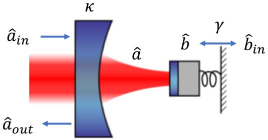
Quantum Electrodynamics and Quantum Optics: Lecture 13

Fall 2024

Ponderomotive squeezing[1]



System Hamiltonian

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_\mathrm{m} \hat{b}^\dagger \hat{b} + \frac{\hbar\omega_c}{L} x_\mathrm{zpf} \hat{a}^\dagger \hat{a} \left(\hat{b} + \hat{b}^\dagger\right)$$

Introducing position and momentum operators $\hat{Q} = \frac{1}{2}(\hat{b} + \hat{b}^\dagger)$, $\hat{P} = \frac{i}{2}(\hat{b} - \hat{b}^\dagger)$, and coupling constant $g_0 = \frac{\omega_c}{L} x_\mathrm{zpf}$, we get

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_\mathrm{m} \left[\hat{P}^2 + \left(\hat{Q} + \frac{g_0}{\Omega_\mathrm{m}} \hat{a}^\dagger \hat{a}\right)^2\right] - \frac{\hbar g_0^2}{\Omega_\mathrm{m}} (\hat{a}^\dagger \hat{a})^2.$$

To isolate the light field dynamics from the mechanics, we introduce *polaron transformation* $\hat{s} = \exp\left(i \frac{g_0}{\Omega_\mathrm{m}} \hat{a}^\dagger \hat{a} \hat{P}\right)$ to transform our Hamiltonian using

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!} \left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \ldots,$$

[1] Quantum Optomechanics - Bowen - Chapter 4

Ponderomotive squeezing

we get

$$\hat{s}^\dagger \hat{Q} \hat{s} = \hat{Q} - \frac{g_0}{\Omega_\mathrm{m}} \hat{a}^\dagger \hat{a}.$$

The polaron transformation cancels the displacement caused by radiation pressure by applying an opposite but equivalent displacement via the unitary operation. The Hamiltonian would thus transform into

$$\hat{H}' = \hat{s}^\dagger \hat{H} \hat{s} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\Omega_\mathrm{m} \left(\hat{P}^2 + \hat{Q}^2\right) - \frac{\hbar g_0^2}{\Omega_\mathrm{m}} (\hat{a}^\dagger \hat{a})^2.$$

We can now linearize it using $\hat{a} \to \alpha + \delta\hat{a}$, so that

$$(\hat{a}^\dagger \hat{a})^2 \to \alpha^4 + 2\alpha^3(\delta\hat{a}^\dagger + \delta\hat{a}) + 4\alpha^2 \delta\hat{a}\delta\hat{a}^\dagger + \alpha^2(\delta\hat{a}^{\dagger 2} + \delta\hat{a}^2)$$

$$\frac{g_0^2}{\Omega_\mathrm{m}} \alpha^2(\delta\hat{a}^{\dagger 2} + \delta\hat{a}^2) \Leftrightarrow S(\xi) = \exp\left(\frac{1}{2}\xi^* \hat{a}^2 + \frac{1}{2}\xi \hat{a}^{\dagger 2}\right),$$

whose unitary evolution leads to a squeezing operator.

Quantum Langevin Equations

We first move to an interaction frame where ω_c is replaced by $\Delta = \omega_c - \omega_L$ where ω_L is the laser driving frequency. To include the dissipation and noise from the environment, we derive

Quantum Langevin equations

$$\dot{\hat{a}}(t) = -\left(i\Delta + \frac{\kappa}{2}\right)\hat{a} - ig_0\hat{a}\left(\hat{b} + \hat{b}^\dagger\right) + \sqrt{\kappa}\hat{a}_{\mathrm{in}}(t),$$

$$\dot{\hat{b}}(t) = -\left(i\Omega_{\mathrm{m}} + \frac{\Gamma_{\mathrm{m}}}{2}\right)\hat{b} - ig_0\hat{a}^\dagger\hat{a} + \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}(t),$$

where Γ_{m} is the mechanical dissipation rate. Here we assume $g_0 \ll \kappa$ in order to linearize the equations $\hat{a} \to \alpha + \hat{a}$. Introducing $g = g_0\sqrt{\alpha}$ and by going to the Fourier domain,

$$\hat{a}[\omega] = \frac{\sqrt{\kappa}\hat{a}_{\mathrm{in}} - ig(\hat{b}[\omega] + \hat{b}^\dagger[\omega])}{i(\Delta - \omega) + \kappa/2},$$

$$\hat{b}[\omega] = \frac{\sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}[\omega]}{i(\Omega_{\mathrm{m}} - \omega) + \Gamma_{\mathrm{m}}/2} - \frac{ig(\hat{a}[\omega] + \hat{a}^\dagger[\omega])}{i(\Omega_{\mathrm{m}} - \omega) + \Gamma_{\mathrm{m}}/2}.$$

Quantum Langevin Equations

The mechanical motion can be expressed as a response to the environmental noise and optical input fluctuations

$$\hat{b}[\omega] = \frac{\sqrt{\Gamma_m}\hat{b}_{\mathrm{in}}[\omega]}{i(\Omega'_{\mathrm{m}} - \omega) + \Gamma'/2} + \frac{ig}{i(\Delta - \omega) + \kappa/2} \frac{-\sqrt{\kappa}\hat{a}_{\mathrm{in}}[\omega]}{i(\Omega'_{\mathrm{m}} - \omega) + \Gamma'/2} + \frac{ig}{-i(\Delta - \omega) + \kappa/2} \frac{-\sqrt{\kappa}\hat{a}^{\dagger}_{\mathrm{in}}[\omega]}{i(\Omega'_{\mathrm{m}} - \omega) + \Gamma'/2}$$

Optical spring and dynamical back-action

where $\Omega'_{\mathrm{m}} = \Omega_{\mathrm{m}} + \delta\Omega_{\mathrm{m}}$ and $\Gamma' = \Gamma_{\mathrm{m}} + \delta\Gamma_{\mathrm{m}}$ with

$$\delta\Omega_{\mathrm{m}} = g^2 \mathrm{Im}\left[\frac{1}{i(\Delta - \Omega_{\mathrm{m}}) + \kappa/2} - \frac{1}{-i(\Delta + \Omega_{\mathrm{m}}) + \kappa/2}\right]$$

$$\delta\Gamma_{\mathrm{m}} = 2g^2 \mathrm{Re}\left[\frac{1}{i(\Delta - \Omega_{\mathrm{m}}) + \kappa/2} - \frac{1}{-i(\Delta + \Omega_{\mathrm{m}}) + \kappa/2}\right].$$

Optical spring and dynamical back-action

Optical spring and dynamical back-action

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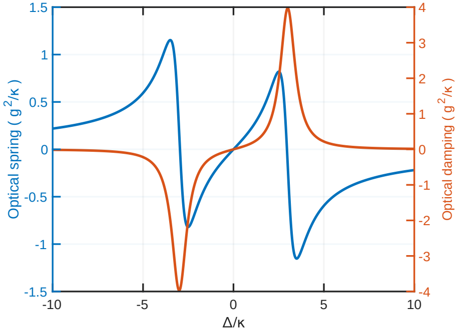


Quasi-static approximation[2]

Here we make a few approximations to simplify the derivation:

Quasi-static approximation

1. $\Delta = 0$: the laser is tuned exactly to the optical cavity resonance frequency.
2. $\kappa \gg \Omega_{\mathrm{m}}$: Bad cavity limit.
3. $\omega \ll \Omega_{\mathrm{m}}$: We are only interested in the quasi-static response, so the resonant response of the mechanical resonator does not play a role.

Under these assumptions together with the input output theorem $\hat{a}_{\mathrm{out}} + \hat{a}_{\mathrm{in}} = \sqrt{\kappa}\hat{a}$,

$$
\begin{aligned}
i\Omega_{\mathrm{m}}\hat{b}[\omega] =& \sqrt{\Gamma_{\mathrm{m}}}\hat{b}_{\mathrm{in}}[\omega] - \frac{2ig}{\sqrt{\kappa}}(\hat{a}_{\mathrm{in}}[\omega] + \hat{a}^{\dagger}{}_{\mathrm{in}}[\omega]) \\
\hat{a}_{\mathrm{out}}[\omega] =& \hat{a}_{\mathrm{in}}[\omega] - \frac{2ig}{\sqrt{\kappa}}(\hat{b}[\omega] + \hat{b}^{\dagger}[\omega]).
\end{aligned}
$$

[2] Safavi-Naeini, Amir H., et al. "Squeezed light from a silicon micromechanical resonator." Nature 500.7461 (2013): 185.

Quasi-static approximation

Define measurement rate $\Gamma_{\mathrm{meas}} \equiv 4g^2/\kappa$, we can rewrite

$$\hat{a}_{\mathrm{out}}[\omega] = \hat{a}_{\mathrm{in}} + \frac{2i\Gamma_{\mathrm{meas}}}{\Omega_{\mathrm{m}}}(\hat{a}_{\mathrm{in}}[\omega] + \hat{a}^{\dagger}{}_{\mathrm{in}}[\omega]) + \frac{\sqrt{\Gamma_{\mathrm{m}}\gamma_{\mathrm{meas}}}}{\Omega_{\mathrm{m}}}(\hat{b}_{\mathrm{in}}[\omega] + \hat{b}^{\dagger}_{\mathrm{in}}[\omega]).$$

Ignoring thermal noise $\Gamma_{\mathrm{m}} = 0$, and dropping the terms of order $(\Gamma_{\mathrm{meas}}/\Omega_{\mathrm{m}})^2$, we can calculate the output optical quadrature power spectral density

Output optical quadrature

$$S^{\mathrm{out}}_{X_\theta X_\theta} = \int_{-\infty}^{+\infty} \mathrm{d}\omega' \left\langle \hat{X}^{\mathrm{out}}_{\theta}[\omega]\hat{X}^{\mathrm{out}}_{\theta}[\omega'] \right\rangle = 1 + \frac{4\Gamma_{\mathrm{meas}}}{\Omega_{\mathrm{m}}}\sin 2\theta,$$

where the quadrature operator is defined as $\hat{X}_\theta = \hat{a}e^{-i\theta} + \hat{a}^{\dagger}e^{i\theta}$. For $\theta = -\pi/4$ we achieve the maximum squeezing with a noise floor strongly dependents on the ratio $\Gamma_{\mathrm{meas}}/\Omega_{\mathrm{m}}$.

Quasi-static approximation

To include the effects from thermal noises, we assume the form of the correlators to be

Thermal correlators

$$\left\langle \hat{b}_{\text{in}}[\omega]\hat{b}_{\text{in}}^{\dagger}[\omega']\right\rangle =(\bar{n}_{\text{th}}[\omega]+1)\delta(\omega+\omega'),$$
$$\left\langle \hat{b}_{\text{in}}^{\dagger}[\omega]\hat{b}_{\text{in}}[\omega']\right\rangle =\bar{n}_{\text{th}}[\omega]\delta(\omega+\omega'),$$

which leads to

$$S_{X_\theta X_\theta}^{\text{out}} = 1 + \frac{4\Gamma_{\text{meas}}}{\Omega_{\text{m}}}\sin 2\theta + \frac{4\Gamma_{\text{meas}}}{\Omega_{\text{m}}}\frac{\bar{n}_{\text{th}}[\omega]}{Q_{\text{m}}}(1-\cos 2\theta).$$

In this model there is no squeezing at $\theta = -\pi/4$ and frequency ω if $\bar{n}_{\text{th}}[\omega] > Q_{\text{m}}$, where Q_{m} is the mechanical quality factor. However some squeezing is always present, but is shifted to other quadratures and the amount of detectable squeezing is reduced at higher temperatures.

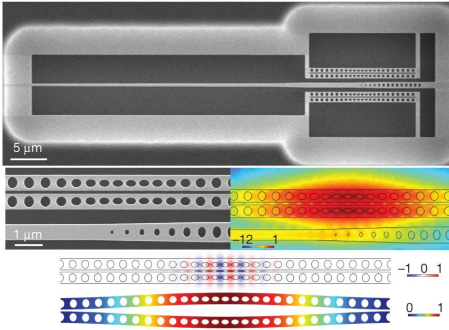
Physical systems[3]

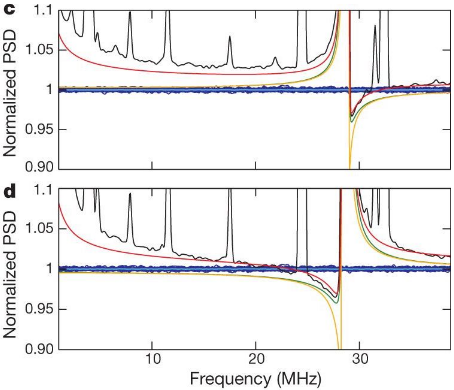
LETTER

doi:10.1038/nature12307

Squeezed light from a silicon micromechanical resonator

Amir H. Safavi-Naeini[1,2]*, Simon Gröblacher[1,2]*, Jeff T. Hill[1,2]*, Jasper Chan[1], Markus Aspelmeyer[3] & Oskar Painter[1,2,4]





[3] Safavi-Naeini, Amir H., et al. "Squeezed light from a silicon micromechanical resonator." Nature 500.7461 (2013): 185.

Physical systems[4]

Strong Optomechanical Squeezing of Light

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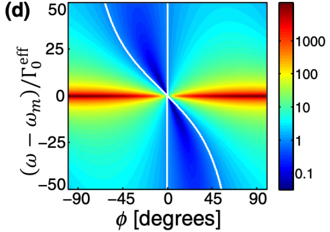
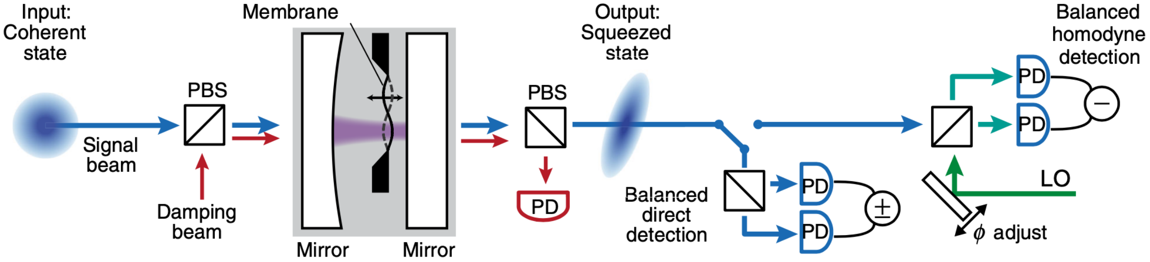




[4] Purdy, Thomas P., et al. "Strong optomechanical squeezing of light." Physical Review X 3.3 (2013): 031012.

Dynamics of mechanical resonator [5]

Taking into account the dynamics of the mechanical resonator while keeping the bad-cavity limit and resonant probing, we can derive the detected spectral density detected by the homodyne as

Homodyne signal

$$S^{\theta}_{II}[\omega] = \frac{1}{2} + 8\Gamma_{\mathrm{m}}^2 |\chi[\omega]|^2 n_{\mathrm{QBA}} (n_{\mathrm{th}} + n_{\mathrm{QBA}} + \frac{1}{2}) \sin^2\theta + 2\Gamma_{\mathrm{m}} \mathrm{Re}[\chi[\omega]] n_{\mathrm{QBA}} \sin 2\theta$$

where $\chi[\omega] = \frac{\Omega_{\mathrm{m}}\Gamma_{\mathrm{m}}}{(\Omega_{\mathrm{m}}^2 - \omega^2 - i\Gamma_{\mathrm{m}}\omega)}$ is the mechanical susceptibility. Here, we introduced the effective optomechanical cooperativity $C_{\mathrm{eff}}[\omega] \equiv \frac{C}{(1-2i\omega/\kappa)^2}$ and $n_{\mathrm{QBA}} = |C_{\mathrm{eff}}|$, with $C \equiv \frac{4g^2}{\kappa\Gamma}$ being the optomechanical cooperativity.

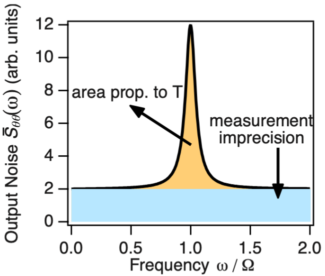
[5] Quantum Optomechanics - Bowen - Chapter 3,4

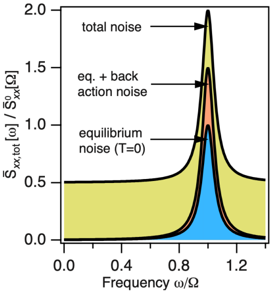
Standard quantum limit[6]

The phase quadrature ($\theta = \frac{\pi}{2}$) homodyne power spectral density is simply

Phase quadrature

$$\begin{aligned} S_{\text{phase}}[\omega] &= \frac{1}{2} + 8\Gamma_{\text{m}}|C_{\text{eff}}|^2|\chi[\omega]|^2 + 4\Gamma_{\text{m}}|C_{\text{eff}}|S_{Q^0Q^0[\omega]} \\ &= \frac{1}{2} + 4\Gamma_{\text{m}}|C_{\text{eff}}|S_{QQ[\omega]} \end{aligned}$$



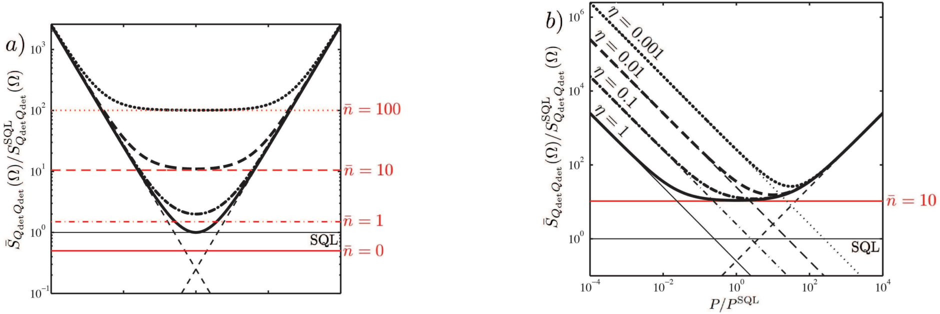


[6] Clerk, Aashish A., et al. "Introduction to quantum noise, measurement, and amplification." Reviews of Modern Physics 82.2 (2010): 1155.

Standard quantum limit[7]

Phase quadrature

$$S_{\text{phase}}[\omega] = \frac{1}{2} + 8\Gamma_{\text{m}}|C_{\text{eff}}|^2|\chi[\omega]|^2 + 4\Gamma_{\text{m}}|C_{\text{eff}}|S_{Q^0Q^0[\omega]}$$
$$= \frac{1}{2} + 4\Gamma_{\text{m}}|C_{\text{eff}}|S_{QQ[\omega]}$$



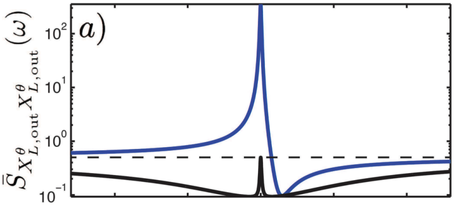
[7] Quantum Optomechanics - Bowen - Chapter 3

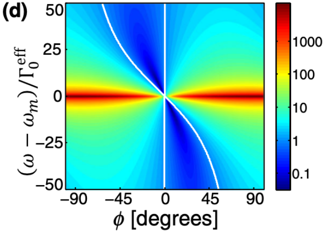
Ponderomotive squeezing

Homodyne signal

$$S_{II}^{\theta}[\omega] = \frac{1}{2} + 8\Gamma_{\mathrm{m}}^{2}|\chi[\omega]|^{2} n_{\mathrm{QBA}}(n_{\mathrm{th}} + n_{\mathrm{QBA}} + \frac{1}{2})\sin^{2}\theta + 2\Gamma_{\mathrm{m}}\mathrm{Re}[\chi[\omega]] n_{\mathrm{QBA}}\sin 2\theta$$

Squeezing when the PSDs of different quadrature angles are probed[8][9].





[8] Quantum Optomechanics - Bowen - Chapter 4

[9] Purdy, Thomas P., et al. "Strong optomechanical squeezing of light." Physical Review X 3.3 (2013): 031012.