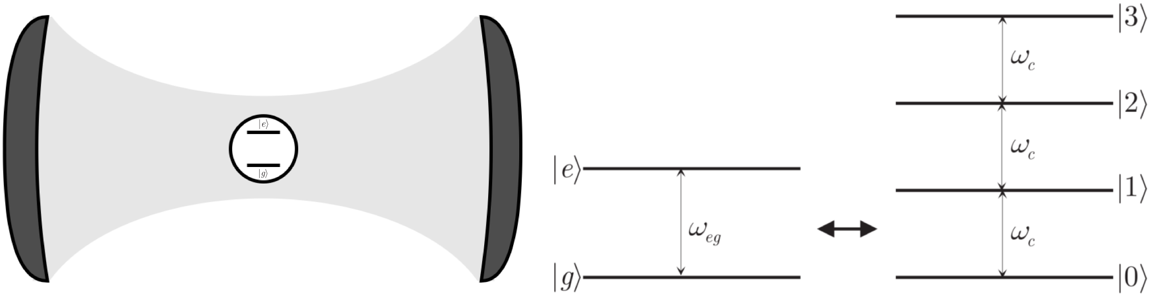
Quantum Electrodynamics and Quantum Optics: Lecture 10

Fall 2024

Coupling an atom to a cavity: Jaynes-Cummings model

A two level system coupled to a cavity (harmonic oscillator)



System Hamiltonian

$$\hat{H} = \underbrace{\frac{\hbar\omega_{\mathrm{eg}}}{2}\hat{\sigma}_z}_{\text{atom}} + \underbrace{\hbar\omega\hat{a}^\dagger\hat{a}}_{\text{cavity}} + \underbrace{\hbar g(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger)}_{\text{atom-cavity coupling}}$$

Jaynes-Cummings model: Uncoupled case[1]

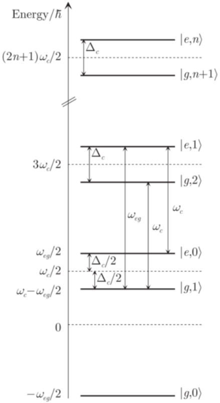


Figure: Uncoupled atom–cavity energy states in the case $\Delta_c > 0$, where $\Delta_c = \omega_{\text{eg}} - \omega_c$.

Apart from the ground state $|g, 0\rangle$, the atom-cavity system forms an infinite ladder of two-level manifolds separated by the cavity frequency ω_c. In each manifold, the states $|e, n\rangle$ and $|g, n+1\rangle$ are separated by $-\hbar\Delta_c$.

[1] Haroche, Serge, and J-M. Raimond. Exploring the quantum: atoms, cavities, and photons. Oxford university press, 2006.

Jaynes-Cummings model: Uncoupled case[1]

Assuming the uncoupled Hamiltonian and finding the eigenstates and eigenenergies:

Eigenstates and eigenenergies

$$\text{Eigenstates}: \; |e,n\rangle, \; |g,n\rangle$$
$$\text{Eigenenergies}: E_{e,n} = \hbar\left(\omega_{\text{eg}}/2 + n\omega_c\right), \; E_{g,n} = \hbar\left(-\omega_{\text{eg}}/2 + n\omega_c\right)$$

Defining $\Delta_c = \omega_{\text{eg}} - \omega_c$:

Eigenenergies

$$\text{Energies}: \; E_{e,n} = \hbar\left(\Delta_c/2 + (n+1/2)\omega_c\right), \; E_{g,n} = \hbar\left(-\Delta_c/2 + (n-1/2)\omega_c\right)$$

In the limit of $\Delta_c \to 0$ eigenstates $|e,n\rangle$ and $|g,n+1\rangle$ are degenerate.

[1] Haroche, Serge, and J-M. Raimond. Exploring the quantum: atoms, cavities, and photons. Oxford university press, 2006.

Dispersive regime of cavity QED: Eigenstate approach[2]

The Hamiltonian can be expressed in the basis formed by $|g, n+1\rangle$ and $|e, n\rangle$:

$$\hat{H} = \begin{bmatrix} \frac{\hbar\omega_{\mathrm{eg}}}{2} + n\hbar\omega & \hbar g\sqrt{n+1} \\ \hbar g\sqrt{n+1} & (n+1)\hbar\omega - \frac{\hbar\omega_{\mathrm{eg}}}{2} \end{bmatrix}$$

Note: Off-diagonal terms are equal

$$\langle g, n+1| \, \hat{\sigma}^- \hat{a}^\dagger \, |e, n\rangle = \sqrt{n+1}$$
$$\langle e, n| \, \hat{\sigma}^+ \hat{a} \, |g, n+1\rangle = \sqrt{n+1}$$

[2] Quantum Optics, Scully, chapter 19.3

Dispersive regime of cavity QED: Eigenstate approach[2]

The Hamiltonian can be diagonalized[1] to obtain atom-field dressed states:

Eigenstates - Eigenenergies

$$
\begin{aligned}
\left|+\right\rangle_n &= \cos\theta_n \left|e,n\right\rangle + \sin\theta_n \left|g,n+1\right\rangle \\
\left|-\right\rangle_n &= -\sin\theta_n \left|e,n\right\rangle + \cos\theta_n \left|g,n+1\right\rangle \\
E_{+\mathrm{n}} &= \hbar\omega n + \frac{\hbar}{2}\omega_{\mathrm{eg}} - \frac{\hbar}{2}(\Omega_n - \Delta) \\
E_{-\mathrm{n}} &= \hbar\omega(n+1) - \frac{\hbar}{2}\omega_{\mathrm{eg}} + \frac{\hbar}{2}(\Omega_n - \Delta)
\end{aligned}
$$

where $\theta_n = \frac{1}{2}\tan^{-1}\left(\frac{g\sqrt{n+1}}{\Delta}\right)$ and $\Omega_n \equiv \sqrt{4g^2(n+1)+\Delta^2}$ is the multiphoton off-resonant Rabi frequency.

[1] Homework 10 Exercise 1

[2] Quantum Optics, Scully, chapter 19.3

Dispersive regime of cavity QED: $4g^2(n+1) \ll \Delta^2$

Dispersive-limit eigenstates and eigenenergies

$$
\begin{aligned}
\left|+\right\rangle_n &\approx \left|e,n\right\rangle \\
\left|-\right\rangle_n &\approx \left|g,n+1\right\rangle \\
E_{\pm\mathrm{n}} &= \hbar\left((n+1)\omega \pm \frac{\Delta}{2}\right) \pm \frac{\hbar g^2(n+1)}{\Delta} \\
E_{g,0} &= -\frac{\hbar\Delta}{2}
\end{aligned}
$$

The eigenenergies assume the above form due to the following approximation in the dispersive limit $\left(\frac{4g^2(n+1)}{\Delta^2} \ll 1\right)$:

Qubit state-dependent shift of the cavity frequency

$$
\frac{\hbar}{2}(\Omega_n - \Delta) = \frac{\hbar}{2}\left(\Delta\sqrt{1+\frac{4g^2(n+1)}{\Delta^2}} - \Delta\right) \approx \hbar\frac{g^2(n+1)}{\Delta}
$$

Dispersive regime of cavity QED: Energy shifts [1]

The energy shifts of the states $|e, n\rangle$ and $|g, n\rangle$ can be calculated for uncoupled and dressed states in dispersive limit:

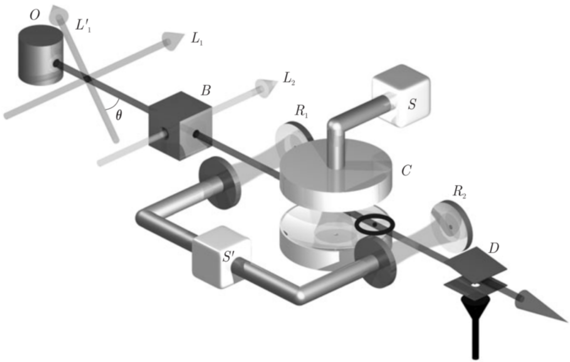
$$\Delta_{\mathrm{e,n}} = \hbar(n+1)\frac{\Omega_0^2}{\Delta_c};\ \Delta_{\mathrm{g,n}} = -\hbar n\frac{\Omega_0^2}{\Delta_c}$$

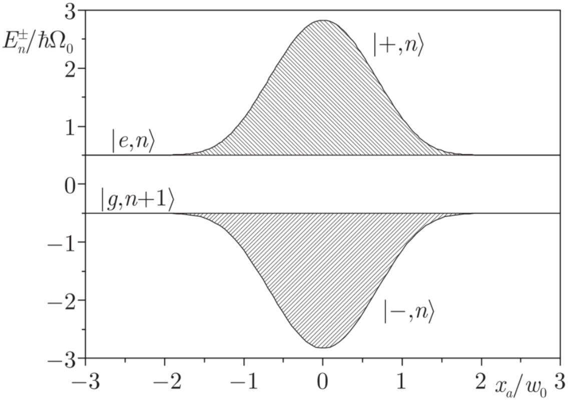
where Ω_0 is the Rabi frequency and Δ_c is the atom-cavity detuning.

The atom behaves in this case as a small speck of dielectric material whose index is greater than 1. As a piece of glass in a light beam, it experiences when entering the cavity a repulsive dipole force, proportional to the gradients of the n-photon field and of the vacuum fluctuations in the cavity. This force tends to expel it from the center of the cavity. To overcome it, the atom loses some of its kinetic energy, precisely balancing the gain in energy of the cavity field and its own gain of “Lamb shift energy”. The residual force remaining when $n = 0$ is a kind of cavity-induced Casimir effect.

[1] Haroche, Serge, and J-M. Raimond. Exploring the quantum: atoms, cavities, and photons. Oxford university press, 2006.

Atom passing through the cavity[1]

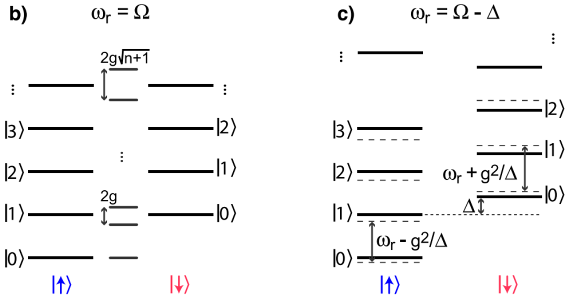




Position of the dressed energies $E_{\pm n}$, in units of $\hbar\Omega_0$, as a function of the atomic position x_a, in units of the cavity mode waist w_0, for $\Delta_c = \Omega_0$ and $n = 30$. The energy origin is taken as $(n + 1/2)\hbar\omega_c$.

[1] Haroche, Serge, and J-M. Raimond. Exploring the quantum: atoms, cavities, and photons. Oxford university press, 2006.

Dispersive regime of cavity QED: Level diagram [3]



Energy spectrum of the uncoupled (left and right) and dressed (center) atom-photon states in the case of zero detuning. The degeneracy of the two-dimensional manifolds of states with $n-1$ quanta is lifted by $2g\sqrt{n+1}$. c) Energy spectrum in the dispersive regime (long dash lines). To second order in g, the level separation is independent of n, but depends on the state of the atom.

[3] Blais, Alexandre, et al. "Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation." Physical Review A 69.6 (2004): 062320.

Double QND Hamiltonian in dispersive regime[3]

More insight into the dispersive regime is gained by making the unitary transformation:

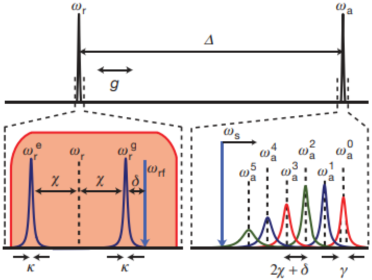
$$\hat{U} = \exp\left[\frac{g}{\Delta}(\hat{a}\hat{\sigma}^{+} - \hat{a}^{\dagger}\hat{\sigma}^{-})\right]$$

Expanding to second order in g:

$$\hat{U}\hat{H}\hat{U}^{\dagger} \approx \hbar\left[\omega + \frac{g^2}{\Delta}\hat{\sigma}_z\right]\hat{a}^{\dagger}\hat{a} + \frac{\hbar}{2}\left[\omega_{\mathrm{eg}} + \frac{g^2}{\Delta}\right]\hat{\sigma}_z$$

[3] Blais, Alexandre, et al. "Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation." Physical Review A 69.6 (2004): 062320.

Double QND Hamiltonian: Spectral response

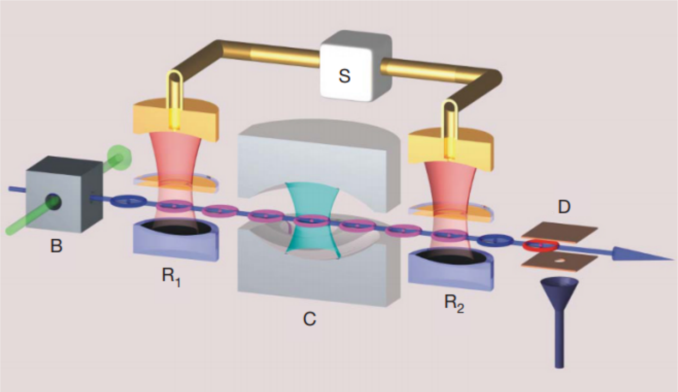


$$\hat{H} = \hbar\omega_r(\hat{a}^\dagger\hat{a} + 1/2) + \hbar\omega_a\hat{\sigma}_z/2 + \hbar\chi(\hat{a}^\dagger\hat{a} + 1/2)\hat{\sigma}_z, \ \chi = \frac{g^2}{\Delta}$$

A double QND Hamiltonian

As is clear from this expression, the atom transition is shifted by $\chi(n+1/2)$. Alternatively, one can interpret the shift as a dispersive shift of the cavity transition by $\chi\sigma_z$. In other words, the atom pulls the cavity frequency by $\pm\chi$.

Example: Birth, life and death of a photon[4]



A thermal beam of rubidium atoms passing through a superconducting microwave cavity in the dispersive limit.

[4] Gleyzes, Sebastien, et al. "Quantum jumps of light recording the birth and death of a photon in a cavity." Nature 446.7133 (2007): 297.

Example: Birth, life and death of a photon[4]

When cavity C contains n photons, the uncoupled atom-cavity states $|e, n\rangle$ and $|g, n\rangle$ evolve into dressed states, shifted respectively, in angular frequency unit, by:$+(\sqrt{\Delta_c + (n+1)g^2} - \Delta_c)/2$ and $-(\sqrt{\Delta_c + ng^2} - \Delta_c)/2$. The difference between these frequencies integrated overtime, yields the phase shift of atom passing the cavity $\Phi(n, \Delta_c)$.

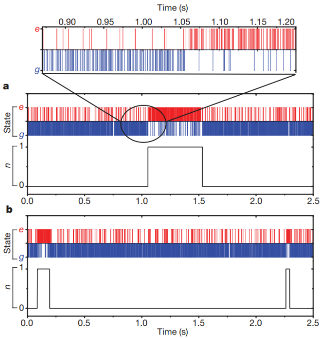
π phase shift per photon

$$\Phi(1, \Delta_c) - \Phi(0, \Delta_c) = \pi$$

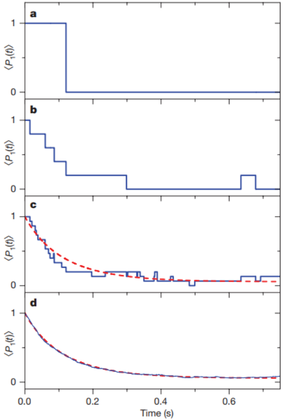
If this condition is met, the Ramsey pulse in R_2 ideally brings the atom into g if $n = 0$ and e if $n = 1$.

[4] Gleyzes, Sebastien, et al. "Quantum jumps of light recording the birth and death of a photon in a cavity." Nature 446.7133 (2007): 297.

Example: Birth, life and death of a photon[4]



QND detection of a single photon (Quantum jumps).



Measurement of the decay of a photon.

[4] Gleyzes, Sebastien, et al. "Quantum jumps of light recording the birth and death of a photon in a cavity." Nature 446.7133 (2007): 297.

Paper for next week's presentation

Reconstruction of non-classical cavity field states with snapshots of their decoherence

Samuel Deléglise[1], Igor Dotsenko[1,2], Clément Sayrin[1], Julien Bernu[1], Michel Brune[1], Jean-Michel Raimond[1] & Serge Haroche[1,2]

Questions for next week's paper

- What are the different time scales in the system, e.g. photon life time, atom life time, atom-photon interaction time?
- How many thermal photons are there in the cavity due to the finite temperature of the environment?
- What is the initial state prepared in the cavity?
- How are the atoms and their states used in this experiment? How does the energy level separation compare to the cavity frequency?
- How is the atomic Ramsey interferometer used to obtain the cavity photon number?