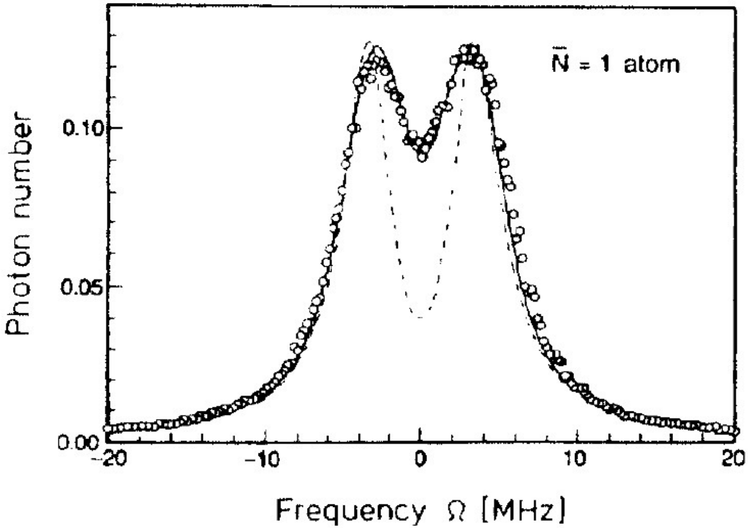
Quantum Electrodynamics and Quantum Optics: Lecture 9

Fall 2024

Classical explanation of normal mode splitting

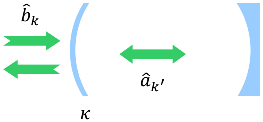


Quantum Optics of Resonators

Cavity Hamiltonian

$$\hat{H} = \sum \underbrace{\hbar\omega_{k'}\hat{a}^{\dagger}{}_{k'}\hat{a}_{k'}}_{\text{cavity modes}} + \sum \hbar\omega_k \hat{b}_k^{\dagger}\hat{b}_k + \hbar\sum_{k,k'} \underbrace{g_{k,k'}(\hat{a}_{k'}\hat{b}_k^{\dagger} + \hat{b}_k\hat{a}^{\dagger}{}_{k'})}_{\text{coupling terms}}$$

Taking the continuity limit $\sum_k |g_k|^2 = \int \mathrm{d}\omega_k |g(\omega_k)|^2 D(\omega_k)$, we define the operators in the frequency domain $[\hat{b}(\omega), \hat{b}^{\dagger}(\omega')] = \delta(\omega - \omega')$, thus:



EOM of $\hat{b}_k$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}_k = -i\omega_k\hat{b}_k - ig_k\hat{a}$$

Forward and backward formal solutions

- For $t_0 < t$, meaning that the process is a forward noise input

$$\hat{b}_k = e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) - ig_k \int_{t_0}^{t} e^{-i\omega_k(t-t')}\hat{a}(t')\mathrm{d}t'$$

- We can also write the process backward with $t_1 > t$ as

$$\hat{b}_k = e^{-i\omega_k(t-t_1)}\hat{b}_k(t_1) + ig_k \int_{t}^{t_1} e^{-i\omega_k(t-t')}\hat{a}(t')\mathrm{d}t'.$$

Optical Resonators

How to derive

Going to the continuum $\sum_k \to \int \mathrm{d}\omega_k D(\omega_k)$, where $D(\omega_k)$ is the density of states (DOS), we can also derive from $\dot{\hat{a}} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{\text{sys}}] - \sum_k g_k \hat{b}_k$ to (forward):

$$\dot{\hat{a}} = - i\omega\hat{a} - i\sum_k g_k e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) - \int_{-\infty}^{+\infty} \mathrm{d}\omega_k |g(\omega_k)|^2 D(\omega_k) \int_{t_0}^{t} e^{-i\omega_k(t-t')}\hat{a}(t')\mathrm{d}t'$$

- Assuming $|g(\omega_k)|^2 = |g|^2$ (first Markov approximation).
- Defining $2\pi|g|^2 D(\omega_k) = \kappa$ and $\hat{a}_{\text{in}} = -i\sum_k g_k e^{-i\omega_k(t-t_0)} b_k(t_0)$.
- Noting that $\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t} = \delta(t)$ and $\int_{t_0}^{t} \delta(t-t')f(t')\mathrm{d}t' = \frac{1}{2}f(t)$.

Forward and backward Quantum Langevin Equations

Quantum Langevin Equation (forward)

$$\dot{\hat{a}} = -i\omega\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\text{in}}$$

$$[\hat{a}_{\text{in}}(t), \hat{a}_{\text{in}}^{\dagger}(t')] = \delta(t - t')$$

One can repeat the same analysis, but now for the **backward** evolution, with $\hat{a}_{\text{out}} = -i\sum_k g_k e^{-i\omega_k(t-t_1)} b_k(t_1)$ and get:

Quantum Langevin Equation (backward)

$$\dot{\hat{a}} = -i\omega\hat{a} + \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{a}_{\text{out}}$$

Input Output Relation[1]

Subtracting the forward and backward Langevin Equations yields:

Input-output Noise Relation

$$\hat{a}_{\text{out}} + \hat{a}_{\text{in}} = \sqrt{\kappa}\hat{a}$$

Note that from the photon number point of view κ represents the energy decay:

$$\frac{d}{dt}\langle \hat{n} \rangle = \frac{d}{dt}\langle \hat{a}^\dagger \hat{a} \rangle = -\kappa \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}_{\text{in}}^\dagger \hat{a}_{\text{in}} \rangle$$

[1] Walls, D. F., Milburn, G. J. "Quantum optics" (2007). Chapter 7

Application : Transmission & Reflection

We can first write down the quantum Langevin equations

$$\partial_t \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \begin{pmatrix} -i\omega_c - \kappa/2 & 0 \\ 0 & i\omega_c - \kappa/2 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} + \sqrt{\kappa} \begin{pmatrix} \hat{a}_{\text{in}} \\ \hat{a}_{\text{in}}^\dagger \end{pmatrix}$$

where $\hat{a}^\dagger(\omega) = [\hat{a}(\omega)]^\dagger$ and solve in the Fourier domain

$$i\omega \begin{pmatrix} \hat{a}(\omega) \\ \hat{a}^\dagger(\omega) \end{pmatrix} = \underbrace{\begin{pmatrix} -i\omega_c - \kappa/2 & 0 \\ 0 & i\omega_c - \kappa/2 \end{pmatrix}}_{\hat{M}} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} + \sqrt{\kappa} \begin{pmatrix} \hat{a}_{\text{in}}(\omega) \\ \hat{a}_{\text{in}}^\dagger(\omega) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \hat{a}(\omega) \\ \hat{a}^\dagger(\omega) \end{pmatrix} = \sqrt{\kappa}[-\hat{M} + i\omega \hat{\mathbb{1}}]^{-1} \begin{pmatrix} \hat{a}_{\text{in}}(\omega) \\ \hat{a}_{\text{in}}^\dagger(\omega) \end{pmatrix}$$

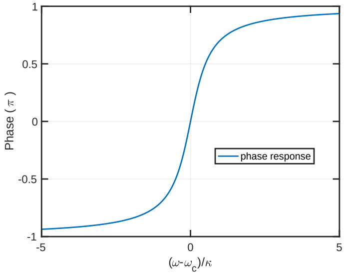
Application : Transmission & Reflection

Cavity transmission / reflection

$$\hat{a}_{\text{out}}(\omega) = \frac{\kappa/2 + i(\omega - \omega_c)}{\kappa/2 - i(\omega - \omega_c)} \hat{a}_{\text{in}}(\omega)$$

$$T(\omega) = \frac{\langle \hat{a}_{\text{out}}(\omega) \rangle}{\langle \hat{a}_{\text{in}}(\omega) \rangle}$$

Here we plot the phase of the transmission.



Separating intrinsic and external loss

The decay κ modeled is very general, the following showcases how to introduce intrinsic and external losses into the equations as $\kappa = \kappa_{\text{ex}} + \kappa_{\text{in}}$.

Modified Quantum Langevin Equation (forward)

$$\dot{\hat{a}} = -i\omega\hat{a} - \frac{\kappa_{\text{ex}} + \kappa_{\text{in}}}{2}\hat{a} + \sqrt{\kappa_{\text{ex}}}\hat{a}_{\text{in}} + \sqrt{\kappa_{\text{in}}}\hat{f}_{\text{in}}$$

$$[\hat{a}_{\text{in}}(t), \hat{a}_{\text{in}}^{\dagger}(t')] = \delta(t - t')$$

$$[\hat{f}_{\text{in}}(t), \hat{f}_{\text{in}}^{\dagger}(t')] = \delta(t - t')$$

Modified Cavity transmission / reflection

$$\hat{a}_{\text{out}}(\omega) = \frac{(\kappa_{\text{ex}} - \kappa_{\text{in}})/2 + i(\omega - \omega_c)}{(\kappa_{\text{ex}} + \kappa_{\text{in}})/2 - i(\omega - \omega_c)}\hat{a}_{\text{in}}(\omega)$$

Purcell Effect in Spontaneous Emission

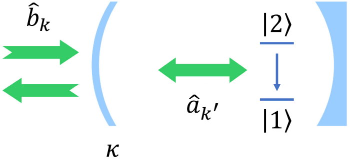
$$\hat{H} = \sum \hbar\omega'_k \hat{a}^\dagger \hat{a} + \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar g(\hat{\sigma}^+ \hat{a} + \hat{\sigma}^- \hat{a}^\dagger)$$
$$+ \sum \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k + \sum_k g_k(\hat{a}\hat{b}_k^\dagger + \hat{b}_k \hat{a}^\dagger)$$

Approximate treatment yields the spontaneous decay rate

$$\Gamma_c = \frac{4}{3\kappa}\frac{|P_{12}|^2\omega}{2\hbar\epsilon_0 V} \propto 2\pi|g|^2\rho(\omega) \propto 2\pi|g|^2 F_{\mathrm{Purcell}}\rho_{\text{free space}}$$

and the density of state $\rho(\omega)$ of resonator is

$$\rho(\omega) = \frac{1}{\pi}\frac{\kappa/2}{(\omega-\omega_c)^2 + (\kappa/2)^2}\frac{1}{V}.$$



Purcell Effect in Spontaneous Emission

The Purcell factor expressed as the Quality factor (Q) and the Mode Volume of the cavity (V)

$$\Gamma_c = \Gamma_0 \cdot \frac{3Q}{4V} \cdot \lambda^3$$

B10. Spontaneous Emission Probabilities at Radio Frequencies. E. M. PURCELL, *Harvard University*.—For nuclear magnetic moment transitions at radio frequencies the probability of spontaneous emission, computed from

$$A_\nu = (8\pi\nu^2/c^3)h\nu(8\pi^3\mu^2/3h^2) \text{ sec.}^{-1},$$

is so small that this process is not effective in bringing a spin system into thermal equilibrium with its surroundings. At 300°K, for $\nu = 10^7$ sec.$^{-1}$, $\mu = 1$ nuclear magneton, the corresponding relaxation time would be 5×10^{21} seconds! However, for a system coupled to a resonant electrical circuit, the factor $8\pi\nu^2/c^3$ no longer gives correctly the number of radiation oscillators per unit volume, in unit frequency range, there being now *one* oscillator in the frequency range ν/Q associated with the circuit. The spontaneous emission probability is thereby increased, and the relaxation time reduced, by a factor $f = 3Q\lambda^3/4\pi^2V$, where V is the volume of the resonator. If a is a dimension characteristic of the circuit so that $V\sim a^3$, and if δ is the skin-depth at frequency ν, $f\sim\lambda^3/a^2\delta$. For a non-resonant circuit $f\sim\lambda^3/a^3$, and for $a<\delta$ it can be shown that $f\sim\lambda^3/a\delta^2$. If small metallic particles, of diameter 10^{-3} cm are mixed with a nuclear-magnetic medium at room temperature, spontaneous emission should establish thermal equilibrium in a time of the order of minutes, for $\nu = 10^7$ sec.$^{-1}$.

Purcell Effect in Spontaneous Emission

Notice that the Purcell effect is an effect that is due to the enhancement of the mode density. The mode density can also be modified to inhibit spontaneous emission by trapping an electron in a Penning trap with dimension comparable to the emission wavelength[2]. One can compute the equations of motion[3] for $\langle \hat{a}^\dagger \hat{a} \rangle$ and $\langle \sigma_z \rangle$:

$$\frac{d}{dt}\langle \hat{a}^\dagger \hat{a} \rangle = -ig\langle \sigma_+ \hat{a} - \hat{a}^\dagger \sigma_- \rangle - \kappa \langle \hat{a}^\dagger \hat{a} \rangle + \kappa \underbrace{\overline{n}_{\text{th}}}_{\approx 0}$$

$$\frac{d}{dt}\langle \sigma_z \rangle = -ig\langle \sigma_+ \hat{a} - \hat{a}^\dagger \sigma_- \rangle$$

Notice that the average of the operator $\langle \sigma_+ \hat{a} - \hat{a}^\dagger \sigma_- \rangle$ has its equation of motion involving the quantity $\langle \hat{a}^\dagger \sigma_z \hat{a} \rangle$. In general, we get an infinite set of equations which may not be analytically solvable, but can be considerably simplified if initially the atom is in the excited state and the field inside the cavity is in the vacuum state.

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[2] Gabrielse,Gerald,and Hans Dehmelt.“Observation of inhibited spontaneous emission.”Physical review letters 55.1 (1985): 67.

[3] Scully, M.O., Zubairy, M.S. "Quantum optics” (1999). Chapter 9, Section 5

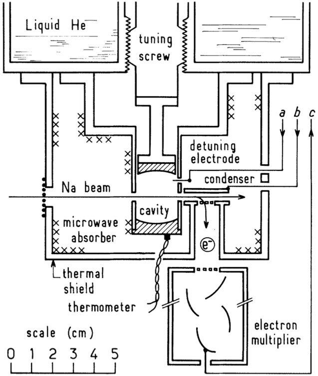
Purcell Effect in Spontaneous Emission: Example[4]

VOLUME 50, NUMBER 24 PHYSICAL REVIEW LETTERS 13 JUNE 1983

Observation of Cavity-Enhanced Single-Atom Spontaneous Emission

P. Goy, J. M. Raimond, M. Gross, and S. Haroche
Laboratoire de Physique de l'Ecole Normale Supérieure, F-75231 Paris Cedex 05, France
(Received 1 April 1983)

It has been observed that the spontaneous-emission lifetime of Rydberg atoms is shortened by a large ratio when these atoms are crossing a high-Q superconducting cavity tuned to resonance with a millimeter-wave transition between adjacent Rydberg states.



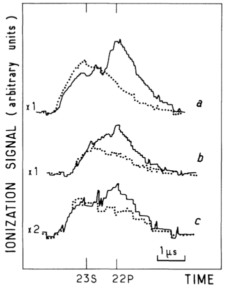


FIG. 3. Cavity-enhanced spontaneous emission signals. Dotted line, off-resonant cavity; full line, resonant cavity. The average numbers of atoms in the cavity are respectively 3.5, 2, and 1.3 in traces a, b, and c. Traces a and c correspond to $23S \rightarrow 22P_{3/2}$, trace b to $23S \rightarrow 22P_{1/2}$.

[4] Goy,Ph.et al.'Observation of cavity-enhanced single-atom spontaneous emission."Phys. Rev. Lett. 50.24 (1983): 1903.

Purcell Effect in Spontaneous Emission: Example[5]

VOLUME 83, NUMBER 7 PHYSICAL REVIEW LETTERS 16 AUGUST 1999

Observing the Quantum Limit of an Electron Cyclotron: QND Measurements of Quantum Jumps between Fock States

S. Peil and G. Gabrielse

Department of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 18 March 1999)

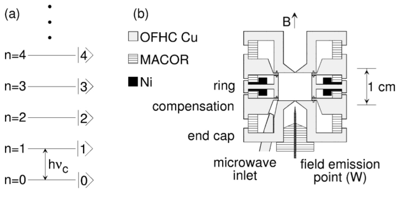


FIG. 1. (a) Energy levels of the one-electron cyclotron oscillator. (b) Electrodes of the cylindrical Penning trap cavity.

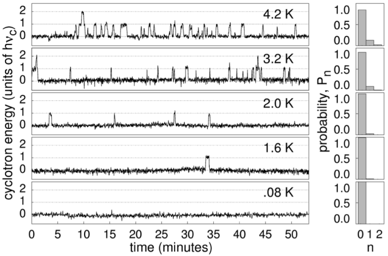


FIG. 2. Quantum jumps between the lowest states of the one-electron cyclotron oscillator decrease in frequency as the cavity temperature is lowered.

[5]Peil, S., and G. Gabrielse. "Observing the quantum limit of an electron cyclotron: QND measurements of quantum jumps between Fock states." Phys. Rev. Lett. 83.7 (1999): 1287.

Paper for this week

Vol 445 | 1 February 2007 | **doi:10.1038/nature05461**

nature

LETTERS

Resolving photon number states in a superconducting circuit

D. I. Schuster[1*], A. A. Houck[1*], J. A. Schreier[1], A. Wallraff[1†], J. M. Gambetta[1], A. Blais[1†], L. Frunzio[1], J. Majer[1], B. Johnson[1], M. H. Devoret[1], S. M. Girvin[1] & R. J. Schoelkopf[1]

Questions for the presenter

- Describe the parameter diagram in Fig 1 and where is the system of the paper in this diagram?
- What is the requirement on the strong dispersive regime?
- Formally describe how the dispersive Hamiltonian emerges in strong dispersive regime.
- What regime is the qubit in, CPB or transmon?
- What is QND measurement?
- What's the photon distribution of different states probed in the experiment?
- How is the state of the cavity prepared?
- Explain the experimental measurement scheme to retrieve Fig 3.
- What determines the width of each photon number peak?