Exercice sheet 5

Construction of the Wiener process

1. Find a basis and a sub-basis for the canonical topology on $\mathbb{R}$.

2. Let (Ω, τ) be a topological space. Let $\mathcal{S} \subset \mathcal{P}(\tau)$ be a totally ordered set (for the partial order being the set inclusion) so that no element of $\mathcal{S}$ has a finite sub-cover for Ω. Show then that $\cup_{S\in\mathcal{S}}$ has no finite sub-cover of Ω.
For $k = 1, \ldots, n$ let $\mathcal{M}_k$ be a finite collection of open sets of some topological space (Ω, τ). For each k, Let $B_k \in \tau$ so that $\mathcal{M}_k \cup \{B_k\}$ covers Ω. Show then that $(\cup_{k=1}^n \mathcal{M}_k) \cup \{\cap_{k=1}^n B_k\}$ covers Ω.

3. Let $\{(\Omega_i, \tau_i)\}_{i\in I}$ is a family of topological spaces. Provide the cartesian product $\prod_{i\in I} \Omega_i$ with the collection τ of sets consisting of arbitrary unions of sets of the form $\prod_{i\in I} V_i$, where all but a finite number of the $V_i's$ are equal to Ω_i.
Show that τ is a topology. For some fixed $i \in I$, let $\pi_i : \prod_{i\in I} \Omega_i \to \Omega_i$, $(x_i)_{i\in I} \mapsto x_i \in \Omega_i$ be the **canonical projection**. Show that the collection of sets $\{\pi_i^{-1}(V) : i \in I \text{ and } V \in \tau_i\}$ is a sub-basis for τ. Prove that the product topology is then the coarsest topology for which all the maps π_i are continuous.

4. The **one-point compactification** of $\mathbb{R}^N$ is defined as the set $\dot{\mathbb{R}}^N := \mathbb{R}^N \cup \{*\}$, where $* \notin \mathbb{R}^N$, provided with the topology τ consisting of the canonical open sets in $\mathbb{R}^N$ together with sets of the form $\{*\} \cup (\mathbb{R}^N \setminus K)$, with $K \subset \mathbb{R}^N$.
Prove that $\dot{\mathbb{R}}^N$ is a compact set. Prove that $f \in C(\dot{\mathbb{R}}^N)$ iff $f = \lambda + g$ with $g \in C_0(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$.
Consider $\Omega_L := \prod_{t>0} \dot{\mathbb{R}}^N$, provided with the product topology, and the set of finite functions $C_{\text{fin}}(\Omega_L)$. Prove that $C_{\text{fin}}(\Omega_L) \subset C(\Omega_L)$ and that $C_{\text{fin}}(\Omega_L)$ is uniformly dense in $C(\Omega_L)$.

5. For a fixed $x_0 \in \mathbb{R}^N$ and for some $F_{f,t_1,\ldots,t_m} \in C_{\text{fin}}(\Omega_L)$, we define

$$I_{x_0}(F_{f,t_1,\ldots,t_m}) := \int_{\mathbb{R}^{mN}} p^{x_0}_{t_1,\ldots,t_m}(x_1, \ldots, x_m) f(x_1, \ldots, x_m) \mu_L(dx_1 \times \ldots \times dx_m),$$

$$p^{x_0}_{t_1,\ldots,t_m}(x_1, \ldots, x_m) = \prod_{k=1}^m \frac{1}{(2\pi(t_k - t_{k-1}))^{N/2}} e^{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}}, \quad t_0 = 0.$$

Check that $I_{x_0}(1) := 1$. For $F_{f,t_1,\ldots,t_m}$ and $0 < t$ with $\{t_1, \ldots, t_m\} \cap \{t\} = \emptyset$, if $G_{f,t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m}(w) := F_{f,t_1,\ldots,t_m}(w)$, check that

$$I_{x_0}(G_{f,t_1,\ldots,t_k,t,t_{k+1},\ldots,t_m}) = I_{x_0}(F_{f,t_1,\ldots,t_m}).$$

or some constant $C > 0$ and for $\epsilon, t > 0$, prove that

$$\mathbb{E}_{x_0,W}(\mathbb{1}_{\{w\in\Omega_L \,:\, |w_t - x_0| > \epsilon\}}) \leq \frac{C}{\epsilon} e^{-\frac{\epsilon^2}{2Nt}}.$$

If we set

$$\rho(\epsilon, \delta) := \text{supp}\{\mu_{x_0,W}(\{w \in \Omega_L \,:\, |w_t - x_0| > \epsilon\}) \,:\, 0 < t \leq \delta\},$$

$$\text{then verify that} \quad \lim_{\delta\to 0^+} \frac{1}{\delta} \rho(\epsilon, \delta) = 0.$$

6. Let $\delta, \epsilon > 0$ and consider a finite set $S \subset \mathbb{R}_+$ so that $\forall t \in S$, $|t - t_{\min(S)}| \leq \delta$. For $t \in S$, let

$$C_{t,\epsilon,S} := \{w \in \Omega_L \,:\, |w_t - w_{\max(S)}| > \frac{\epsilon}{2}\},$$

$$D_{t,\epsilon,S} := \{w \in \Omega_L \,:\, |w_t - w_{\min(S)}| > \epsilon \text{ and } \forall s \in S \text{ with } s < t,\, |w_s - w_{\min(S)}| \leq \epsilon\}.$$

Show then that $\forall t \in S$,

$$\mu_{x_0,W}(C_{t,\epsilon,S} \cap D_{t,\epsilon,S}) \leq \rho(\frac{\epsilon}{2}, \delta)\mu_{x_0,W}(D_{t,\epsilon,S}).$$

7. Let $\delta, \epsilon > 0$ and consider a finite set $S \subset \mathbb{R}_+$ so that $\forall t \in S$, $|t - t_{\min(S)}| \leq \delta$. Let

$$A_{\epsilon,S} := \{w \in \Omega_L \,:\, \exists s \in S \text{ s.t. } |w_s - w_{\min(S)}| > \epsilon\}.$$

Prove then that

$$\mu_{x_0,W}(A_{\epsilon,S}) \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

8. Let $\delta, \epsilon > 0$ and consider $0 < t_0 < t_1$ with $t_1 - t_0 \leq \delta$. Define

$$E_{t_0,t_1,\epsilon} := \{w \in \Omega_L \,:\, \exists s, t \in [t_0, t_1] \text{ s.t. } |w_s - w_t| > 2\epsilon\}.$$

Prove then that

$$\mu_{x_0,W}(E_{t_0,t_1,\epsilon}) \leq 2\rho(\frac{\epsilon}{2}, \delta).$$

9.

10.