Quantum Electrodynamics and Quantum Optics: Lecture 1

Fall 2024

Quantized Harmonic Oscillator[1]

Classical Hamiltonian of a harmonic oscillator:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2x^2$$

We can also express the state of the H.O. via a single complex variable:

$$\alpha(t) = c\left(x(t) + \frac{i\dot{x}(t)}{\omega}\right)$$

where c is a constant. In this case:

$$\begin{aligned}
\frac{\partial}{dt}\alpha(t) &= \dot{\alpha}(t) = c\left(\dot{x} + \frac{i}{\omega}\ddot{x}(t)\right) \\
&\overset{\ddot{x}=-\omega^2x}{=} c\left(\dot{x} - i\omega x\right) = -i\omega c\left(x + \frac{i}{\omega}\dot{x}\right) \\
\partial_t\alpha &= -i\omega\alpha(t)
\end{aligned}$$

[1] Cohen–Tannoudji C., Diu B., Laloe F. (2017) Mécanique quantique - Tome 3

Quantized Harmonic Oscillator

This new variable can be used to express the energy of the H.O. as:

$$E = \frac{m\omega^2}{4c^2}\left(\alpha^*\alpha + \alpha\alpha^*\right).$$

The quantization of the harmonic oscillator

The quantization of the H.O. proceeds by normalizing the hamiltonian with $\frac{\hbar\omega}{2} \equiv \frac{m\omega^2}{4c^2}$, and replacing α and α^* by $\alpha \to \hat{a}$ and $\alpha^* \to \hat{a}^\dagger$ which are the annihilation and creation operators with the following rules

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \qquad \hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \qquad \left[\hat{a},\hat{a}^\dagger\right] = 1.$$

The procedure yields

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right). \tag{1}$$

Quantized Harmonic Oscillator

Note the corresponding relation between creation annihilation operators and position momentum operators:

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + i\hat{p}\right) \\ \hat{a}^{\dagger} &= \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} - i\hat{p}\right) \end{aligned}$$

Or

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^{\dagger}\right) \\ \hat{p} &= \sqrt{\frac{m\hbar\omega}{2}}\frac{\left(\hat{a} - \hat{a}^{\dagger}\right)}{i} \end{aligned}$$

Quantized Harmonic Oscillator

Thus: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$,

Heisenberg equations of motion

$$\begin{aligned}\frac{d}{dt}\hat{x} &= \frac{1}{i\hbar}\left[\hat{x}, \hat{H}\right] = \frac{\hat{p}}{m} \\ \frac{d}{dt}\hat{p} &= \frac{1}{i\hbar}\left[\hat{p}, \hat{H}\right] = -m\omega^2\hat{x}\end{aligned}$$

In order to derive the wave function from the Schrödinger equation, we recall that $\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$. Take $\left|0\right\rangle$ for an example:

$$\hat{a}\left|0\right\rangle = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{p} - im\omega\hat{x}\right)\left|0\right\rangle = 0$$

Quantized Harmonic Oscillator

As $\psi_0(x) = \langle x|0\rangle$, where x is the position basis $\{|x\rangle\}$, with $\langle x|\,\hat{p}\,|0\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_0$ and $\langle x|\,\hat{x}\,|0\rangle = x\psi_0(x)$, we have

$$\begin{aligned}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} - im\omega x\right)\psi_0(x) &= 0 \\ \Rightarrow \quad \psi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}.\end{aligned}$$

We can also use the operator definition to derive the wavefunctions of the excited states through

$$\begin{aligned}\psi_n(x) &= \langle x\,|n\rangle = \frac{1}{\sqrt{n}}\langle x|\,\hat{a}^\dagger\,|n-1\rangle \\ &= \frac{1}{\sqrt{2nm\hbar\omega}}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)\psi_{n-1}(x).\end{aligned}$$

Quantized Harmonic Oscillator

Computation of HO wave fct.

In[◦]:= `SetQuantumAliases[]`

Out[◦]= `SetQuantumAliases[]`

In[◦]:= `ClearAll[ψ0, ψ1, q]`

Definition of the ground state

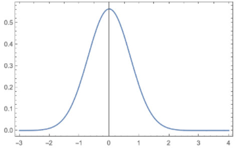
In[◦]:= `ψ0[q_] = 1/ (π)^(1/4) * Exp[-q^2 / 2]`

Out[◦]= $\frac{e^{-\frac{q^2}{2}}}{\pi^{1/4}}$

- Ground state wave function

In[◦]:= `Plot[ψ0[q] * Conjugate[ψ0[q]], {q, -3, 4}, PlotRange → All, Frame → True, Axes → FalseFrameLabel → {"Δp", "|S11|^2 (dB)"}]`

Out[◦]=



- Compute next higher wave function using the operator differential operator correspondence. Notably,

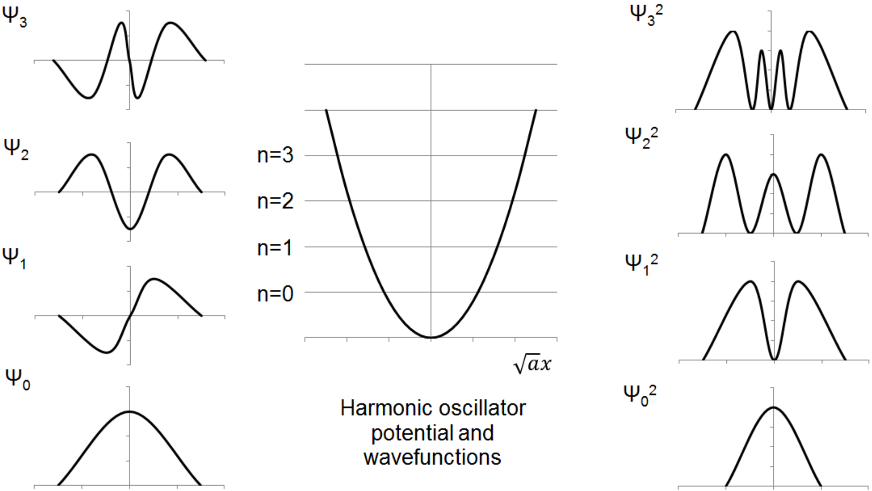
We apply for the creation operator the operator correspondence:

$$a^\dagger \left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \to 1/\sqrt{m} * \left(q - \frac{\delta}{\delta q}\right)\psi\_n = \sqrt{n+1}\,\psi\_n+1$$

In[◦]:= $\psi1[q\_] = \frac{1}{\sqrt{1}} * \frac{1}{\sqrt{2}} * (q * \psi0[q] - D[\psi0[q], q])$

Out[◦]= $\frac{\sqrt{2}\, e^{-\frac{q^2}{2}}\, q}{\pi^{1/4}}$

Quantized Harmonic Oscillator



Harmonic oscillator potential and wavefunctions

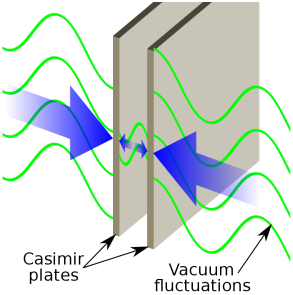
Fock States

The eigenstates of the quantized harmonic oscillator Hamiltonian are:

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle$$

For the vacuum state $|0\rangle$,

$$\hat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle \quad \Rightarrow \quad E_0 = \frac{\hbar\omega}{2}$$



which yields the vacuum energy of a harmonic oscillator (this energy leads to the Casimir force[2]).

[2] Casimir, Hendrick BG. "On the attraction between two perfectly conducting plates." Proc. Kon. Ned. Akad. Wet.. Vol. 51. 1948.

Effects Due to The Vacuum Energy

M. Planck's "second theory" derives the zero-point energy[3]

Thermal energy

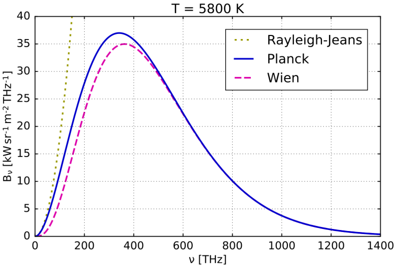
$$U = \frac{h\nu}{e^{h\nu/kT} - 1} + \frac{1}{2}h\nu$$

which marked the birth of the concept of *zero-point energy*. From that, Planck would have obtained the correct spectral energy density

Spectral energy density

$$\rho(\nu) = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT} - 1} + \frac{4\pi h\nu^3}{c^3}$$

which would give the spectrum shown in the right figure.



[3] Planck, Max. "Über die Begründung des Gesetzes der schwarzen Strahlung." Annalen der physik 342.4 (1912): 642-656.

Quantization of the Electromagnetic Field

Recall the Maxwell Equations (no sources)

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \qquad \text{[Law of induction]}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{[No monopole]}$$

$$\vec{\nabla} \cdot D = 0 (= \rho) \qquad \text{[Gauss law (no charge)]}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \left(= \frac{\partial \vec{D}}{\partial t} + J_f\right) \qquad \text{[Biot-Savart law (no current)]}$$

Introducing the vector potential $\vec{A}(\vec{r}, t)$ in Coulomb gauge

$$\left(\vec{\nabla} \cdot \vec{A} = 0\right) \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\nabla^2 \vec{A}(\vec{r}, t) - \frac{1}{c^2} \partial_t^2 \vec{A}(\vec{r}, t) = 0 \qquad \vec{E} = -\partial_t \vec{A}$$

Quantization of the Electromagnetic Field

From classical electrodynamics (travelling wave)

$$A(\vec{r},t) = \sum_k \frac{\vec{\epsilon}_k}{i\omega_k} E_k^{\text{vac}} \alpha_k(t) e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + c.c.$$

$$E(\vec{r},t) = \sum_k \vec{\epsilon}_k E_k^{\text{vac}} \alpha_k(t) e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + c.c.$$

$$H(\vec{r},t) = \frac{1}{\mu_0} \sum_k \frac{\vec{k} \times \vec{\epsilon}_k}{\omega_k} E_k^{\text{vac}} \alpha_k(t) e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + c.c.$$

Here α_k is the classical component of vector potential. Define:

$$E_k^{\text{vac}} \equiv \left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2} \quad \text{vacuum field}$$

where V is the spatial volume the plane wave occupies (periodic boundary conditions $k = k_m = 2\pi m/L$).

Quantization of the Electromagnetic Field

Quantization proceeds by identifying $\alpha_k \to \hat{a}$, $\alpha_k^* \to \hat{a}^\dagger$. Notice that

$$H_k = \frac{1}{2} \int_V \left(\varepsilon_0 |\vec{E}_k|^2 + \mu_0 |\vec{H}_k|^2 \right) d^3 r.$$

In this case,

$$\begin{aligned} H_k &= \frac{\hbar\omega_k}{2} \left(\alpha_k \alpha_k^* + \alpha_k \alpha_k^* \right) && \text{classically} \\ &\Downarrow \\ \hat{H}_k &= \frac{\hbar\omega_k}{2} \left(\hat{a}_k \hat{a}_k^\dagger + \hat{a}_k \hat{a}_k^\dagger \right) && \text{quantum mechanically} \end{aligned}$$

which does not give the correct quantized Hamiltonian.

Quantization of the Electromagnetic Field

Instead, quantization now proceeds by first replacing the fields with operators and applying the **symmetrization postulate**:

$$H_k = \frac{1}{2}\int_V \varepsilon_0 \vec{E}\cdot\vec{E}^* + \mu_0 \vec{H}\cdot\vec{H}^* = \frac{1}{2}\int_V \frac{\varepsilon_0}{2}\left(\vec{E}\cdot\vec{E}^* + \vec{E}^*\cdot\vec{E}\right) + \dots$$

This then yields

$$\hat{H}_k = \frac{1}{2}\int_V \frac{\varepsilon_0}{2}\left(\hat{E}\hat{E}^\dagger + \hat{E}^\dagger\hat{E}\right) + \frac{\mu_0}{2}\left(\hat{H}\hat{H}^\dagger + \hat{H}^\dagger\hat{H}\right)$$

inserting the field expressions leads to,

$$\hat{H} = \sum_k \hbar\omega_k\left(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}\right)$$

Quantization inside a cavity[4]

Cavity field solution, assuming linearly polarized in the x-direction

$$E_x(z,t) = \sum_k \left(E_k \alpha_k e^{-i\omega_k t} - E_k \alpha_k^* e^{+i\omega_k t}\right) \sin(kz)$$

$$H_y(z,t) = -i\varepsilon_0 c \sum_k \left(E_k \alpha_k e^{-i\omega_k t} - E_k \alpha_k^* e^{+i\omega_k t}\right) \cos(kz)$$

where $E_k = \sqrt{\frac{\hbar\omega_k}{\varepsilon_0 V}}$ is the vacuum field (notice the subtle difference to the travelling wave E_k^{vac}).

With the same notation, the energy takes a particularly simple form

$$H_k = \int \frac{1}{2}\varepsilon_0 |\vec{E}|^2 + \frac{1}{2}\mu_0 |\vec{H}|^2 = \frac{\varepsilon_0}{2}\int dV \left(|\vec{E}|^2 + c^2|\vec{B}|^2\right)$$

$$= \int \frac{\hbar\omega_k}{2V}\left(2|\alpha_k|^2\right) dV = \frac{\hbar\omega_k}{2}\left(\alpha_k\alpha_k^* + \alpha_k^*\alpha_k\right)$$

[4] derivation taken from "Quantum Mechanics Part III", Cohen-Tannoudji

Quantization inside a cavity

The quantization now proceeds by setting $\alpha_k \to \hat{a}_k$, $\alpha_k^* \to \hat{a}_k^\dagger$, we have

$$\hat{H}_k = \frac{\hbar\omega_k}{2}\left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) = \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \underbrace{\frac{\hbar\omega_k}{2}}_{\text{zero-point energy}}$$

As for the commutation relations between $E_j(\vec{r},t)$ and $H_k(\vec{r},t)$, we insert the operators:

$$\vec{E}(\vec{r},t) = \sum_k \vec{\epsilon}_k E_k^{\text{vac}} \hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} - c.c.$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\sum_k \frac{\vec{k}\times\vec{\epsilon}_k}{\omega_k} E_k^{\text{vac}} \hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} - c.c.$$

and recall the commutation relations

$$\left[\hat{a}_{k,x}, \hat{a}_{k',x'}\right] = \left[\hat{a}_{k,x}^\dagger, \hat{a}_{k',x'}^\dagger\right] = 0 \quad \text{and} \quad \left[\hat{a}_{k,x}, \hat{a}_{k',x'}^\dagger\right] = \delta_{k,k'}\delta_{x,x'},$$

Quantization inside a cavity

We can then derive

Commutation relations between electric and magnetic field

$$\left[\hat{E}_j(\vec{r},t),\hat{H}_j(\vec{r}',t)\right] = 0 \quad \Rightarrow \hat{E}_j,\hat{H}_j \text{ can be measured simultaneously}$$

$$\left[\hat{E}_j(\vec{r},t),\hat{H}_k(\vec{r}',t)\right] = -i\hbar c^2 \sum_l \epsilon_{jkl} \frac{\partial}{\partial l} \delta^{(3)}(\vec{r}-\vec{r}')$$

$$\Rightarrow \hat{E}_j,\hat{H}_{k(\neq j)} \text{ cannot be measured simultaneously}$$

where $i,j,k = x,y,z$ and ϵ_{jkl} is the Levi-Civita symbol which is antisymmetric in all the indices.

Momentum of light

Recall the momentum for the transverse wave:

$$\vec{P}_{\text{trans}} = -\sum_k \varepsilon_0 \int \vec{E}_k(\vec{r},t) \times \vec{B}_k(\vec{r},t) dV$$

applying symmetrization and inserting the operators,

Quantization of momentum

$$\vec{P}_{\text{trans}} = \varepsilon_0 \sum_k \frac{\omega}{4\left(\frac{\varepsilon_0 \omega}{2\hbar}\right)} \vec{k} \left[\alpha_k^* \alpha_k + \alpha_k \alpha_k^*\right]$$

$$\hat{P}_{\text{trans}} = \sum_k \frac{\hbar \vec{k}}{2} \left[\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger\right] = \sum_k \hbar \vec{k} \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) = \sum_k \hbar \vec{k} \hat{a}_k^\dagger \hat{a}_k$$

which is a very intuitive result as $\hat{P}_{\text{trans}} |n_k\rangle = \hbar \vec{k} n_k |n_k\rangle$. The vacuum fluctuations are canceled out by the opposite momentum.

Quantization Procedure via Lagrangian Mechanics[5]

First, one needs to obtain the system's Lagrangian as a function of generalised coordinates x_i and their time derivatives $\dot{x}_i$

$$\mathcal{L}(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) = T - V \tag{2}$$

where $T = \frac{1}{2}\sum m_i(\dot{x}_i)^2$ is the system's kinetic energy, and $V = V(x_1, \ldots, x_n)$ - potential energy. The system's equations of motion are then recovered as Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i} \tag{3}$$

The next step is to introduce the <u>conjugate momenta</u> p_i of the coordinates x_i

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \tag{4}$$

and the Hamiltonian, by performing Legendre transform

$$H = \sum_i \dot{x}_i p_i - \mathcal{L} \tag{5}$$

[5] Cohen–Tannoudji C., Diu B., Laloe F. (2017) Mécanique quantique - Tome 3 - Complement A_{XVIII}

Quantization Procedure via Lagrangian Mechanics

We now substitute the canonically conjugate coordinates x_i and momenta p_i with operators $\hat{x}_i$ and $\hat{p}_i$, imposing the canonical commutation relation

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{6}$$

The Hamiltonian operator is then obtained from the **symmetrised** (e.g. $x_i p_i = (x_i p_i + p_i x_i)/2$) classical Hamiltonian:

$$\hat{H} = H_{sym}(\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n) \tag{7}$$