

Exercice sheet 4

Brownian Motions & probability

1. The heat kernel is given by

$$u(t, x) = \frac{1}{(2\pi Dt)^{3/2}} \exp\left(-\frac{1}{2Dt}(x \cdot x)\right).$$

Verify that $\partial_t u(t, x) = \frac{D}{2} \Delta u(t, x)$ with $\lim_{t \rightarrow 0^+} u(t, x) = \delta_0^{(3)}$. Conclude, that if $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ is continuous and polynomially bounded, $(u * f)(t)$ is also a solution to the heat equation with $u(0, x) = f(x)$. Compute the solution with $u(0, x) = 0 \vee x$ in the case $(t, x) \in \mathbb{R}^2$.

Substitute t by it and D by $(m\hbar)^{-1}$ in the heat kernel. Verify that now we have a solution $v(t, x)$ for the Schrödinger equation. Prove that if $f \in \mathcal{S}(\mathbb{R}^3)$, $(v * f)(t)$ is a solution with initial value $f(x)$. What happens when $f(x)$ is of compact support?

2. Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable on a probability space (Ω, Σ, μ) together with its cumulative probability distribution $F_X(x) := \mu(X^{-1}] - \infty, x]$.

Prove that $F_X(x)$ is **càdlàg**, i.e.

- $\forall x, y \in \mathbb{R}, x \leq y$ implies $0 \leq F_X(x) \leq F_X(y) \leq 1$,
- $\forall x \in \mathbb{R}, \lim_{y \rightarrow x^+} F_X(y) = F_X(x)$,
- $\forall x \in \mathbb{R}, \lim_{y \rightarrow x^-} F_X(y) \leq F_X(x)$.

3. Prove that any real number $x \in]0, 1]$ may be uniquely expressed as $x = \sum_{n \geq 1} \frac{x_n}{3^n}$ where $x_n \in \{0, 1, 2\}$ and $|\{n \geq 1 : x_n \neq 0\}| = |\mathbb{N}^*|$.

The **Cantor set** $\mathcal{C}$ is then defined as the set of $x \in]0, 1]$ so that $x_n \in \{0, 2\}$. Prove that $\mu_L(\mathcal{C}) = 0$ and that there is a bijection from $\mathcal{C}$ to $]0, 1]$.

For $x \in]0, 1] \setminus \mathcal{C}$, denote by N_x as the smallest $n \geq 1$, so that $x_n = 1$. The **Cantor function** is defined as

$$c(x) := \begin{cases} \frac{1}{2} \sum_{n \geq 1} \frac{x_n}{2^n} & \text{if } x \in \mathcal{C}, \\ \frac{1}{2^{N_x}} + \frac{1}{2} \sum_{n=1}^{N_x-1} \frac{x_n}{2^n} & \text{otherwise} \end{cases}.$$

Prove that $c(x)$ is a continuous and increasing function on $]0, 1]$. Prove $c(x)$ is constant μ_L -a.e. and conclude from these properties, that there is no Lebesgue summable function $d(x)$, so that $c(x) = \int_{[0, x]} d(s) \mu_L(ds)$.

4. Let $\Omega := \{\pm 1\}^{\mathbb{N}^*}$ and consider the map

$$f : \Omega \rightarrow [0, 1], \quad w \mapsto \sum_{n \geq 1} \frac{1 + w(n)}{2^{n+1}}.$$

Show that this map is surjective and that $\Sigma := \{f^{-1}(E) : E \in \Sigma_L\}$, where Σ_L is the Lebesgue σ -algebra is one as well. Show that $\mu(E) := \mu_L(f(E))$ defines a probability measure on Σ .

For a finite sequence $t \in \{\pm 1\}^n$, define $A_t = \{w \in \{\pm 1\}^{\mathbb{N}^*} : \forall k = 1, \dots, n, w(k) = t(k)\}$. Prove that $A_t \in \Sigma$ with $\mu(A_t) = \frac{1}{2^n}$.

For the random walk $\{S_n\}_{n \in \mathbb{N}^*}$, verify the equalities $\mathbb{E}(S_n) = 0$, $\mathbb{V}(S_n) = n$ and $\mu(S_n^{-1}\{n - 2k\}) = 2^{-n} \binom{n}{k}$ for $k = 0, \dots, n$.

5. For the random walk $\{S_n\}_{n \in \mathbb{N}^*}$ defined on (Ω, Σ, μ) in the previous exercise, show that $R \in \Sigma$, where

$$R := \{w \in \Omega : \exists n \geq 1 \text{ s.t. } S_n(w) = 0\}.$$

Show then that $\mu(R) = 1$. Imitate the construction of the last exercise to build a 2-dimensional random walk.

6. Let X be some random variable defined on a probability space (Ω, Σ, μ) . Prove that the characteristic function $\Phi_X(t) := \mathbb{E}(\exp(itX))$ is bounded and continuous. Conclude that $\varphi_{\Phi_X(t)} \in \mathcal{S}'(\mathbb{R})$ and that $\mathcal{F}(\varphi_{\Phi_X(t)})(f) = \sqrt{2\pi} \mathbb{E}(f(X))$ for any $f \in \mathcal{S}(\mathbb{R})$.
7. Let (Ω, Σ, μ) be a probability space and let X be a random variable defined on it. Let Σ_B be the σ -algebra generated by the open intervals of (R) . For an open interval $I \subset \mathbb{R}$, define $\mu_X(I) := \mu(X^{-1}I)$. Prove that μ_X may be uniquely extended to Σ_B and that for any $f \in C_b(\mathbb{R}, \mathbb{R})$, $\mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x) \mu_X(dx)$.

8. Define

$$r(x) := \begin{cases} \left(e^{ix} - 1 - ix + \frac{x^2}{2}\right) x^{-2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Show that $r(x)$ is continuous and bounded on $\mathbb{R}$. Use this to show, that for some random variable X on some probability space (Ω, Σ, μ) with $X \in \mathcal{L}^1(\Omega, \mu) \cap \mathcal{L}^2(\Omega, \mu)$ and $\mathbb{E}(X) = 0$, the characteristic function

$$\mathbb{E}(e^{itX}) = 1 - \frac{t^2}{2} \mathbb{V}(X) + t^2 h(t),$$

with $\lim_{t \rightarrow 0} h(t) = 0$ and $h(t)$ is bounded.

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