Solutions to exercice sheet 3

Fourier transforms

1. Prove that if an associative multiplication $\times$ on $\mathcal{S}'(\mathbb{R})$ satisfies

- $\forall \varphi \in \mathcal{S}'(\mathbb{R}), \quad \varphi_0 \times \varphi = \varphi_0$ and $\varphi \times \varphi_1 = \varphi = \varphi_1 \times \varphi$,
- for any polynomially bounded and μ_L-measurable functions f, g,
$$\varphi_f \times \varphi_g = \varphi_{fg} \quad \text{and} \quad \varphi_{f+g} = \varphi_f + \varphi_g,$$
- for any polynomially bounded and μ_L-measurable functions f, g,
$$f(x) = \int_0^x g(t)\mu_L(dt) \quad \Rightarrow \quad D\varphi_f = \varphi_g,$$
- $\forall \varphi, \eta \in \mathcal{S}'(\mathbb{R}), \quad D(\varphi \times \eta) = (D\varphi) \times \eta + \varphi \times (D\eta),$

then $\varphi_x \delta_0 = \varphi_0 = \delta_0 \times \varphi_x$ and $\varphi_x \times \text{P.v.}(\frac{1}{x}) = \varphi_1$, where $\delta_0 = D\varphi_\Theta$ and $\text{P.v.}(\frac{1}{x}) = D^2(x(\ln(|x|) - 1)$.

Denote the Heavyside function $1_{\mathbb{R}_+}(x)$ by $\Theta(x)$. One has

$$\begin{aligned}
\varphi_0 = \varphi_\Theta - \varphi_\Theta &= D(\varphi_{0\vee x}) - \varphi_\Theta = D(\varphi_x \times \varphi_\Theta) - \varphi_\Theta \\
&= D(\varphi_x) \times \varphi_\Theta + \varphi_x \times D(\varphi_\Theta) - \varphi_\Theta \\
&= \varphi_1 \times \varphi_\Theta + \varphi_x \times \delta_0 - \varphi_\Theta = \varphi_x \times \delta_0,
\end{aligned}$$

$$\begin{aligned}
\varphi_0 = \varphi_\Theta - \varphi_\Theta &= D(\varphi_{0\vee x}) - \varphi_\Theta = D(\varphi_\Theta \times \varphi_x) - \varphi_\Theta \\
&= D(\varphi_\Theta) \times \varphi_x + \varphi_\Theta \times D(\varphi_x) - \varphi_\Theta \\
&= \delta_0 \times \varphi_x + \varphi_\Theta \times \varphi_1 - \varphi_\Theta = \delta_0 \times \varphi_x,
\end{aligned}$$

$$\begin{aligned}
\varphi_x \times \text{p.v.}(\frac{1}{x}) &= \varphi_x \times D^2(\varphi_{x(\ln(|x|)-1)}) \\
&= D(\varphi_x \times D\varphi_{x(\ln(|x|)-1)}) - D(\varphi_x) \times D\varphi_{x(\ln(|x|)-1)} \\
&= D^2(\varphi_x \times \varphi_{x(\ln(|x|)-1)}) - D((D\varphi_x) \times \varphi_{x(\ln(|x|)-1)}) - \varphi_1 \times D\varphi_{x(\ln(|x|)-1)} \\
&= D^2(\varphi_{x^2(\ln(|x|)-1)}) - D(\varphi_1 \times \varphi_{x(\ln(|x|)-1)}) - \varphi_1 \times D\varphi_{x(\ln(|x|)-1)} \\
&= D(\varphi_{2x(\ln(|x|)-1)+x}) - \varphi_1 \times D(\varphi_{x(\ln(|x|)-1)}) - \varphi_1 \times D\varphi_{x(\ln(|x|)-1)} \\
&= D(2\varphi_{x(\ln(|x|)-1))} + \varphi_x) - 2D(\varphi_{x(\ln(|x|)-1)}) = D\varphi_x = \varphi_1.
\end{aligned}$$

2. Prove the **Leibnitz integral rule**:

Let $U \subset \mathbb{R}$ be open, and let (X, Σ, μ) be a measure space. Suppose $f : U \times X \to \mathbb{K}$ satisfies:

- For each $t \in U$, $f \in \mathcal{L}^1(X, \mu)$,
- there is a μ-null set $N \in \Sigma$, so that $\partial_t f(t, x)$ exists $\forall t \in U$ and $\forall x \in X \setminus N$,
- $\forall t \in U$, $\exists r > 0$, $\exists g_{t,r} \in \mathcal{L}^1(X, \mu)$, s.t. $|\partial_s f(s, x)| \leq g_{t,r}$ $\forall s \in]t - r, t + r[$ and $\forall x \in X \setminus N$.

Then $\frac{d}{dt} \int_X f(t, x) d\mu = \int_{X \setminus N} \partial_t f(t, x) d\mu$.

Use the Leibnitz integral rule to show, that $\partial^\alpha \widehat{f(x)}(p) = \widehat{(-ix)^\alpha f}(x)(p)$ and that $(ip)^\alpha \widehat{f(x)}(p) = \widehat{\partial^\alpha f(x)}$.

By the first property, $I(t) := \int_X f d\mu$ exists for all $t \in U$.
Let $t \in U$ be fixed and consider $h \in \mathbb{R}$ with $|h| < r$, with r given by the third property. Then by Rolle's theorem and for a μ-null set $N \in \Sigma$ given by the second property, one has

$$\frac{1}{h} \left(I(t + h) - I(t) \right) = \int_X \frac{1}{h} \left(f(t + h, x) - f(t, x) \right) d\mu$$

$$= \int_{X \setminus N} \frac{1}{h} \left(f(t + h, x) - f(t, x) \right) d\mu = \int_{X \setminus N} (\partial_t f)(\tau, x) d\mu.$$

for some $|\tau| < h$. By the third criterion, the integrand is bounded by the $\mathcal{L}^1$-function $g_{r,t}$ on all of $X \setminus N$, so that one may apply dominated convergence to get

$$\frac{d}{dt} I(t) = \lim_{h \to 0} \int_{X \setminus N} \frac{1}{h} \left(f(t + h, x) - f(t, x) \right) d\mu = \int_{X \setminus N} \partial_t f(t, x) d\mu.$$

If $f \in \mathcal{S}(\mathbb{R}^N)$ and if $\delta \in \mathbb{N}^N_{=1}$, then for $k \cdot \delta \in \mathbb{R}$ and fixed values of $k \cdot (\overline{1} - \delta)$, one has that $\exp(-ik \cdot x) f(x)$ satisfies all the criterions for Leibnitz's integral rule (note that N may be even chosen to be the empty set and r as large as desired). Hence,

$$\partial^\delta \hat{f}(p) = \frac{d}{d(\delta \cdot k)} \int_{\mathbb{R}^N} \exp(-ik \cdot x) f(x) \mu_L(dx)$$

$$= \int_{\mathbb{R}^N} \partial_k^\delta \exp(-ik \cdot x) f(x) \mu_L(dx) = \int_{\mathbb{R}^N} (-ix)^\delta \exp(-ik \cdot x) f(x) \mu_L(dx).$$

By iteration on δ, one has the announced result for any $\alpha \in \mathbb{N}^N$.
Applying integration by parts yields the second equality.

3. Prove that the Fourier transform is a continuous endomorphism on $\mathcal{S}(\mathbb{R}^N)$. (Hint: use continuity of the maps ∂^α and $(-ix)^\alpha$ previously proven.)

For $f \in \mathcal{S}(\mathbb{R}^N)$, one has

$$\left| \hat{f}(p) \right| \leq \int_{\mathbb{R}^N} |\exp(-ip \cdot x) f(x)| \mu_L(dx)$$

$$\leq \int_{\mathbb{R}^N} (1 + x \cdot x)^{-N} |(1 + x \cdot x)^N f(x)| \mu_L(dx) \leq \left(\int_{\mathbb{R}^N} (1 + x \cdot x)^{-N} \mu_L(dx) \right) |\!|\!| f |\!|\!|_N.$$

Therefore, $\|\hat{f}\|_\infty \leq \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)\|\!|f|\!\|_N$. By continuity of the maps ∂^α and x^β (see previous exercice sheet), one has for $n\in\mathbb{N}$ and $\alpha\in\mathbb{N}^N_{\leq n}$,

$$\begin{aligned}
\|(1+p\cdot p)^n\partial^\alpha\hat{f}\|_\infty &= \|\widehat{(1-\nabla\cdot\nabla)^n\left((-ix)^\alpha f(x)\right)}\|_\infty \\
&\leq \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)\|\!|(1-\nabla\cdot\nabla)^n\left((-ix)^\alpha f(x)\right)|\!\|_N \\
&= \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)\left|\!\left|\!\left|\left(\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}\Delta^k\right)\left((-ix)^\alpha f(x)\right)\right|\!\right|\!\right|_N \\
&\leq \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)\sum_{k=0}^n\binom{n}{k}\|\!|\Delta^k\left((-ix)^\alpha f(x)\right)|\!\|_N \\
&= \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)\sum_{k=0}^n\binom{n}{k}\left|\!\left|\!\left|\left(\sum_{\beta\in\mathbb{N}^N_{=k}}\frac{k!}{\beta!}\partial^{2\beta}\right)\left((-ix)^\alpha f(x)\right)\right|\!\right|\!\right|_N \\
&\leq \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)\sum_{\beta\in\mathbb{N}^N_{\leq n}}\binom{n}{|\beta|}\frac{|\beta|!}{\beta!}\|\!|\partial^{2\beta}\left((-ix)^\alpha f(x)\right)|\!\|_N \\
&\leq \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)\sum_{\gamma\in\mathbb{N}^{N+1}_{=n}}\frac{n!}{\gamma!}\|\!|(-ix)^\alpha f(x)|\!\|_{N+2n} \\
&\leq \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)(N+1)^n\binom{N+n}{n}\alpha!\|\!|f(x)|\!\|_{N+2n+|\alpha|}.
\end{aligned}$$

This shows, that for a given $n\in\mathbb{N}$ and a fixed $\alpha\in\mathbb{N}^N$, $(1+p\cdot p)^n\partial^\alpha\hat{f}$ remains bounded in $p\in\mathbb{N}$. Therefore, $\hat{f}(p)\in\mathcal{S}(\mathbb{R}^N)$.
Taking the supremum over all $p\in\mathbb{N}$ and $\alpha\in\mathbb{N}^N$ yields

$$\left|\!\left|\!\left|\hat{f}\right|\!\right|\!\right|_n \leq \left(\int_{\mathbb{R}^N}(1+x\cdot x)^{-N}\mu_L(dx)\right)(N+1)^n\binom{N+n}{n}n!\|\!|f(x)|\!\|_{N+3n},$$

which shows continuity of $\mathcal{F}$ for $\tau_{\mathcal{S}}$.

4. Use a contour integral to prove that

$$\mathcal{F}(\exp(-\frac{x\cdot x}{2}))(p) = \exp(-\frac{p\cdot p}{2}).$$

Consider then for $\epsilon>0$ the integrals

$$I_\epsilon(x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^{2N}} f(y)\exp(ik\cdot(x-y))\exp(-\epsilon^2\frac{k\cdot k}{2})\mu_L(dk\times dy).$$

Use Fubini and dominated convergence, to prove that

$$\mathcal{F}\circ\mathcal{F}^* = \mathbb{1}_{\mathcal{S}(\mathbb{R}^N)} = \mathcal{F}^*\circ\mathcal{F}.$$

The function $\exp(-\frac{z^2}{2})$ is certainly analytical over the whole complex plane (hence it is an entire function). Any integration $\oint_\gamma f(z)dz$ over a closed contour $\gamma\subset\mathbb{C}$ is therefore going to be 0. In particular, this is true if $\gamma = \gamma_{h1} \cup \gamma_{v1} \cup \gamma_{h2} \cup \gamma_{v2}$ with $\gamma_{h1} := [-R, R]$, $\gamma_{h2} := ia + [-R, R]$, $\gamma_{v1} := [R, R + ia]$ and $\gamma_{v2} := [-R + ia, -R]$. It is however easy to see, that $\left| \oint_{\gamma_{vj}} \exp(-z^2/2)dz \right| \leq \exp(-R^2/2)|a| \exp(a^2/2)$ for $j = 1, 2$, so that for a fixed value of a $\lim_{R\to\infty} \oint_{\gamma_{vj}} \exp(-z^2/2)dz = 0$ for $j = 1, 2$ and consequently,

$$\lim_{R\to\infty} \int_{-R}^{R} \exp(-\frac{x^2}{2})dx = \lim_{R\to\infty} \int_{-R}^{R} \exp(-\frac{(x + ia)^2}{2})dx.$$

Now, consider the one-dimensional Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-ikx) \exp(-\frac{x^2}{2})\mu_L(dx) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \exp(-ikx) \exp(-\frac{x^2}{2})dx$$

$$= \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \exp(-\frac{1}{2}(x^2 + 2ikx))dx = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \exp(-\frac{1}{2}(x + ik)^2 - \frac{k^2}{2})dx$$

$$= \lim_{R\to\infty} \exp(-\frac{k^2}{2}) \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \exp(-\frac{1}{2}x^2)dx.$$

To conclude, it remains to show, that $I := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\frac{1}{2}x^2)\mu_L(dx) = 1$. In order to do this, we compute

$$I^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-\frac{1}{2}x^2)\mu_L(dx) \int_{\mathbb{R}} \exp(-\frac{1}{2}y^2)\mu_L(dy)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(-\frac{1}{2}(x^2 + y^2))\mu_L(dx \times dy) = \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_0^{2\pi} \exp(-\frac{1}{2}r^2) r \mu_L(dr \times d\theta)$$

$$= \int_{\mathbb{R}_+} \exp(-\frac{1}{2}r^2) r \mu_L(dr) = -\exp(-r^2/2)\Big|_0^\infty = 1.$$

From this we derive the multi-dimensional case

$$\mathcal{F}(\exp(-\frac{x \cdot x}{2}))(p) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp(-ip \cdot x) \exp(-\frac{x \cdot x}{2})\mu_L(dx)$$

$$= \prod_{j=1}^{N} \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}^N} \exp(-ip_j x_j) \exp(-\frac{x_j^2}{2})\mu_L(dx_j) = \exp(-\frac{p \cdot p}{2}).$$

For $\epsilon > 0$, consider now the auxiliary integrals

$$I_\epsilon(x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} f(y) \exp(ik \cdot (x - y)) \exp(-\epsilon^2 \frac{k \cdot k}{2})\mu_L(dk \times dy).$$

The integrand is certainly in $L^1(\mathbb{R}^{2N}, \mu_L)$ for all $\epsilon > 0$. We may apply Fubini's theorem to chose to integrate over y first and obtain

$$I_\epsilon(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \hat{f}(k) \exp(ik \cdot x) \exp(-\epsilon^2 \frac{k \cdot k}{2})\mu_L(dk).$$

Since $\hat{f} \in \mathcal{S}(\mathbb{R}^N)$, we have that the integrand is in $L^1(\mathbb{R}^N, \mu_L)$ for all $\epsilon > 0$ again and is bounded bounded by $|\hat{f}(k)|$. Applying the dominated convergence theorem, one gets

$$\lim_{\epsilon\to 0^+} I_\epsilon(x) = \mathcal{F}^*(\hat{f})(x) = (\mathcal{F}^* \circ \mathcal{F})(f)(x).$$

If, on the other hand, one first integrates the defining integral of $I_\epsilon(x)$ over k, one gets

$$\begin{aligned} I_\epsilon(x) &:= \frac{1}{(2\pi\epsilon^2)^{N/2}} \int_{\mathbb{R}^N} f(y) \exp\left(-\frac{(x-y)\cdot(x-y)}{2\epsilon^2}\right)\mu_L(dy) \\ &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x-\epsilon u) \exp\left(-\frac{u\cdot u}{2}\right)\mu_L(du). \end{aligned}$$

These integrands are bounded by $\|f\|_\infty \exp(-|u|^2/2)$ and applying dominated convergence again, one obtains

$$\lim_{\epsilon\to 0^+} I_\epsilon(x) = f(x) = (\mathcal{F}^* \circ \mathcal{F})(f)(x).$$

The equality $\mathcal{F}\circ\mathcal{F}^* = \mathbb{1}_{\mathcal{S}(\mathbb{R}^N)}$ is obtained by the same argument after substitution of k by $-k$.

5. For a given $\epsilon > 0$ define the following functions

$$\begin{aligned} \hat{\Delta}^C_{\epsilon,\pm} &:= \frac{-1}{(2\pi)^2}\frac{1}{(p_0 \mp i\epsilon + E_p)(p_0 \mp i\epsilon - E_p)}, \\ \hat{\Delta}^F_{\epsilon,\pm} &:= \frac{-1}{(2\pi)^2}\frac{1}{(p_0 \mp i\epsilon + E_p)(p_0 \pm i\epsilon - E_p)}, \end{aligned}$$

where $E_p = \sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}$. Show that $\hat{\Delta}^C_\pm := \lim_{\epsilon\to 0^+} \varphi_{\hat{\Delta}^C_{\epsilon,\pm}}$ and $\hat{\Delta}^F_\pm := \lim_{\epsilon\to 0^+} \varphi_{\hat{\Delta}^C_{\epsilon,\pm}}$ exist in $\mathcal{S}'(\mathbb{R}^4)$.
Show then that $(\Box + m^2)\Delta^C_\pm = -\delta_0^{(4)} = (\Box - m^2)\Delta^F_\pm$.

By definition, for a given $\epsilon > 0$, one has for $f \in \mathcal{S}(\mathbb{R}^4)$,

$$\begin{aligned} \varphi_{\hat{\Delta}^C_{\epsilon,\pm}}(\check{f}) &= \int_{\mathbb{R}^4} \hat{\Delta}^C_{\epsilon,\pm}(p)\check{f}(p)\mu_L(dp) \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{-1}{(p_0 \mp i\epsilon + E_p)(p_0 \mp i\epsilon - E_p)}\check{f}(p)\mu_L(dp) \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \left(\frac{1}{(p_0 \mp i\epsilon + E_p)} - \frac{1}{(p_0 \mp i\epsilon - E_p)}\right)\check{f}(p)\frac{\mu_L(dp)}{2E_p} \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \left(\log(p_0 \mp i\epsilon - E_p) - \log(p_0 \mp i\epsilon + E_p)\right)\frac{\partial \check{f}(p)}{\partial p_0} dp_0 \frac{d^3p}{2E_p} \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} (p_0 \mp i\epsilon + E_p)\left(\log(p_0 \mp i\epsilon + E_p) - 1\right)\frac{\partial^2 \check{f}(p)}{(\partial p_0)^2} dp_0 \frac{d^3p}{2E_p} \\ &\quad - \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} (p_0 \mp i\epsilon - E_p)\left(\log(p_0 \mp i\epsilon - E_p) - 1\right)\frac{\partial^2 \check{f}(p)}{(\partial p_0)^2} dp_0 \frac{d^3p}{2E_p}, \end{aligned}$$

where $\log(z) = \ln(|z|) + \text{p.arg.}(z)$. In this form, the limit $\epsilon \to 0^+$ may be taken and

$$\hat{\Delta}^C_\pm(\hat{f}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} g^C_\pm(p)\frac{\partial^2 \check{f}(p)}{(\partial p_0)^2}dp_0\frac{d^3p}{2E_p}$$

$$\text{with } g^C_\pm(p) = (p_0+E_p)\Big(\ln(|p_0+E_p|) - 1 \pm i\frac{\pi}{2}(\text{sgn}(p_0+E_p)-1)\Big)$$
$$-(p_0-E_p)\Big(\ln(|p_0-E_p|) - 1 \pm i\frac{\pi}{2}(\text{sgn}(p_0-E_p)-1)\Big).$$

This function is piece-wise continuous, polynomially bounded and with jumps on the hyperbolic naps $p_0 = \pm E_p$. $\hat{\Delta}^C_\pm$ are therefore tempered distributions, which by construction are weak* limits of the tempered distributions $\varphi_{\hat{\Delta}^C_{\epsilon,\pm}}$.
Since multiplication of tempered distributions by continuous and polynomially bounded functions are operations which are also weak*-continuous, one has for any $f \in \mathcal{S}(\mathbb{R}^N)$,

$$(p_0^2 - E_p^2)\hat{\Delta}^C_\pm(\check{f}) = \hat{\Delta}^C_\pm((p_0^2-E_p^2)\check{f}) = \lim_{\epsilon\to 0^+}\varphi_{\hat{\Delta}^C_{\epsilon,\pm}}((p_0^2-E_p^2)\check{f})$$
$$= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\frac{-1}{(p_0 \mp i\epsilon + E_p)(p_0 \mp i\epsilon - E_p)}(p_0^2-E_p^2)\check{f}(p)\mu_L(dp)$$
$$= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}(\mathbb{1}_{\{p\in\mathbb{R}^4 \,:\, p_0^2-E_p^2=0\}} - 1)\check{f}(p)\mu_L(dp).$$

Since the set $\{p \in \mathbb{R}^4 \,:\, p_0^2 - E_p^2 = 0\}$ is Lebesgue negligible, one has, after a Fourier transform, that

$$\Box\Delta^C_\pm = \delta^{(4)}_0.$$

Similarly,

$$\hat{\Delta}^F_\pm(\check{f}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} g^F_\pm(p)\frac{\partial^2 \check{f}(p)}{(\partial p_0)^2}dp_0\frac{d^3p}{2E_p}$$

$$\text{with } g^F_\pm(p) = (p_0+E_p)\Big(\ln(|p_0+E_p|) - 1 \pm i\frac{\pi}{2}(\text{sgn}(p_0+E_p)-1)\Big)$$
$$-(p_0-E_p)\Big(\ln(|p_0-E_p|) - 1 \mp i\frac{\pi}{2}(\text{sgn}(p_0-E_p)-1)\Big).$$

6. Consider the Schrödinger equation $i\partial_t\psi(t) = -\frac{1}{2m}\Delta\Psi(t)$ together with an $L^2(\mathbb{R}^3,\mu_L)$ solution $\Psi(t)$ to it, i.e.:

- $(1+p_1^2+p_2^2+p_3^2)\widehat{\Psi(0)}(p) \in L^2(\mathbb{R}^3,\mu_L)$,
- $\forall t \in \mathbb{R}, \quad \widehat{\Psi(t)}(p) = \exp(-i\frac{t(p_1^2+p_2^2+p_3^2)}{2m})\widehat{\Psi(0)}(p)$.

Suppose $\text{supp}(\Psi(0)) \subset B(0,r)$. What may one conclude on $\text{supp}(\Psi(t))$?
(Hint: use Schwarz's Paley & Wiener theorem).

Since $(1+p_1^2+p_2^2+p_3^2)\widehat{\Psi(0)}(p) \in L^2(\mathbb{R}^3,\mu_L)$, then both $\widehat{\Psi(0)}(p) \in L^2(\mathbb{R}^3,\mu_L)$ and $(p_1^2+p_2^2+p_3^2)\widehat{\Psi(0)}(p) \in L^2(\mathbb{R}^3,\mu_L)$.
Therefore, $\Psi(0) \in L^2(\mathbb{R}^3,\mu_L)$ by inverse Fourier transform and $\varphi_{\Psi(0)}$ is a tempered distribution whose support lies in $B(0,r)$ if $\text{supp}(\Psi(0)) \subset B(0,r)$.

Schwartz's Paley & Wiener theorem hence applies and $\mathcal{L}(\Psi(0), q)(p)$ is an entire fonction $F(p+iy)$ for which there is some $n \in \mathbb{N}$ so that $|F(p+iy)| \leq C_{n,\Psi}(1+|p+iy|)^n e^{r|y|}$. Obviously, $F(p+i0) = \widehat{\Psi(0)}(p)$.
Set then $F_t(p+iy) := \exp(\frac{-it}{2m}(p+iy)\cdot(p+iy))F(p+iy)$, which is clearly the only analytical extension of $F_t(p+i0) = \widehat{\Psi(t)}(p)$. Clearly, $F_t(p+iq)$ is entire and $|F_t(p+iq)| \leq C_{n,\Psi}(1+|p+iy|)^n e^{(r+\frac{t}{m}|p|)|y|}$. Note that the exponential term is not of the type $e^{r'|y|}$: since $\Psi(0)$ is compact, $\widehat{\Psi(t)}(p)$ is entire and does not vanish outside any ball $B(0,r')$ for any $r' > 0$.

7. Consider the relativistic Fourier transform $\hat{f}(p) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \exp(-ip^t\eta x)f(x)\mu_L(dx)$. For a fixed $f \in \mathcal{S}(\mathbb{R}^4)$, let $C_f := \{y \in \mathbb{R}^4 \,:\, \forall x \in \operatorname{supp}(f),\, y^t\eta x \leq 0\}$.
Show that C_f is convex and that on $p+iy \in \mathbb{R}+i\mathring{C}_f$, $\mathcal{L}(f,y)(p)$ is well-defined, holomorphic and obeys the same estimates as in Paley & Wiener's theorem, but for the exponential term.

Let $y, y' \in C_f$. Then, for $t \in [0,1]$ and $x \in \operatorname{supp}(f)$, $(ty+(1-t)y')^t\eta x = ty^t\eta x + (1-t){y'}^t\eta x \leq 0$, so that $ty+(1-t)y' \in C_f$ again.
Now by definition, for $p+iy \in \mathbb{R}+iC_f$, one has

$$\begin{aligned}\mathcal{L}(f,y)(p) &= \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \exp(-i(p+iy)\cdot x)f(x)\mu_L(dx)\\ &= \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \exp(-ip\cdot x)\exp(y\cdot x)f(x)\mu_L(dx)\end{aligned}$$

and $y \cdot x$ being negative for all $x \in \operatorname{supp}(f)$, this yields an integrand which is again a Schwartz function. The Leibnitz integral criterion is therefore valid for both variables p and y and C_f being convex, differentiation with respect to y is well-defined. A direct computation gives for some $\delta \in \mathbb{N}^N_{=1}$

$$\begin{aligned}\partial_p^\delta \mathcal{L}(f,y)(p) &= \partial_p^\delta \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \exp(-ip\cdot x)\exp(y\cdot x)f(x)\mu_L(dx)\\ &= \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \partial_p^\delta \exp(-ip\cdot x)\exp(y\cdot x)f(x)\mu_L(dx)\\ &= \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} (-i\delta\cdot x)\exp(-ip\cdot x)\exp(y\cdot x)f(x)\mu_L(dx)\\ &= -i\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \partial_y^\delta \exp(-ip\cdot x)\exp(y\cdot x)f(x)\mu_L(dx)\\ &= -i\partial_y^\delta \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \exp(-ip\cdot x)\exp(y\cdot x)f(x)\mu_L(dx) = -i\partial_y^\delta\mathcal{L}(f,y)(p),\end{aligned}$$

which is just the Cauchy-Riemann equation for $\mathcal{L}(f,y)(p)$ and proves it being holomorphic.
Let z^α be some monomial expression for $\alpha \in \mathbb{N}^N$ and $z = p+iy \in \mathbb{R}^N + iC_f$. We

then have by partiel integration:

$$\begin{aligned}
|z^\alpha \mathcal{L}(f,y)(p)| &= \Big|\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} z^\alpha \exp(-ip\cdot x)\exp(y\cdot x) f(x)\mu_L(dx)\Big| \\
&= \Big|\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \left((i)^\alpha \partial_x^\alpha \exp(-ip\cdot x)\exp(y\cdot x)\right) f(x)\mu_L(dx)\Big| \\
&= \Big|\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \exp(-ip\cdot x)\exp(y\cdot x)(-i)^\alpha \partial_x^\alpha f(x)\mu_L(dx)\Big| \\
&\leq \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} \frac{1}{(1+x\cdot x)^{N+|\alpha|}}\mu_L(dx)|||f|||_{N+|\alpha|}.
\end{aligned}$$

Let us abbreviate this last number by $C_{f,|\alpha|}$ and observe, that for a given $n \in \mathbb{N}$,

$$(1+|z|)^n \leq (1+\sum_{k=1}^N |z_k|)^n = \sum_{\alpha\in\mathbb{N}^N_{\leq n}} \frac{n!}{(n-|\alpha|)!\alpha!}|z^\alpha|,$$

$$\text{and} \qquad \sum_{\alpha\in\mathbb{N}^N_{\leq n}} \frac{n!}{(n-|\alpha|)!\alpha!} = (1+N)^n.$$

Hence,

$$|\mathcal{L}(f,y)(p)| \leq C_{f,n}(1+N)^n \frac{1}{(1+|p+iy|)^n}.$$

8. Use the previous exercice to show, that for $f \in \mathcal{S}(\mathbb{R}^4)$ with $\sup(f) \subset \mathbb{R}_\pm \times \mathbb{R}^3$, $\mathcal{L}(f,y)(p)$ is holomorphic in $p_0 + iy$ if $y \in \mathbb{R}^*_\mp \times \{(0,0,0)\}$.
Show that if $f \in \mathcal{S}(\mathbb{R}^4)$ with $\sup(f) \subset \mathbb{R}_\mp \times \mathbb{R}^3$, $\mathcal{L}^*(f,y)(p) := \mathcal{L}(f,-y)(-p)$ is holomorphic in $p_0 + iy$ if $y \in \mathbb{R}^*_\mp \times \{(0,0,0)\}$.
Use this to show, that $\Delta^C_\pm$ have causal supports.

Part I:
If $\sup(f) \subset \mathbb{R}_\pm \times \mathbb{R}^3$ and if $y \in \mathbb{R}_\mp \times \{(0,0,0)\}$, then $x\cdot y \leq 0$ for all $x \in \operatorname{supp}(f)$ and $f(x)\exp(-i(p+iy)\cdot x)$ is a Schwartz function. By the previous exercice, $\mathcal{L}(f,y)(p)$ is well-defined and holomorphic for $y_0 \in \mathbb{R}^*_\mp$.
If $\sup(f) \subset \mathbb{R}_\mp \times \mathbb{R}^3$ and if $y \in \mathbb{R}_\mp \times \{(0,0,0)\}$, then $x\cdot y \geq 0$ for all $x \in \operatorname{supp}(f)$ and $f(x)\exp(i(p+iy)\cdot x)$ is a Schwartz function. By the previous exercice, $\mathcal{L}^*(f,y)(p)$ is well-defined and holomorphic for $y_0 \in \mathbb{R}^*_\mp$.

The same holds for the relativistic Fourier transform once one replaces $x\cdot(p+iy)$ by $x^t\eta(p+iy)$. Applying $\Delta^C_\pm$ to such a fonction yields

$$\begin{aligned}
\Delta^C_\pm(f) = \varphi_{\hat{\Delta}^C_\pm}(\check{f}(p)) &= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\frac{-1}{(p_0\mp i\epsilon+E_p)(p_0\mp i\epsilon-E_p)}\check{f}(p)\mu_L(dp) \\
&= \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\left(\frac{1}{(p_0\mp i\epsilon+E_p)}-\frac{1}{(p_0\mp i\epsilon-E_p)}\right)\check{f}(p)\frac{\mu_L(dp)}{2E_p}.
\end{aligned}$$

It has already been proven that this limit exists and defines a tempered distribution. For a given $\epsilon > 0$ this integral certainly is well-defined by the fast decrease of $\check{f}$.

Therefore,

$$\Delta_{\pm}^{C}(f) = \varphi_{\hat{\Delta}_{\pm}^{C}}(\check{f}(p)) = \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3}\int_{\mathbb{R}} \left(\frac{1}{(p_0 \mp i\epsilon + E_p)} - \frac{1}{(p_0 \mp i\epsilon - E_p)} \right) \check{f}(p)dp_0 \frac{d^3p}{2E_p}.$$

Note that $\check{f}(p) = \mathcal{L}^*(f,0)(p)$. The integral in p_0 can now be computed by a contour integral with $y_0 \in \mathbb{R}_{\mp}$: we chose a path $\gamma_r = [-r,r] \cup \gamma_{C_r}$, where γ_{C_r} is the anti-trigonometric oriented path along the lower semi-circle centered in 0 of radius r in the case of $x_0 < 0$ (in the case $x_0 > 0$, take for γ_{C_r} the trigonometric oriented path along the upper semi-circle centered in 0 of radius r). Along this path we have

$$\oint_{\gamma_r} \left(\frac{1}{(z \mp i\epsilon + E_p)} - \frac{1}{(z \mp i\epsilon - E_p)} \right) \mathcal{L}(f,-y)(-p)dz$$
$$= \int_{-r}^{r} \left(\frac{1}{(p_0 \mp i\epsilon + E_p)} - \frac{1}{(p_0 \mp i\epsilon - E_p)} \right) \check{f}(p)dp_0$$
$$- \int_0^{\pi} \left(\frac{1}{(re^{-i\theta} \mp i\epsilon + E_p)} - \frac{1}{(re^{-i\theta} \mp i\epsilon - E_p)} \right) \mathcal{L}^*(f,-r\sin(\theta))(r\cos(\theta))ire^{-i\theta}d\theta.$$

Since $|\mathcal{L}^*(f,-r\sin(\theta))(r\cos(\theta))| \le C_{2,f}(1+r)^{-2}$, the second integral goes to 0 if r tends to infinity. But note, that the contour integral is 0 since all the poles lie outside the path γ_r. Hence, $\Delta_{\pm}^{C}(f) = 0$ if $\mathrm{supp}(f) \subset \mathbb{R}_{\mp} \times \mathbb{R}^3$.

Part *II*:
Suppose now that the support of f is such that $x \in \mathrm{supp}(f) \iff n^t\eta x < 0$ for some $n \in \mathbb{R}^4$ with $n^t\eta n = 1$, $n_0 \in \mathbb{R}_{\pm}$. There is then a Lorentz transformation $\Lambda \in O(1,3)$ so that $n' := \Lambda n = (\pm 1,0,0,0)$ and the function $g(x') := f(\Lambda^{-1}x')$ is such that $\mathrm{supp}(g) \subset \mathbb{R}_{\mp}^{*} \times \mathbb{R}^3$. Note also, that

$$\check{g}(p') = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \exp(ix'^t\eta p')g(x')\mu_L(dx') = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \exp(ix'^t\eta p')f(\Lambda^{-1}x')\mu_L(dx')$$
$$= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \exp(ix^t\eta p)f(x)\mu_L(dx) = \check{f}(p) = \check{f}(\Lambda^{-1}p').$$

Therefore,

$$\Delta_{\pm}^{C}(f) = \varphi_{\hat{\Delta}_{\pm}^{C}}(\check{f}(p)) = \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{-1}{(p_0 \mp i\epsilon + E_p)(p_0 \mp i\epsilon - E_p)} \check{f}(p)\mu_L(dp)$$
$$= \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{-1}{p^t\eta p \mp 2i\epsilon p_0 - \epsilon^2 - m^2}\check{f}(p)\mu_L(dp)$$
$$= \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{-1}{p^t\eta p - 2i\epsilon(\Lambda n)^t\eta p - \epsilon^2 - m^2}\check{f}(p)\mu_L(dp)$$
$$= \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{-1}{(\Lambda^{-1}p')^t\eta(\Lambda^{-1}p') - 2i\epsilon(\Lambda n)^t\eta(\Lambda^{-1}p') - \epsilon^2 - m^2}\check{f}(\Lambda^{-1}p')|\Lambda^{-1}|\mu_L(dp')$$
$$= \lim_{\epsilon\to 0^+} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \frac{-1}{p'^t\eta p' - 2i\epsilon(\Lambda^2 n)^t\eta p' - \epsilon^2 - m^2}\check{g}(p')\mu_L(dp').$$

The poles of this integrand for the variable p_0' lie at

$$i\epsilon(\Lambda^2 n)^t\eta e_0 \pm \sqrt{-\epsilon^2((\Lambda^2 n)\eta e_0)^2 + \epsilon^2 + E_{p'}^2 + 2i\epsilon(\Lambda^2 n)^t\eta p_s'},$$

where $e_0 := (1,0,0,0)$ and $p'_s := p' - p'_0 e_0$. The imaginary part of these new poles are

$$\epsilon(\Lambda^2 n)^t \eta e_0 \pm \sqrt{\frac{\sqrt{\left(-\epsilon^2((\Lambda^2 n)\eta e_0)^2+\epsilon^2+E_{p'}^2\right)^2+(2i\epsilon(\Lambda^2 n)^t\eta p'_s)^2}+\epsilon^2((\Lambda^2 n)\eta e_0)^2-\epsilon^2-E_{p'}^2}{2}}.$$

For these numbers we have the estimations

$$\begin{aligned}
|\epsilon(\Lambda^2 n)^t \eta e_0| &> \sqrt{\frac{\sqrt{\left(-\epsilon^2((\Lambda^2 n)\eta e_0)^2+\epsilon^2+E_{p'}^2\right)^2+(2\epsilon(\Lambda^2 n)^t\eta p'_s)^2}+\epsilon^2((\Lambda^2 n)\eta e_0)^2-\epsilon^2-E_{p'}^2}{2}} \\
\iff (\epsilon(\Lambda^2 n)^t \eta e_0)^2 &> \frac{\sqrt{\left(-\epsilon^2((\Lambda^2 n)\eta e_0)^2+\epsilon^2+E_{p'}^2\right)^2+(2\epsilon(\Lambda^2 n)^t\eta p'_s)^2}+\epsilon^2((\Lambda^2 n)\eta e_0)^2-\epsilon^2-E_{p'}^2}{2} \\
\iff (\epsilon(\Lambda^2 n)^t \eta e_0)^2 + \epsilon^2 + E_{p'}^2 &> \sqrt{\left(-\epsilon^2((\Lambda^2 n)\eta e_0)^2+\epsilon^2+E_{p'}^2\right)^2+(2\epsilon(\Lambda^2 n)^t\eta p'_s)^2} \\
\iff \left(\epsilon^2(\Lambda^2 n)^t \eta e_0)^2 + \epsilon^2 + E_{p'}^2\right)^2 &> \left(-\epsilon^2((\Lambda^2 n)\eta e_0)^2 + \epsilon^2 + E_{p'}^2\right)^2 + \left(2\epsilon(\Lambda^2 n)^t\eta p'_s\right)^2 \\
\iff \epsilon^2\left((\Lambda^2 n)^t\eta e_0\right)^2(\epsilon^2+E_{p'}^2) &> \left(\epsilon(\Lambda^2 n)^t\eta p'_s\right)^2.
\end{aligned}$$

This last inequality certainly holds: indeed, since $n^t\eta n = 1$ and since Λ^2 is again a Lorentz transformation, one has that $(\Lambda^2 n)^t\eta(\Lambda^2 n) = 1$. Therefore, if we denote $\Lambda^2 n - (\Lambda^2 n)_0 e_0$ by $(\Lambda^2 n)_s$, $\left((\Lambda^2 n)^t\eta e_0\right)^2 = 1 + |(\Lambda^2 n)_s|^2$ and we get

$$\left((\Lambda^2 n)^t\eta e_0\right)^2(\epsilon^2+E_{p'}^2) > \left((\Lambda^2 n)^t\eta e_0\right)^2 E_{p'}^2 \geq |(\Lambda^2 n)_s|^2|p'_s|^2 \geq \left((\Lambda^2 n)^t\eta p'_s\right)^2.$$

Hence the imaginary parts of the poles

$$i\epsilon(\Lambda^2 n)^t\eta e_0 \pm \sqrt{-\epsilon^2((\Lambda^2 n)\eta e_0)^2+\epsilon^2+E_{p'}^2+2i\epsilon(\Lambda^2 n)^t\eta p'_s}$$

have the same sign as $\epsilon(\Lambda^2 n)^t\eta e_0$. This sign is the same as $\mathrm{sgn}((\Lambda^2 n)_0) = \mathrm{sgn}(n_0)$. In conclusion, the poles of

$$\lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\frac{-1}{p'^t\eta p' - 2i\epsilon(\Lambda^2 n)^t\eta p' - \epsilon^2 - m^2}\check{g}(p')\mu_L(dp')$$

lie in the upper (lower) complex plane and we are in the same situation as in part I: after performing a contour integral in the lower (upper) complex plane in order to compute de integral over p_0, we get $\Delta^C_\pm(f) = 0$.

<u>Part *III*:</u>
Having established that $\Delta^C_\pm(f) = 0$ for any test function f so that $x \in \mathrm{supp}(f) \iff n^t\eta x < 0$ for some $n \in \mathbb{R}^4$ with $n^t\eta n = 1$, $n_0 \in \mathbb{R}_\pm$, we can conclude that by definition of the support for distributions, $\mathrm{supp}(\Delta^C_\pm) \subset LC_\pm$, where $LC_\pm$ are the future (past) light cones.

9. A solution to the Cauchy problem

$$(\Box + m^2)\varphi = 0, \quad \varphi(0) = f \text{ and } (\partial_t\varphi)(0) = f_t, \quad f, f_t \in \mathcal{S}'(\mathbb{R}^3)$$

is a function $\varphi : \mathbb{R} \to \mathcal{S}'(\mathbb{R}^3)$, so that $\lim_{\tau\to 0}\frac{\varphi(t+\tau)-\varphi(t)}{\tau} = \varphi_t(t)$ and $\lim_{\tau\to 0}\frac{\varphi_t(t+\tau)-\varphi_t(t)}{\tau} = \varphi_{tt}(t)$ exist in the weak*-topology and so that $\varphi_{tt}(t) - \Delta\varphi(t) + m^2\varphi(t) = 0$.
Use Schwartz's Paley & Wiener theorem to show, that if both f and f_t have compact support, then $\varphi(t)$ has causal support.
What happens if one considers positive and negative energy frequencies separately?
(Hint: $\sqrt{a+ib} = \pm\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right)$)

Let us find a function $\mathbb{R} \ni t \mapsto \varphi(t) \in \mathcal{S}'(\mathbb{R}^3)$ which is a solution to $(\Box + m^2)\varphi = 0$ and satisfies the initial conditions $\varphi(0) = f$ and $\varphi_t(0) = f_t$, where both $f(x)$ and $f_t(x)$ are compactly supported tempered distributions.
We know then by Schwartz's Paley & Wiener theorem, that $\hat{f}$ and $\widehat{f_t}$ are (integration against) entire functions satisfying

$$|\hat{f}(p+iq)|,\ |\widehat{f_t}(p+iq)| \leq C_N(1+|p+iq|)^N \exp(r|q|),$$

where $C_N > 0$ is some constant and $r > 0$ is the radius of some open ball $B(0,r)$ containing the supports of both f and f_t.
For each fixed t, the Fourier transform $\hat{\varphi}(t)$ satisfies the equation

$$\left(\frac{d^2}{dt^2} + p\cdot p + m^2\right)\hat{\varphi}(t) = 0 \qquad \Longleftrightarrow \qquad \frac{d^2}{dt^2}\hat{\varphi}(t) = -(p\cdot p + m^2)\hat{\varphi}(t),$$
$$\hat{\varphi}(0) = \varphi_{\hat{f}(p)},\ \partial_t\hat{\varphi}(0) = \varphi_{\widehat{f_t}(p)}.$$

For a fixed value for p, the general solution to this second order linear differential equation becomes

$$\hat{\varphi}(t) = \varphi_{\cos(E_pt)\hat{f}(p)+\frac{1}{E_p}\sin(E_pt)\widehat{f_t}(p)},$$

where $E_p = \sqrt{p\cdot p + m^2}$. This last function is not entire in p because the square root is not holomorphic in $z = 0$. But, once the trigonometric functions are expanded in their series, it yields

$$\begin{aligned}
&\cos(E_pt)\hat{f}(p) + \frac{1}{E_p}\sin(E_pt)\widehat{f_t}(p) \\
&= \left(\sum_{n\geq 0}(-1)^n\frac{(E_pt)^{2n}}{(2n)!}\right)\hat{f}(p) + \frac{1}{E_p}\left(\sum_{n\geq 0}(-1)^n\frac{(E_pt)^{2n+1}}{(2n+1)!}\right)\widehat{f_t}(p) \\
&= \left(\sum_{n\geq 0}(-1)^n\frac{(m^2+p\cdot p)^n t^{2n}}{(2n)!}\right)\hat{f}(p) + \left(\sum_{n\geq 0}(-1)^n\frac{(p\cdot p+m^2)^n t^{2n+1}}{(2n+1)!}\right)\widehat{f_t}(p),
\end{aligned}$$

displaying explicitly the entirety of the solution in the variable p.
It remains to be shown that the module of this solution is exponentially bounded. In order to do so, we write $\hat{\varphi}(t) = \varphi_F$ with

$$\begin{aligned}
F(p+iq) = {} & \frac{e^{iE_{p+iq}t}}{2E_{p+iq}}\left(E_{p+iq}\hat{f}(p+iq) - i\widehat{f_t}(p+iq)\right) \\
& + \frac{e^{-iE_{p+iq}t}}{2E_{p+iq}}\left(E_{p+iq}\hat{f}(p+iq) + i\widehat{f_t}(p+iq)\right)
\end{aligned}$$

where $E_{p+iq} = \sqrt{m^2 + p\cdot p - q\cdot q + 2ip\cdot q}$. Observe now, that $|e^{iE_{p+iq}t}| \leq e^{t|\mathrm{Im}(E_{p+iq})|}$

and that

$$
\begin{aligned}
|\mathrm{Im}(E_{p+iq})| \leq |q| \quad &\iff \quad |\mathrm{Im}(\sqrt{m^2 + p\cdot p - q\cdot q + 2ip\cdot q})| \leq |q| \\
&\iff \quad \left|\sqrt{\frac{\sqrt{(m^2+|p|^2-|q|^2)^2+4(p\cdot q)^2}-(m^2+|p|^2-|q|^2)}{2}}\right| \leq |q| \\
&\iff \quad \frac{\sqrt{(m^2+|p|^2-|q|^2)^2+4(p\cdot q)^2}-(m^2+|p|^2-|q|^2)}{2} \leq |q|^2 \\
&\iff \quad \sqrt{(m^2+|p|^2-|q|^2)^2+4(p\cdot q)^2} \leq m^2+|p|^2+|q|^2 \\
&\iff \quad (m^2+|p|^2-|q|^2)^2+4(p\cdot q)^2 \leq (m^2+|p|^2+|q|^2)^2 \\
&\iff \quad -2(m^2+|p|^2)|q|^2+|q|^4+4(p\cdot q)^2 \leq 2(m^2+|p|^2)|q|^2+|q|^4 \\
&\iff \quad (p\cdot q)^2 \leq m^2|q|^2+|q|^2|p|^2,
\end{aligned}
$$

which is always the case. Hence,

$$
\begin{aligned}
|F(z)| &\leq \frac{1}{2}(|e^{iE_{p+iq}t}|+|e^{-iE_{p+iq}t}|)\left(|\hat{f}(p+iq)|+|E_{p+iq}\widehat{f_t}(p+iq)|\right) \\
&\leq e^{|q|t}C_N(1+|p+iq|)^N(1+|E_{p+iq}|)e^{r|q|} \\
&\leq e^{|q|t}C_N(1+|p+iq|)^N(1+m+|p+iq|)e^{r|q|} \\
&\leq C_N(1+m)(1+|p+iq|)^{N+1}e^{|q|(r+t)},
\end{aligned}
$$

which shows that at time t, the support of $\varphi(t)$ is a subset of $B(0, r+t)$. The solution is hence causal.
Note that the solution can be split into negative and positive energy parts:

$$
\begin{aligned}
\text{negative part:} \quad & \frac{e^{iE_{p+iq}t}}{2E_{p+iq}}\left(E_{p+iq}\hat{f}(p+iq) - i\widehat{f_t}(p+iq)\right), \\
\text{positive part:} \quad & \frac{e^{-iE_{p+iq}t}}{2E_{p+iq}}\left(E_{p+iq}\hat{f}(p+iq) + i\widehat{f_t}(p+iq)\right).
\end{aligned}
$$

None of these are holomorphic, even for $t = 0$ (unless $f_t = 0$). Consequently, none of them can have compact support. Even in the case where $f_t = 0$, and f has compact support, the positive part with frequency will have non-compact support at any given time $t > 0$. It is therefore relativistically inconsistent to view $\varphi(t)$ as a wave function. This is one of the reasons one needs to consider them as fields and step away from wave-functions when aspiring to a relativistic theory of quantum physics.

10. Let $f, f_t \in \mathcal{S}(\mathbb{R}^3)$ and show, that the solution obtained in this case by the previous exercice reads

$$
\lim_{k\to\infty}\left((D_t\Delta_+^C) * (d_k(t)f) + \Delta_+^C * (d_k(t)f_t)\right),
$$

where $(d_k)_{k\in\mathbb{N}^*} \subset \mathcal{S}(\mathbb{R})$ is a Dirac sequence.
What happens when one substitutes Δ_+^C by Δ_-^C or $\Delta_\pm^F$?
(Hints: compute the Fourier transform of the convolution of a tempered distribution with a test function. Apply Paley & Wiener's theorem to the Dirac sequence and use a contour integral on p_0.)

For a tempered distribution $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ and test functions $f, g \in \mathcal{S}(\mathbb{R}^N)$, one has

$$\begin{aligned}\mathcal{F}(\varphi * f)(g) = (\varphi * f)(\hat{g}) = \varphi(P^t(f) * \hat{g}) = \varphi(\mathcal{F} \circ \mathcal{F}^*(P^t(f) * \hat{g}))\\ = (2\pi)^{N/2}\varphi(\mathcal{F}(\hat{f}g)) = (2\pi)^{N/2}\mathcal{F}(\varphi)(\hat{f}g),\\ \Rightarrow \quad \widehat{\varphi * f} = (2\pi)^{N/2}\hat{f}\hat{\varphi}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{F}\left((D_t\Delta_+^C) * (d_k(t)f)\right) = (2\pi)^2\widehat{d_k f}(p_0, p)\mathcal{F}(D_t\Delta_+^C) = (2\pi)^2\widehat{d_k f}(p_0, p) i p_0 \hat{\Delta}_+^C,\\ \mathcal{F}\left(\Delta_+^C * (d_k(t)f_t)\right) = (2\pi)^2\widehat{d_k f_t}(p_0, p)\hat{\Delta}_+^C.\end{aligned}$$

Note that the functions $d_k f$ and $d_k f_t$ have separated variables, so that

$$\widehat{d_k f}(p_0, p) = \widehat{d_k}(p_0)\hat{f}(p) \quad \text{and} \quad \widehat{d_k f_t}(p_0, p) = \widehat{d_k}(p_0)\widehat{f_t}(p).$$

By Paley & Wiener's theorem, one has for fixed $k \in \mathbb{N}^*$, that $\widehat{d_k}$ is an entire function satisfying

$$\forall n \in \mathbb{N},\ \exists C_{k,n} \text{ s.t.} \quad |\widehat{d_k}(p_0 + iq_0)| \leq C_{k,n}(1 + |p_0 + iq_0|)^{-n} e^{\frac{1}{k}|q_0|},$$

since $\mathrm{supp}(d_k) \subset B(0, \frac{1}{k})$. Using the definition of $\hat{\Delta}_+^C$ one gets for a fixed $k \in \mathbb{N}^*$

$$\mathcal{F}\left((D_t\Delta_+^C) * (d_k(t)f) + \Delta_+^C * (d_k(t)f_t)\right) = \lim_{\epsilon \to 0^+} \varphi_{F_\epsilon(p_0,p)} \quad \text{with}$$

$$F_\epsilon(p_0, p) = \left(\frac{\widehat{d_k}(p_0)}{(p_0 - \epsilon + E_p)} - \frac{\widehat{d_k}(p_0)}{(p_0 - i\epsilon - E_p)}\right)\frac{i}{2E_p}\left(p_0\hat{f}(p) - i\widehat{f_t}(p)\right).$$

This is a bona fide Schwartz function if $\epsilon > 0$. Continuity of the Fourier transform for the weak*-topology implies

$$(D_t\Delta_+^C) * (d_k(t)f) + \Delta_+^C * (d_k(t)f_t) = \lim_{\epsilon \to 0^+} \varphi_{\check{F}_\epsilon(t,x)} \quad \text{with}$$

$$\check{F}_\epsilon(t, x) = \frac{i}{(2\pi)^2}\int_{\mathbb{R}^4}\left(\frac{\widehat{d_k}(p_0)}{(p_0 - i\epsilon + E_p)} - \frac{\widehat{d_k}(p_0)}{(p_0 - i\epsilon - E_p)}\right)\frac{e^{itp_0 - ix\cdot p}}{2E_p}\left(p_0\hat{f}(p) - i\widehat{f_t}(p)\right) dp_0 d^3p.$$

By Fubini's theorem, one may proceed with the integration over p_0 first. In this variable the integrand is holomorphic and we shall use a countour integral.
For a fixed $(t, x) \in \mathbb{R}^4$ and $t > 0$ chose a $k \in \mathbb{N}^*$ large enough so that $\frac{1}{k} < t$. For such a k, $e^{\frac{1}{k}|q_0| - tq_0}$ will decay exponentially fast to 0, if $q_0 > 0$, i.e. if we chose a contour along p_0 and closing it anti-clockwise along the upper semi-circle in the complex plane. This contour integral will hence pick the poles $p_0 + iq_0 = \pm E_p + i\epsilon$.
For a fixed $(t, x) \in \mathbb{R}^4$ and $t < 0$ chose a $k \in \mathbb{N}^*$ large enough so that $\frac{1}{k} < -t$. For such a k, $e^{\frac{1}{k}|q_0| - tq_0}$ will decay exponentially fast to 0, if $q_0 < 0$, i.e. if we chose a contour along p_0 and closing it clockwise along the lower semi-circle in the complex plane. This contour integral will hence pick no poles. Consequently,

$$\lim_{\epsilon \to 0^+} \check{F}_\epsilon(t, x) = \Theta(t)\frac{1}{2\pi}\int_{\mathbb{R}^3}\left(\widehat{d_k}(-E_p)e^{-itE_p}\left(E_p\hat{f}(p) + i\widehat{f_t}(p)\right) + \widehat{d_k}(E_p)e^{itE_p}\left(E_p\hat{f}(p) - i\widehat{f_t}(p)\right)\right)\frac{e^{-ix\cdot p}d^3p}{2E_p}.$$

Taking the limit when k goes to ∞ and using the fact that for a Dirac sequence $(d_k)_{k\in\mathbb{N}^*}$, $\lim_k \hat{d}_k(p_0) = \frac{1}{\sqrt{2\pi}}$, we get

$$\begin{aligned}
&\lim_k \left((D_t\Delta_+^C) * (d_k(t)f) + \Delta_+^C * (d_k(t)f_t)\right)\\
&= \frac{\Theta(t)}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(e^{itE_p}\left(E_p\hat{f}(p) - i\widehat{f_t}(p)\right) + e^{-itE_p}\left(E_p\hat{f}(p) + i\widehat{f_t}(p)\right)\right) \frac{e^{-ix\cdot p}d^3p}{2E_p},
\end{aligned}$$

which for $t > 0$ is just the spatial Fourier transform of the solution found in the previous exercice.
If one uses Δ_-^C instead, one gets the poles in the lower complex half-plane, meaning that the contour integrals must be reversed and one has

$$\begin{aligned}
&\lim_k \left((D_t\Delta_-^C) * (d_k(t)f) + \Delta_-^C * (d_k(t)f_t)\right)\\
&= \frac{-\Theta(-t)}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(e^{itE_p}\left(E_p\hat{f}(p) - i\widehat{f_t}(p)\right) + e^{-itE_p}\left(E_p\hat{f}(p) + i\widehat{f_t}(p)\right)\right) \frac{e^{-ix\cdot p}d^3p}{2E_p}.
\end{aligned}$$

Finally, using $\Delta_\pm^F$, one has to close the contour anti-clockwise in the upper (clockwise in the lower) semi-circle for $t > 0$ and clockwise in the lower (anti-clockwise in the upper) semi-circle for $t < 0$. In doing so, one selects the positive energy part and propagates it in the future, whereas the negative energy part is propagated in the past:

$$\begin{aligned}
&\lim_k \left((D_t\Delta_\pm^F) * (d_k(t)f) + \Delta_p^F m * (d_k(t)f_t)\right)\\
&= \frac{\Theta(\pm t)}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-itE_p}\left(E_p\hat{f}(p) + i\widehat{f_t}(p)\right) \frac{e^{-ix\cdot p}d^3p}{2E_p}\\
&-\frac{\Theta(\mp t)}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{itE_p}\left(E_p\hat{f}(p) - i\widehat{f_t}(p)\right) \frac{e^{-ix\cdot p}d^3p}{2E_p}.
\end{aligned}$$

As already noticed in the previous exercice, separating the positive and negative energy parts results in two non-causal solutions. This is consistent with the non-causal support of $\Delta_\pm^F$.

Remark: In physics, the computation in this exercice is usually denoted by

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Delta(x-y)\overleftrightarrow{\partial_{y_0}} f(y)d^3y\\
&:= \int_{\mathbb{R}^3} \left(\Delta(x-y)\partial_{y_0}f(y) - f(y)\partial_{y_0}\Delta(x-y)\right) d^3y\\
&= \int_{\mathbb{R}^3} \left(\Delta(x-y)\partial_{y_0}f(y) + f(y)\partial_{x_0}\Delta(x-y)\right) d^3y,
\end{aligned}$$

where Δ stands for $\Delta_\pm^C$ or $\Delta_\pm^F$. Setting aside the exact meaning of the variables in $\Delta(x-y)$, there is still the problem of integrating over $\mathbb{R}^3$: this should be an integral over $\mathbb{R}^4$, since Δ is a tempered distribution in $\mathbb{R}^4$. This problem may be dealt with by inserting a Dirac sequence $(d_k(y_0))_{k\in\mathbb{N}^*}$:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Delta(x-y)\overleftrightarrow{\partial_{y_0}} f(y) d^3y \\
&= \lim_k \int_{\mathbb{R}^3} d_k(y_0) \left(\Delta(x-y)\partial_{y_0} f(y) + f(y)\partial_{x_0}\Delta(x-y)\right) d^4y \\
&= \lim_k \left(\Delta * (d_k \partial_t f + (D_t\Delta) * (d_k f))\right),
\end{aligned}$$

which is the mathematical rigourous expression.