\geq 1} \frac{\bar{y}_n}{2^n} = c(\bar{y}) = c(y).$$

Now, an increasing and surjective function $f :]0,1] \to]0,1]$ is also continuous. Indeed, for $x \in]0,1[$, define

$$f(x^-) := \sup\{f(y) \,:\, y < x\} \quad \text{and} \quad f(x^+) := \inf\{f(y) \,:\, y > x\}.$$

By monotonicity of f one obviously has $f(x-) \leq f(x) \leq f(x^+)$ and $f(x-) < f(x^+)$ would imply that the non-empty set $f^{-1}(]f(x^-), f(x^+)[\setminus\{f(x)\}) = \emptyset$, contradicting the surjectivity of f. Thus, $f(x-) = f(x) = f(x^+)$, proving the continuity of f and hence of $c(x)$.

Now, $c(x)$ is also locally constant μ_L-a.e.: for any $n \in \mathbb{N}$ $c(x)$, by definition will take only a discrete set of values on $\mathring{E}_n$. By continuity, this implies that $c(x)$ can only be locally constant on any E_n. Since the union of the$E_n's$ yield back the whole interval $]0, 1]$ but for the Cantor set, we find that $c(x)$ is locally constant μ_L-a.e..

If $f \in \mathcal{L}^1(]0, 1], \mu_L)$ were a function so that $\forall x \in]0, 1]$, $c(x) = \int_0^x f(t)\mu_L(dt)$, then since $c(x)$ is increasing, we should have $f(x) \geq 0$ μ_L-a.e.. Since $c(x)$ is locally constant μ_L-a.e., $f(x)$should be 0 μ_L-a.e.. Therefore, we would have $c(x) = 0$ μ-a.e., which is obviously a contradiction.

4. Let $\Omega := \{\pm 1\}^{\mathbb{N}^*}$ and consider the map

$$f : \Omega \to [0, 1], \quad w \mapsto \sum_{n\geq 1} \frac{1 + w(n)}{2^{n+1}}.$$

Show that this map is surjective and that $\Sigma := \{f^{-1}(E) : E \in \Sigma_L\}$, where Σ_L is the Lebesgue σ-algebra is also a σ-algebra. Show that $\mu(E) := \mu_L(f(E))$ defines a probability measure on Σ.
For a finite sequence $t \in \{\pm 1\}^n$, define $A_t = \{w \in \{\pm 1\}^{\mathbb{N}^*} : \forall k = 1, \ldots, n,\ w(k) = t(k)\}$. Prove that $A_t \in \Sigma$ with $\mu(A_t) = \frac{1}{2^n}$.
For the random walk $\{S_n\}_{n\in\mathbb{N}^*}$, verify the equalities $\mathbb{E}(S_n) = 0$, $\mathbb{V}(S_n) = n$ and $\mu(S_n^{-1}\{n - 2k\}) = 2^{-n}\binom{n}{k}$ for $k = 0, \ldots, n$.

f is surjective, because all $x \in [0, 1]$ can be expanded as a binary sequence $(x_n)_{n\in\mathbb{N}^*} \in \{0, 1\}^{\mathbb{N}^*}$. But then, $f(w) = x$ with $w = (w_n)_{n\in\mathbb{N}^*}$, $w_n = 2x_n - 1$. Let us check that $\Sigma := \{f^{-1}(E) : E \in \Sigma_L\}$ is a σ-algebra:

- $\emptyset = f^{-1}(\emptyset)$, $\Omega = f^{-1}([0, 1]) \in \Sigma$.
- If $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$, then $\exists\{F_n\} \subset \Sigma_L$, so that $\forall n \in \mathbb{N}$, $E_n = f^{-1}(F_n)$. But then, $\cup_n F_n \in \Sigma_L$ for Σ_L is a σ-algebra and
$$\Sigma \ni f^{-1}(\cup_n F_n) = \cup_n f^{-1}(F_n) = \cup_n E_n.$$
- If $E \in \Sigma$ then there is some $F \in \Sigma_L$ so that $E = f^{-1}(F)$. Since Σ_L is a σ-algebra, $[0, 1] \setminus F \in \Sigma_L$ and
$$\Sigma \ni f^{-1}([0, 1] \setminus F) = f^{-1}([0, 1]) \setminus f^{-1}(F) = \Omega \setminus E.$$

All these observations are direct consequences of the definition of pre-images:

$$f^{-1}(\cup_n A_n) := \{w \in \Omega : f(w) \in \cup_n A_n\} = \cup_n\{w \in \Omega : f(w) \in A_n\} = \cup_n f^{-1}(A_n),$$
$$f^{-1}(\cap_n A_n) := \{w \in \Omega : f(w) \in \cap_n A_n\} = \cap_n\{w \in \Omega : f(w) \in A_n\} = \cap_n f^{-1}(A_n).$$

Because f is surjective, $f(f^{-1}(E)) = E$ for any $E \subset [0,1]$ (actually, $f^{-1} : \mathcal{P}([0,1]) \to \mathcal{P}(\Omega)$ is injective). Hence, if $\mu : \Sigma \to [0,1]$ is defined by $\mu(E) := \mu_L(f(E))$, then μ is a probability measure:

- $\mu(\emptyset) = \mu_L(f(\emptyset)) = 0$ and $\mu(\Omega) = \mu_L(f(\Omega)) = \mu_L([0,1]) = 1$.
- If $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ is a family of mutually disjoint sets, then for $n \neq m$, $f(E_n) \cap f(E_m) = \emptyset$ and $\{f(E_n)\}_{n\in\mathbb{N}} \subset \Sigma_L$ are mutually disjoint as well. Thus

$$\mu(\cup_n E_n) = \mu_L(\cup_n f(E_n)) = \sum_{n\geq 0} \mu_L(f(E_n)) = \sum_{n\geq 0} \mu(E_n).$$

For a finite sequence $t \in \{\pm 1\}^n$, if $A_t = \{w \in \{\pm 1\}^{\mathbb{N}^*} : \forall k = 1, \ldots, n,\ w(k) = t(k)\}$, then $f(a_t)$ is the set of numbers in $[0,1]$ whose binary sequences have their first n decimals equal to that of $f(t)$. Thus, $f(A_t) = [f(t), f(t) + \frac{1}{2^n}] \in \Sigma_L$ and $A_t \in \Sigma$ with $\mu(A_t) = \mu_L(f(A_t)) = \mu_L([f(t), f(t) + \frac{1}{2^n}]) = \frac{1}{2^n}$.

The random walk $\{S_n\}_{n\in\mathbb{N}^*}$, given by $S_n(w) := \sum_{k=1}^n w(k)$ is a family of random variables defined on Ω, since per definition, for a given $n \in \mathbb{N}^*$, $\mathrm{im}(S_n) = \{l \in \mathbb{Z} : \exists k = 1, \ldots, n \text{ s.t } l = -n + 2k\}$ and for a given $l \in \mathbb{Z}$, one has

$$S_n^{-1}\{l\} = \{w \in \Omega : \sum_{k=1}^n w(k) = l\} = \cup_{\substack{t\in\{\pm\}^n \text{ s.t.} \\ \sum_{k=1}^n t(k) = l}} A_t.$$

This last set is a finite union of sets in Σ and is thus again a set in Σ. This shows that all S_n's are Σ-measurable functions.
For a given $l = -n + 2k$ with $k = 0, 1, \ldots, n$, if $t \in \{\pm\}^n$ is such that $\sum_{k=0}^n t(k) = l$, then t must be a sequence of k plus steps and $n-k$ minus steps. There are $\binom{n}{k}$ such t's, each of which have a measure equal to $\frac{1}{2^n}$, and so,

$$\mathbb{E}(\mathbb{1}_{S_n = l}) = \binom{n}{k}\frac{1}{2^n} = \binom{n}{\frac{n+l}{2}}\frac{1}{2^n}.$$

Therefore,

$$\mathbb{E}(S_n) = \sum_{k=0}^n (-n+2k)\mathbb{E}(\mathbb{1}_{S_n = -n+2k}) = \sum_{k=0}^n (2k-n)\binom{n}{k}\frac{1}{2^n}$$

$$= \frac{1}{2^n}\left(\sum_{k=0}^n 2k\binom{n}{k}\right) - \frac{n}{2^n}\left(\sum_{k=1}^n \binom{n}{k}\right) = \frac{1}{2^{n-1}}\left(\sum_{k=1}^n n\binom{n-1}{k-1}\right) - \frac{n}{2^n}2^n$$

$$= \frac{1}{2^{n-1}} n 2^{n-1} - n = 0,$$

$$
\begin{aligned}
\mathbb{V}(S_n) &= \mathbb{E}((S_n - \mathbb{E}(S_n))^2) = \mathbb{E}(S_n^2) = \sum_{k=0}^{n} (-n+2k)^2 \mathbb{E}(\mathbb{1}_{S_n=-n+2k}) \\
&= \sum_{k=0}^{n} (2k-n)^2 \binom{n}{k} \frac{1}{2^n} = \sum_{k=0}^{n} (4k^2 - 4kn + n^2) \binom{n}{k} \frac{1}{2^n} \\
&= \frac{1}{2^n} \left(4\sum_{k=0}^{n} k^2 \binom{n}{k} - 4n \sum_{k=0}^{n} k \binom{n}{k} + n^2 \sum_{k=0}^{n} \binom{n}{k} \right) \\
&= \frac{1}{2^n} \left(4\sum_{k=1}^{n} nk \binom{n-1}{k-1} - 4n^2 \sum_{k=1}^{n} \binom{n-1}{k-1} + n^2 2^n \right) \\
&= \frac{1}{2^n} \left(4n^2 + 4n \sum_{k=1}^{n-1} k \left(\binom{n}{k} - \binom{n-1}{k} \right) - 4n^2 2^{n-1} + n^2 2^n \right) \\
&= \frac{1}{2^n} \left(4n^2 + 4n \sum_{k=1}^{n-1} \left(n\binom{n-1}{k-1} - (n-1)\binom{n-2}{k-1} \right) - 4n^2 2^{n-1} + n^2 2^n \right) \\
&= \frac{1}{2^n} \left(4n^2 + 4n^2 2^{n-1} - 4n^2 - 4n(n-1) \sum_{k=1}^{n-1} \binom{n-2}{k-1} - 4n^2 2^{n-1} + n^2 2^n \right) \\
&= \frac{1}{2^n} \left(-4n(n-1) \sum_{k=1}^{n-1} \binom{n-2}{k-1} + n^2 2^n \right) = \frac{1}{2^n} \left(-4n(n-1) 2^{n-2} + n^2 2^n \right) \\
&= \frac{1}{2^n} \left(-n(n-1) 2^n + n^2 2^n \right) = \frac{1}{2^n} n 2^n = n.
\end{aligned}
$$

5. For the random walk $\{S_n\}_{n\in\mathbb{N}^*}$ defined on (Ω, Σ, μ) in the previous exercice, show that $R \in \Sigma$, where

$$R := \{w \in \Omega \,:\, \exists n \geq 2 \text{ s.t. } S_n(w) = 0\}.$$

Show then that $\mu(R) = 1$. Imitate the construction of the last exercice to build a 2-dimensional random walk.

If $w \in R$ then this means, that there is some $n \geq 1$ so that the finite sequence $(w_1, w_2, \ldots, w_n)$ has as many pluses than minuses. Let us call such a finite sequence a balanced sequence. Then,

$$R = \cup_{\substack{t\in\{\pm\}^n, \\ t \text{ balanced and} \\ n\in 2\mathbb{N}^*}} A_t.$$

This is a countable union of sets A_t, all of which are measurable, so that $R \in \Sigma$.

To find the measure of R, observe that all $w \in \Omega$ starting with a $+$ and so that $S_n(w) \leq 0$ at a fixed number of steps n are certainly all elements of R. Starting at $+$, the possible negative values for S_n are $-n + 2k$ with $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Here, k represent the number of pluses in a path $w \in \Omega$, so that $S_1(w) = 1$ and $S_n(w) = -n + 2k \leq 0$. There are $\binom{n-1}{k-1}$ such w's, each having a probability of $\frac{1}{2^n}$. The probability $p_{n,-}$ of $S_1(w) = 1$ and $S_n(w) \leq 0$ is hence

$$p_{n,-} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^n} \binom{n-1}{k-1}.$$

We need also to add here all the w's, so that $S_1(w) = 1$, $S_n(w) \geq 1$, and $S_m(w) = 0$ for some $1 < m < n$. Observe that for such a w corresponds exactly one $\bar{w} \in \Omega$, equal to w up to m and with opposite steps from there on to n. For such a $\bar{w}$, one has $S_1(\bar{w}) = 1$ and $S_n(\bar{w}) = -n + 2l$, with $l = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$. One has again that the probability $p_{n,+}$ of all those samples $\bar{w}$ is

$$p_{n,+} = \sum_{l=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2^n} \binom{n-1}{l-1}.$$

Adding these two probabilities gives

$$p_{n,-} + p_{n,+} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^n} \binom{n-1}{k-1} + \sum_{l=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2^n} \binom{n-1}{l-1}$$

$$= \frac{1}{2^n} \left(\sum_{k=1}^{n-1} \binom{n-1}{k-1} - \binom{n-1}{n-1} \right) = \frac{1}{2^n} (2^{n-1} - 1) = \frac{1}{2} - \frac{1}{2^n}.$$

The same reasoning may now be made for all w's starting with a minus for $n = 1$ and we have that the probability of all $w \in \Omega$ making the random walk $\{S_k\}_{1 \leq k \leq n}$ touch or cross the value 0 is $p_n = 1 - \frac{1}{2^{n-1}}$. Taking the limit for $n \to \infty$ yields that the "return home" probability $\mu(R) = 1$.

To build a 2-dimensional random walk, chose as the sample space $\Omega := \{(\pm, \pm)^{\mathbb{N}^*}\}$ and consider the map

$$f : \Omega \to [0, 1]^2,$$

$$\Omega \ni w \mapsto f(w) := (\sum_{n \geq 1} \frac{w_n \cdot (1, 0)}{2^n}, \sum_{n \geq 1} \frac{w_n \cdot (0, 1)}{2^n}).$$

This is again a surjection and one may define $\Sigma := f^{-1}(\Sigma_L)$ and for $E \in \Sigma$, $\mu(E) := \mu_L(f(E))$.

6. Let X be some random variable defined on a probability space (Ω, Σ, μ). Prove that the characteristic function $\Phi_X(t) := \mathbb{E}(\exp(itX))$ is bounded and continuous. Conclude that $\varphi_{\Phi_X(t)} \in \mathcal{S}'(\mathbb{R})$ and that $\mathcal{F}(\varphi_{\Phi_X(t)})(f) = \sqrt{2\pi}\mathbb{E}(f(X))$ for any $f \in \mathcal{S}(\mathbb{R})$.

By definition,

$$\Phi_X(t) = \mathbb{E}(\exp(itX)) = \int_\Omega \exp(itX(w)) d\mu(w)$$

$$\Rightarrow \quad \left| \Phi_X(t) \right| \leq \int_\Omega |\exp(itX(w))| d\mu(w) = \int_\Omega d\mu = 1.$$

Since for all values of t and all $w \in \Omega$, $|\exp(itX)| \leq 1$, the family $\{\exp(itX)\}_{t\in\mathbb{R}}$ is a family of $\mathcal{L}^1$-bounded measurable functions on (Ω, Σ, μ). For a fixed $w \in \Omega$, one clearly has $\lim_{s\to t} \exp(isX(w)) = \exp(itX(w))$, so that we may invoke dominated convergence to get

$$\lim_{s\to t} \Phi_X(t) = \lim_{s\to t} \int_\Omega \exp(isX) d\mu$$

$$= \int_\Omega \lim_{s\to t} \exp(isX) d\mu = \int_\Omega \exp(itX) d\mu = \Phi_X(t).$$

Clearly, $\varphi_{\Phi_X(t)}$ is then a tempered distribution and if $f \in \mathcal{S}(\mathbb{R})$

$$\mathcal{F}(\varphi_{\Phi_X(t)})(f) = \int_\mathbb{R} \Phi_X(t) \hat{f}(t) dt$$

$$= \int_\mathbb{R} \int_\Omega \exp(itX) \hat{f}(t) d\mu dt = \int_\Omega \int_\mathbb{R} \exp(itX) \hat{f}(t) dt d\mu,$$

where the interchange of integration is justified by Fubini's theorem, since $\exp(itX)\hat{f}(t) \in \mathcal{L}^1(\Omega \times \mathbb{R}, \mu \times \mu_L)$.
Now, for a given $w \in \Omega$,

$$\int_\mathbb{R} \exp(itX(w)) \hat{f}(t) dt = \sqrt{2\pi} \mathcal{F}^*(\hat{f})(X(w)) = \sqrt{2\pi} f(X(w)).$$

It remains to integrate this function on Ω with respect to the probability measure μ which yields the desired result.

7. Let (Ω, Σ, μ) be a probability space and let X be a random variable defined on it. Let Σ_B be the σ-algebra generated by the open intervals of (R). For an open interval $I \subset \mathbb{R}$, define $\mu_X(I) := \mu(X^{-1}I)$. Prove that μ_X may be uniquely extended to Σ_B and that for any $f \in C_b(\mathbb{R}, \mathbb{R})$, $\mathbb{E}(f(X)) = \int_\mathbb{R} f(x) \mu_X(dx)$.

Define

$$\Sigma_X := \{E \subset \Omega \,:\, \exists F \in \Sigma_B \text{ s.t.} \, E = X^{-1}(F)\}.$$

Since $X^{-1}(F \cup G) = X^{-1}(F) \cup X^{-1}(G)$, $X^{-1}(\emptyset) = \emptyset$ and $X^{-1}(F \setminus G) = X^{-1}(F) \setminus X^{-1}(G)$, we have that Σ_X is a σ-algebra.
Since σ_B is the smallest σ-algebra on $\mathcal{R}$ containing all open sets of $\mathbb{R}$ and since all sets of the form $X^{-1}(]a, \infty[) \in \Sigma$ by measurability of X, we conclude that $\Sigma_X \subset \Sigma$ and thus, μ is well-defined on Σ_X.

The function

$$\Sigma_B \ni E \mapsto \mu_X(E) := \mu(X^{-1}(E))$$

is then a well-defined probability measure on Σ_B. For some $f \in C_b(\mathbb{R}, \mathbb{R}_+)$, we have by definition

$$\mathbb{E}(f(X)) = \int_\Omega f(X) d\mu$$

$$= \sup\{\sum_{y\in\mathrm{Im}(s)} y\mu(s^{-1}\{y\}) \,:\, s : \Omega \to \mathbb{R},\, 0 \leq s \leq f(X),\, s \text{ is a } \Sigma\text{-simple function}\}$$

8. Define

$$r(x) := \begin{cases} \left(e^{ix} - 1 - ix + \frac{x^2}{2}\right) x^{-2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Show that $r(x)$ is continuous and bounded on $\mathbb{R}$. Use this to show, that for some random variable X on some probability space (Ω, Σ, μ) with $X \in \mathcal{L}^1(\Omega, \mu) \cap \mathcal{L}^2(\Omega, \mu)$ and $\mathbb{E}(X) = 0$, the characteristic function

$$\mathbb{E}(e^{itX}) = 1 - \frac{t^2}{2}\mathbb{V}(X) + t^2 h(t),$$

with $\lim_{t\to 0} h(t) = 0$ and $h(t)$ is bounded.

Continuity of $r(x)$ may only be problematic for $x = 0$. By Bernoulli's rule, one has

$$\lim_{x\to 0} r(x) = \lim_{x\to 0} \frac{e^{ix} - 1 - ix + \frac{x^2}{2}}{x^2} = \lim_{x\to 0} \frac{ie^{ix} - i + x}{2x} = \lim_{x\to 0} \frac{-e^{ix} + 1}{2} = 0 = r(0).$$

Hence, $r(x)$ is continuous on $\mathbb{R}$.
By continuity of $r(x)$, the number $M := \max\{|r(x)| : |x| \leq 1\}$ certainly exists since $[-1, 1]$ is compact. For $|x| > 1$, one has

$$|r(x)| = |\frac{e^{ix} - 1}{x^2} - \frac{i}{x} + \frac{1}{2x^2}| < 4,$$

so that $\forall x \in \mathbb{R}$, $|r(x)| \leq (4 \vee M)$.

We use this to compute

$$\begin{aligned} \mathbb{E}(e^{itX}) &= \mathbb{E}\left(1 + itX - \frac{t^2X^2}{2} + t^2X^2 r(tX)\right) \\ &= 1 + it\mathbb{E}(X) - \frac{t^2}{2}\mathbb{V}(X) + t^2 \underbrace{\mathbb{E}(X^2 r(tX))}_{h(t)}, \end{aligned}$$

and it remains to show, that $h(t)$ is well-defined, bounded and that $\lim_{t\to 0} h(t) = 0$. By the previous part of this exercice, $r(x)$ is bounded, by B, say, so that $|X^2 r(tX)| \leq X^2 B$ which is in $\mathcal{L}^1(\Omega, \mu)$. This shows that $h(t)$ is well-defined, bounded by $B\mathbb{V}(X)$, and by dominated convergence,

$$\lim_{t\to 0} \mathbb{E}(X^2 r(tX)) = \mathbb{E}(\lim_{t\to 0} X^2 r(tX)) = \mathbb{E}(X^2 r(0)) = 0.$$

□