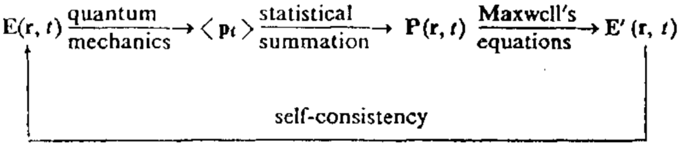
Quantum Electrodynamics and Quantum Optics: Lecture 8

Fall 2024

Semiclassical atom light interaction: Maxwell-Schrödinger equations [1]



Atomic polarization : bridging quantum mechanical and classical descriptions

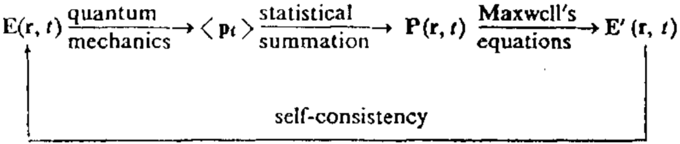
The medium is described by its susceptibility $\chi(\tau)$

$$\mathbf{P}(z,t) = \varepsilon_0 \int_{-\infty}^{t} \chi(t-\tau)\mathbf{E}(z,\tau)d\tau$$

On the other hand $\mathbf{P}(z,t) = N\langle\hat{\mathbf{p}}\rangle$ where $\hat{\mathbf{p}} = q\hat{\mathbf{r}}$ and N is the density of dipoles. For a state $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$, we denote the dipole matrix element $\langle 1|\, q\hat{\mathbf{r}}\,|2\rangle = \mathbf{p}_{12} = \mathbf{p}_{21}^*$. Note that $\hat{\mathbf{r}}$ is actually a measure of distance (not position), hence the diagonal elements are $\mathbf{p}_{ii} = 0$.

[1] Laser physics; Sargent, Scully, Lamb

Semiclassical atom light interaction: Maxwell-Schrödinger equations [2]



Atomic polarization : bridging quantum mechanical and classical descriptions

We compute $\hat{\mathbf{p}} = q\hat{\mathbf{r}}$ using $|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle$ and $\langle 1| q\hat{\mathbf{r}} |2\rangle = \mathbf{p}_{12} = \mathbf{p}_{21}^*$,

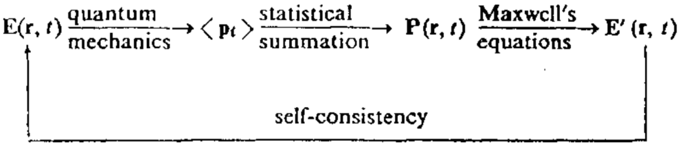
$$\langle q\hat{\mathbf{r}} \rangle = \mathrm{Tr}\{\rho \hat{\mathbf{p}}\} = \langle \psi | \hat{\mathbf{p}} | \psi \rangle = c_1^* c_2 \mathbf{p}_{12} + c_2^* c_1 \mathbf{p}_{21} = \rho_{21}\mathbf{p}_{12} + \rho_{12}\mathbf{p}_{21}$$

with density matrix

$$\hat{\rho} = |\psi\rangle \langle \psi| = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix}$$

[2] Laser physics; Sargent, Scully, Lamb

Semiclassical atom light interaction: Maxwell-Schrödinger equations [3]



Helmholtz equation : semi-classical evolution

Propagation of electromagnetic waves in a medium is governed by the Helmholtz equation:

$$\underbrace{\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}}_{\text{Homogeneous part describing plane waves}} = \underbrace{\mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}_{\text{Interaction with the medium}}$$

The quantum mechanical part enters in the polarization.

[3] Laser physics; Sargent, Scully, Lamb

Evolution of density matrix

Von Neumann equation

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}]$$

To derive a realistic refractive index and susceptibility, we need to account for the physical dissipation. This dissipation can be introduced with a spontaneous emission model, which at this stage corresponds to the addition of an "ad-hoc" decay of the density matrix (formally through the master equation).

$$\Gamma_{12} = A_{12} = \frac{1}{4\pi\varepsilon_0}\frac{4\omega^3|\mathbf{p}_{12}|}{3\hbar c^3}$$

is the rate of spontaneous emission (derived later in the lecture) and we have the

Master equation in Linblad form

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \Gamma_{12}\left(\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \frac{1}{2}\left\{\hat{\rho}, \hat{\sigma}_+\hat{\sigma}_-\right\}\right)$$

Evolution of density matrix

Projecting the master equation onto the states $|1\rangle, |2\rangle$, with $\hat{H} = \frac{\hbar\Delta}{2}\hat{\sigma}_z + q\hat{\mathbf{r}}\cdot\mathbf{E}$ in the rotating frame with detuning $\Delta = \omega_{12} - \omega$, we obtain the

Optical Bloch equations

$$
\begin{aligned}
\frac{\mathrm{d}\rho_{11}}{\mathrm{d}t} &= -\frac{i}{\hbar}(\mathbf{p}_{12}\cdot\mathbf{E})\rho_{12} + \text{c.c.} + \Gamma_{12}\rho_{22} \\
\frac{\mathrm{d}\rho_{22}}{\mathrm{d}t} &= +\frac{i}{\hbar}(\mathbf{p}_{12}\cdot\mathbf{E})\rho_{12} + \text{c.c.} - \Gamma_{12}\rho_{22} = -\frac{\mathrm{d}\rho_{11}}{\mathrm{d}t} \\
\frac{\mathrm{d}\rho_{12}}{\mathrm{d}t} &= -i\Delta\rho_{12} + \frac{i}{\hbar}(\mathbf{p}_{12}\cdot\mathbf{E})(\rho_{11} - \rho_{22}) - \frac{\Gamma_{12}}{2}\rho_{12}
\end{aligned}
$$

We can extract the polarization in the steady-state $\dot{\rho}_{ij} = 0$ from $\rho_{12}(\omega)$ (recall $\mathbf{p}_{12} = \langle 1|\, q\hat{\mathbf{r}}\, |2\rangle = -\langle 1|\, e\hat{\mathbf{r}}\, |2\rangle$)

$$
\rho_{12}(\omega) = \frac{i}{\hbar}\frac{\mathbf{p}_{12}\cdot\mathbf{E}}{\Gamma_{12} + i\Delta}(\rho_{11} - \rho_{22})
$$

Evolution of density matrix

In the Fourier domain, using $\mathbf{p}_{21} = \mathbf{p}_{12}^*$ and for N atoms,

$$\mathbf{P}(\omega) = \varepsilon_0 \chi(\omega) \mathbf{E} = N\langle \hat{\mathbf{p}} \rangle = N\mathbf{p}_{12}^* \rho_{12}(\omega) + \text{c.c.} = \frac{i}{\hbar} \frac{N|\mathbf{p}_{12}|^2 \mathbf{E}}{\Gamma_{12} + i\Delta} (\rho_{11} - \rho_{22})$$

thus we obtain for the susceptibility $\chi = \chi'_{\text{Re}} + i\chi''_{\text{Im}}$

$$\chi'_{\text{Re}} = N \frac{|\mathbf{p}_{12}|^2}{\varepsilon_0 \hbar} \frac{\Delta}{\Gamma_{12}^2 + \Delta^2} (\rho_{11} - \rho_{22})$$

$$\chi''_{\text{Im}} = N \frac{|\mathbf{p}_{12}|^2}{\varepsilon_0 \hbar} \frac{\Gamma_{12}}{\Gamma_{12}^2 + \Delta^2} (\rho_{11} - \rho_{22})$$

Semiclassical model : derivation[4]

Plane wave solution

Considering light linearly polarized along $\boldsymbol{\epsilon}$ and introducing slowly varying amplitudes $\mathcal{E}(z,t)$ and $\mathcal{P}(z,t)$ with phase $\phi(z,t)$

$$\mathbf{E}(z,t) = \boldsymbol{\epsilon}\mathcal{E}(z,t)e^{-i(\omega t - kz + \phi(z,t))} + \text{c.c.} = \boldsymbol{\epsilon}E(z,t)$$

$$\mathbf{P}(z,t) = \boldsymbol{\epsilon}\mathcal{P}(z,t)e^{-i(\omega t - kz + \phi(z,t))} + \text{c.c.} = \boldsymbol{\epsilon}P(z,t)$$

$$\mathcal{P}(z,t) = \varepsilon_0 \mathcal{E}\chi(\omega) = \varepsilon_0 \mathcal{E}\left(\chi'_{\text{Re}} + i\chi''_{\text{Im}}\right)$$

We assume slowly varying amplitudes and phase, i.e.

$$\begin{cases} \frac{\partial \mathcal{E}}{\partial t} \ll \omega \mathcal{E}, & \frac{\partial \mathcal{P}}{\partial t} \ll \omega \mathcal{P}, & \frac{\partial \phi}{\partial t} \ll \omega \\ \frac{\partial \mathcal{E}}{\partial z} \ll k \mathcal{E}, & \frac{\partial \mathcal{P}}{\partial z} \ll k \mathcal{P}, & \frac{\partial \phi}{\partial z} \ll k \end{cases} \tag{1}$$

We will derive the corrections to the linear dispersion relation $\omega \simeq ck$

[4] Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Chapter 5, Section 4

Semiclassical model : derivation

Consider the Helmholtz equation

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(-\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{E} = -\mu_0 \frac{\partial^2}{\partial t^2}\mathbf{P}(z,t)$$

Applying the slowly varying envelope approximation on the first part with $\omega = ck$ yields :

$$\left(-\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) E \simeq -2ikE + \cancel{\left(-\frac{\partial \mathcal{E}}{\partial z} + i\mathcal{E}\frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} - \frac{i\mathcal{E}}{c}\frac{\partial \phi}{\partial t}\right) e^{-i(\omega t - kz + \phi(z,t))}}$$

thus the total left-hand side is

$$-2ik\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) E = -2ik\left(\frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + i\left(k - \frac{\omega}{c} - \frac{\partial \phi}{\partial z} - \frac{1}{c}\frac{\partial \phi}{\partial t}\right)\mathcal{E}\right) e^{-i(\omega t - kz + \phi(z,t))}$$

Semiclassical model : derivation

Applying the same slowly varying envelope approximation for the right-hand side of Helmholtz equation yields

$$-\mu_0 \frac{\partial^2 P}{\partial t^2} = -\frac{1}{\varepsilon_0 c^2} \frac{\partial}{\partial t} \left(\left(-i\omega \mathcal{P} + \cancel{\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial \phi}{\partial t}} \mathcal{P} \right) e^{-i(\omega t - kz + \phi(z,t))} \right)$$
$$= i\omega \frac{1}{\varepsilon_0 c^2} \left(-i\omega \mathcal{P} + \cancel{\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial \phi}{\partial t}} \mathcal{P} \right) e^{-i(\omega t - kz + \phi(z,t))} = \frac{\omega^2}{\varepsilon_0 c^2} \mathcal{P} e^{-i(\omega t - kz + \phi(z,t))}$$

Overall, if we simplify the phase we arrive to

$$\frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + i\left(k - \frac{\omega}{c} - \frac{\partial \phi}{\partial z} - \frac{1}{c}\frac{\partial \phi}{\partial t}\right)\mathcal{E} = i\frac{\omega^2}{2k\varepsilon_0 c^2}\mathcal{P} \approx i\frac{\omega}{2\varepsilon_0 c}\mathcal{P}$$

Note that we do not use the linear dispersion on the "$k - \frac{\omega}{c}$" term to allow us to derive the corrections to this relation.

Semiclassical model : effects of complex susceptibility

Taking the real and imaginary part ($\mathcal{P}$ is complex in general) and using $\mathcal{P} = \varepsilon_0 \mathcal{E} \chi(\omega) = \varepsilon_0 \mathcal{E} \left(\chi'_{\mathrm{Re}} + i\chi''_{\mathrm{Im}}\right)$ gives

Physical meaning of the real and imaginary parts of susceptibility

$$\text{(absorption/gain)} : \frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = -\frac{k}{2\varepsilon_0}\mathrm{Im}\mathcal{P} = -\frac{k}{2} \cdot \chi''_{\mathrm{Im}} \cdot \mathcal{E}$$

$$\text{(dispersion)} : \frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial \phi}{\partial t} = k - \frac{\omega}{c} - \frac{\omega}{2\varepsilon_0 c}\frac{\mathrm{Re}\mathcal{P}}{\mathcal{E}} = k - \left(1 + \frac{\chi'_{\mathrm{Re}}}{2}\right)\frac{\omega}{c}$$

The term $g(\omega) = -\frac{k}{2} \cdot \chi''_{\mathrm{Im}}(\omega)$ is the gain coefficient. We also have the corrected dispersion relation when we assume a constant phase ϕ :

$$k = \left(1 + \frac{\chi'_{\mathrm{Re}}}{2}\right)\frac{\omega}{c}$$

Semiclassical model : complex refractive index

If we consider a complex refractive index given by $n^2(\omega) = 1 + \chi$, up to first order approximation we have:

$$n(\omega) = n' + in'' = \sqrt{1 + \chi(\omega)} \approx 1 + \frac{\chi'_{\text{Re}}}{2} + i\frac{\chi'_{\text{Im}}}{2}$$

Hence $n' \approx 1 + \frac{\chi''_{\text{Re}}}{2}$ and $n'' \approx \frac{\chi''_{\text{Im}}}{2}$. Coming back to the dispersion relation, we see that the real part of the refractive complex index corresponds to the classical refractive index which

$$n' = \left(1 + \frac{\chi'_{\text{Re}}}{2}\right) = c\frac{k}{\omega} = \frac{c}{v_p}$$

We recover the classical connection between the refractive index and the ratio of the speed of light in vacuum c to the phase velocity $v_p = \frac{\omega}{k}$ in the medium.

Semiclassical model : complex susceptibility

Solving the Bloch equations gives us

Susceptibility

$$\chi'_{\mathrm{Re}} = N \frac{|\mathbf{p}_{12}|^2}{\varepsilon_0 \hbar} \frac{\Delta}{\Gamma_{12}^2 + \Delta^2} (\rho_{11} - \rho_{22})$$

$$\chi''_{\mathrm{Im}} = N \frac{|\mathbf{p}_{12}|^2}{\varepsilon_0 \hbar} \frac{\Gamma_{12}}{\Gamma_{12}^2 + \Delta^2} (\rho_{11} - \rho_{22})$$

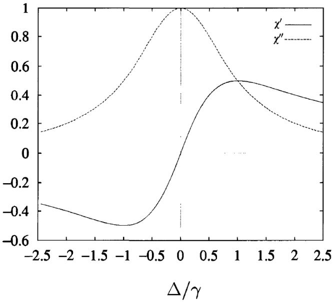


Figure: Real and imaginary part of the susceptibility

Generalization : Susceptibility of multilevel atoms

$$\chi = \frac{\mathcal{P}}{\varepsilon_0 \mathcal{E}} = \frac{2}{\varepsilon_0 \mathcal{E}} \left(\rho_{12} p_{21} + \rho_{13} p_{31}\right) e^{i\omega_0 t}$$

Slow and Fast light

Dispersion relation $\omega = \omega(k)$ and assuming $|\chi'_{\text{Re}}(\omega)| \gg |\chi''_{\text{Im}}(\omega)|$ such that the refractive index is $n(\omega) \approx n'(\omega) = \frac{c}{v_p} = \frac{ck}{\omega}$.

$$\text{Phase velocity : } v_p = \frac{\omega}{k}$$

$$\text{Group velocity : } v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1}$$

Eliminate k and rewrite group velocity in terms of refractive index

$$v_g = \frac{c}{n_g}, \text{ where } n_g = n(\omega) + \omega\frac{dn(\omega)}{d\omega} \text{ is the "Group Index"}$$

Now, we go back to the definition of polarizability

$$P(\omega) = \varepsilon_0 \chi(\omega) E \implies n(\omega) = \sqrt{1 + \chi'_{\text{Re}}(\omega)} \approx 1 + \frac{\chi'_{\text{Re}}(\omega)}{2}$$

By modifying $\chi(\omega)$, you can modify $n(\omega)$ and the group velocity of the light in the medium.

Slow and Fast light

We can engineer the $\chi(\omega)$ of the system such that the group velocity is-

- $v_g \ll c$ - "Slow light", Eg. EIT[5] gives[6] $v_g \approx$ 17 m/s
- $v_g > c$ or $v_g < 0$ - "Fast or Advanced light", Anomalous Dispersion by two Raman gain resonances [7]

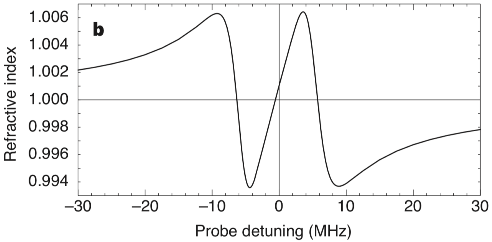


Figure: Refractive index profile. The steepness of the slope at resonance is inversely proportional to the group velocity of transmitted light

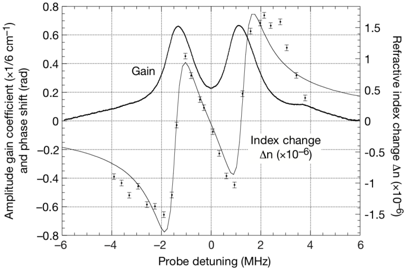


Figure: Measured refractive index and gain coefficient.

[5]Substitute numerical values in HW7.5 from Ch. 5 of Fast light, slow light and Left-Handed light, PW Milonni, 2005

[6]Hau, L. et al. Light speed reduction to 17 metres per second in an ultracold atomic gas. Nature 397, 594-598 (1999)

[7]Gain-assisted superluminal light propagation, L. J. Wang, A. Kuzmich & A. Dogariu, *Nature* **406** (2000)

Slow and Fast light

What about causality?[8]

- Phase velocity: $v_p > c$ allowed
- Group velocity: $v_g > c$ allowed. Not the same as velocity of information travel. Can be explained by Classical theory of wave propagation.
- Velocity of energy transfer: $v_E = |S(\omega)|/u(\omega)$, where S is the Poynting vector and u the Energy density. One can show that $v_E \leq c$. More interpretive than measurable.

[8] Fast light, slow light and Left-Handed light, PW Milonni, 2005

Quantum theory of atom-field interaction

The quantized electic field is

$$\hat{\mathbf{E}}(\mathbf{r},t) = \mathcal{E}_k \varepsilon_{\mathrm{ZPF}} \left(\hat{a}_k e^{-i\omega_0 t + i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_k^\dagger e^{+i\omega_0 t - i\mathbf{k}\cdot\mathbf{r}} \right),$$

where $\varepsilon_{\mathrm{ZPF}} = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V_k}}$ and $\mathcal{E}_k$ is the polarization

$$\hat{H}_{\mathrm{int}} = e\hat{\mathbf{r}} \cdot \hat{\mathbf{E}} = e\hat{\mathbf{r}} \cdot \mathcal{E}_k \varepsilon_{\mathrm{ZPF}} \left(\hat{a}_k e^{-i\omega_0 t + i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_k^\dagger e^{+i\omega_0 t - i\mathbf{k}\cdot\mathbf{r}} \right)$$

$$\begin{aligned} e\hat{\mathbf{r}} &= e\mathbb{1}\hat{\mathbf{r}}\mathbb{1} = e(|1\rangle\langle 1| + |2\rangle\langle 2|)\hat{\mathbf{r}}(|1\rangle\langle 1| + |2\rangle\langle 2|) \\ &= e\underbrace{\langle 1|\hat{\mathbf{r}}|2\rangle}_{\mathbf{p}_{12}}\underbrace{|2\rangle\langle 1|}_{\hat{\sigma}^+} + e\underbrace{\langle 1|\hat{\mathbf{r}}|1\rangle}_{\text{zero}}|2\rangle\langle 1| + e\underbrace{\langle 2|\hat{\mathbf{r}}|1\rangle}_{\mathbf{p}_{21}=\mathbf{p}_{12}}\underbrace{|1\rangle\langle 2|}_{\hat{\sigma}^-} \end{aligned}$$

Atom-field interaction

$$\hat{H}_{\mathrm{int}} = (\mathbf{p}_{12}\hat{\sigma}^+ + \mathbf{p}_{21}\hat{\sigma}^-) \cdot \mathcal{E}_k \varepsilon_{\mathrm{ZPF}} \left(\hat{a}_k e^{-i\omega_0 t} + \hat{a}_k^\dagger e^{+i\omega_0 t} \right)$$

$$\hat{\sigma}^+\hat{a}_k|1,n\rangle = |2,n-1\rangle \quad , \quad \hat{\sigma}^-\hat{a}_k|2,n\rangle = |1,n-1\rangle \quad , \quad \Delta E = \hbar(\omega_{12} + \omega_0)$$

Quantum theory of atom-field interaction

Rotating wave approximation

$$\hat{H}_{\text{int}} = \hbar\omega_{12}\frac{\hat{\sigma}_z}{2} + \hbar\sum_k g_k(\hat{\sigma}^+\hat{a}_k + \hat{\sigma}^-\hat{a}_k^\dagger)$$

$$g_k = \frac{\mathbf{p}_{12}\cdot\mathcal{E}_k\varepsilon_{\text{ZPF}}}{\hbar} = \mathbf{p}_{12}\cdot\mathcal{E}_k\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V_k}}\frac{1}{\hbar}$$

Rabi frequency becomes photon-number dependent in the quantum mechanical atom-field interaction.

$$\langle 1,n|\,\hbar g_k\hat{\sigma}^-\hat{a}_k^\dagger\,|2,n-1\rangle = \langle 1|\,\hat{\sigma}^-\,|2\rangle\,\langle n|\,\hat{a}_k^\dagger\,|n-1\rangle\,\hbar g_k = \hbar\sqrt{n}g_k$$

$$\langle 2,n+1|\,\hbar g_k\hat{\sigma}^+\hat{a}_k\,|1,n\rangle = \langle 2|\,\hat{\sigma}^+\,|1\rangle\,\langle n-1|\,\hat{a}_k\,|n\rangle\,\hbar g_k = \hbar\sqrt{n}g_k$$

However $\hat{\sigma}^+\hat{a}_k^\dagger$ and $\hat{\sigma}^-\hat{a}_k$ do not conserve excitation :

$$\hat{\sigma}^+\hat{a}_k^\dagger\,|1,n\rangle = \sqrt{n}\,|2,n+1\rangle \text{ thus } \Delta E = -\hbar(\omega+\omega_{12})$$
$$\hat{\sigma}^-\hat{a}_k\,|2,n\rangle = \sqrt{n}\,|1,n-1\rangle \text{ thus } \Delta E = +\hbar(\omega+\omega_{12})$$

Quantum theory of atom-field interaction : state evolution

Consider the manifold spanned by $\{|1\rangle|n+1\rangle, |2\rangle|n\rangle\}$ and ω_0 as laser frequency, $E_1 = \hbar\omega_1, E_2 = \hbar\omega_2$, where a generic state in this subspace can be written as:

$$\left|\Psi(t)\right\rangle = c_1(t)|1\rangle|n+1\rangle + c_2(t)|2\rangle|n\rangle$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}c_1 & = -ig\sqrt{n+1}c_2 e^{+i(\omega_0-\omega_1)t} \\ \frac{\mathrm{d}}{\mathrm{d}t}c_2 & = -ig\sqrt{n+1}c_1 e^{-i(\omega_0-\omega_2)t} \end{cases}$$

As for the semi-classical case, we have Rabi oscillations, but now with a frequency :

$$\Omega_n^2 = \Delta^2 + 4g^2(n+1)$$

Now, we will perform a change of frame into the interaction picture, using Baker-Campbell-Hausdorff formula

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} \approx \hat{B} + \alpha[\hat{A},\hat{B}] + \alpha^2[\hat{A},[\hat{A},\hat{B}]]/2!$$

Quantum theory of atom-field interaction : interaction picture

The time evolution of $|\Psi\rangle$ under the bare Hamiltonian $\hat{H}_0$

$$i\hbar\partial_t |\Psi\rangle = \hat{H}_0 |\Psi\rangle \ , \ \left|\Psi(t)\right\rangle = \left|\Psi(0)\right\rangle e^{-i\hat{H}_0 t/\hbar}$$

Moving to **interaction picture**, we remove this time evolution from the state and put it on the operators instead

$$\langle\Psi| \hat{H}_{\text{int}} |\Psi\rangle = \langle\Psi_0| e^{+i\hat{H}_0 t/\hbar}\hat{H}_{\text{int}} e^{-i\hat{H}_0 t/\hbar} |\Psi_0\rangle$$

The operators now transform as the Hamiltonian $\langle\Psi| \hat{a} |\Psi\rangle = \langle\Psi_0| e^{+i\hat{H}_0 t/\hbar}\hat{a} e^{-i\hat{H}_0 t/\hbar} |\Psi_0\rangle$

$$\begin{cases} \hat{a} & \rightarrow e^{i\omega_0 \hat{a}^\dagger \hat{a} t}\, \hat{a}\, e^{-i\omega_0 \hat{a}^\dagger \hat{a} t} & = \hat{a} e^{-i\omega_0 t} \\ \hat{\sigma}^+ & \rightarrow e^{i\omega_{12}\hat{\sigma}_z t/2}\, \hat{\sigma}^+\, e^{-i\omega_{12}\hat{\sigma}_z t/2} & = \hat{\sigma}^+ e^{-i\omega_{12} t} \end{cases}$$

Quantum theory of atom-field interaction : dressed states

The Hamiltonian in the interaction picture is for $\Delta = \omega_0 - \omega_{12}$

Interaction Hamiltonian in rotating frame

$$\hat{H}_{\text{int}} = \hbar g \left(\hat{\sigma}^+ \hat{a} e^{-i\Delta t} + \hat{\sigma}^- \hat{a}^\dagger e^{+i\Delta t} \right)$$

Dressed states of atom-field interaction

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|n,2\rangle + |n+1,1\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|n,2\rangle - |n+1,1\rangle)$$

Spontaneous emission: Wigner-Weisskopf Theory

Assume that

$$|\psi(t=0)\rangle = \sum_k |0_k, 2\rangle$$

$$|\psi(t)\rangle = \sum_k c_1 |0_k, 2\rangle + \underbrace{\sum_k c_{2,k} |1_k, 1\rangle}_{\text{all possible modes k of the field}}$$

Thus we obtain

$$\begin{cases} \dot{c}_1 & = -i \sum_k g_k e^{i(\omega_{12}-\omega_k)t} c_{2,k} \\ \dot{c}_{2,k} & = i g_k e^{-i(\omega_{12}-\omega_k)t} c_1 \end{cases}$$

Eliminating $c_{2,k} = -i g_k(r_0) \int_0^t \mathrm{d}t' e^{-i(\omega_{12}-\omega_k)t} c_1(t')$:

$$\dot{c}_1 = \sum_k |g_k(r)|^2 \int_0^t \mathrm{d}t' e^{i(\omega_{12}-\omega_k)(t-t')} c_1(t')$$

Spontaneous emission: Wigner-Weisskopf Theory

Next we introduce the **density of states** (note that $\omega = k \cdot c$ and the factor 2 due to polarizations):

$$\begin{aligned}\sum_k \Rightarrow & 2\frac{V}{(2\pi)^3}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \int \mathrm{d}k \cdot k^2 \\ = & 2\frac{V}{8\pi^3 c^3}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \int \mathrm{d}\omega_k \cdot \omega_k^2\end{aligned}$$

Note that $|g(\vec{r})|^2 = \frac{\hbar\omega_k}{2\varepsilon_0 V}|\mathbf{p}_{12}|^2 \cos^2\theta$ where θ is the angle between $\mathcal{E}_k$ and $\mathbf{p}_{12}$. Hence one obtains:

$$\dot{c}_(t) = -\frac{4|\mathbf{p}_{12}|^2}{(2\pi)^2 6\hbar\varepsilon_0 c^3}\int_0^{\infty} \mathrm{d}\omega_k \omega_k^3 \int_0^t \mathrm{d}t' e^{i(\omega_{12}-\omega_k)(t-t')} c_1(t')$$

Spontaneous emission: Wigner-Weisskopf Theory

Note that $\int_{-\infty}^{+\infty} \mathrm{d}\omega_k e^{-i(\omega_{12}-\omega_k)(t-t')} = 2\pi\delta(t-t')$, and $\int_0^t \delta(t-t')\mathrm{d}t' = \frac{1}{2}$, we have:

$$\dot{c}_1(t) = -\left[\frac{1}{4\pi\varepsilon_0}\frac{4\omega^3|\mathbf{p}_{12}|^2}{3\hbar c^3}\right]\frac{1}{2}c_1(t)$$

Spontaneous emission rate

$$\Gamma_{12} = \frac{1}{4\pi\varepsilon_0}\frac{4\omega^3|\mathbf{p}_{12}|^2}{3\hbar c^3}$$

Paper for next week's presentation

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Quantum theory of a bandpass Purcell filter for qubit readout

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The measurement fidelity of superconducting transmon and Xmon qubits is partially limited by the qubit energy relaxation through the resonator into the transmission line, which is also known as the Purcell effect. One way to suppress this energy relaxation is to employ a filter which impedes microwave propagation at the qubit frequency. We present semiclassical and quantum analyses for the bandpass Purcell filter realized by E. Jeffrey *et al.* [Phys. Rev. Lett. **112**, 190504 (2014)]. For typical experimental parameters, the bandpass filter suppresses the qubit relaxation rate by up to two orders of magnitude while maintaining the same measurement rate. We also show that in the presence of a microwave drive the qubit relaxation rate further decreases with increasing drive strength.

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