Quantum Electrodynamics and Quantum Optics: Lecture 7

Fall 2024

Description of atom-field interaction

Semi-classical

$\mathbf{E}(t, \mathbf{r}), \psi(t, \mathbf{r})$ **E** is a vector

Quantum

$\hat{\mathbf{E}}(t, \mathbf{r}), \psi(t, \mathbf{r})$ $\hat{\mathbf{E}}$ is an Operator

Quantum model predicts all effects e.g., Wigner-Weisskopf model of spontaneous emission, Lamb shift.

Semi-Classical model

Schrodinger equation: $\hat{H}\psi_j = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right)\psi_j = i\hbar\frac{\partial}{\partial t}\psi_j$

The action of EM field is given by the Lorentz force $F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, which modifies the Hamlitonian as

$$\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2 + qU(\mathbf{r}, t) + V(\mathbf{r})$$

where **A** is the vector potential and U is the Coulomb potential

Gauge transform

Hamiltonian can be rewritten from local (phase) gauge invariance:

$$\psi'(\mathbf{r}, t) = e^{i\chi(\mathbf{r},t)}\psi(\mathbf{r}, t)$$

$$\mathbf{A}' \leftarrow \mathbf{A} + \frac{\hbar}{e}\nabla\chi, \quad U' \leftarrow U - \frac{\hbar}{e}\partial_t\chi$$

In the Coulomb gauge, the free field U will vanish, leaving only the static Coulomb potential V of the atom.

Dipole approximation and $\mathbf{r}.\mathbf{E}$ Hamiltonian

$$\left(-\frac{\hbar^2}{2m}(\nabla - \frac{ie\mathbf{A}(\mathbf{r},t)}{\hbar})^2 + V(\mathbf{r})\right)\ \psi = i\hbar\partial_t\psi$$

Note that $|\psi(\mathbf{r})|^2$ is localised around a_0 (Bohr radius) and $a_0 \ll \lambda$. Therefore we have:

$$\mathbf{A}(\mathbf{r}+\mathbf{r}_0,t) = \mathbf{A}(0,t)e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{r}_0)} \approx \mathbf{A}(0,t)e^{i\mathbf{k}\cdot\mathbf{r}_0}$$

Applying the approximation:

$$\left(-\frac{\hbar^2}{2m}(\nabla - \frac{ie\mathbf{A}(\mathbf{r}_0,t)}{\hbar})^2 + V(\mathbf{r})\right)\psi(\mathbf{r}) = i\hbar\partial_t\psi(\mathbf{r})$$

Light-matter interaction Hamiltonian

Local gauge transformation

Consider gauge transform of a wavefunction: $\psi(\mathbf{r},t) = \phi(\mathbf{r},t)e^{i\chi(\mathbf{r},t)}$, which does not affect the probability, i.e. $|\psi|^2 = |\phi|^2$.
We choose gauge $\chi(\mathbf{r},t) = q/\hbar \mathbf{A}(r_0,t)\cdot\mathbf{r}$ in such a way that inserting the wavefunction $\psi(\mathbf{r},t)$ in Schrödinger equation yields:

$$i\hbar\left(\frac{iq}{\hbar}\right)\underbrace{\partial_t \mathbf{A}(r_0,t)}_{\mathbf{E}(r_0,t)}\cdot\mathbf{r}\phi(\mathbf{r},t) + \partial_t\phi(\mathbf{r},t) = \underbrace{\left(\frac{\hat{p}^2}{2m} + V(\mathbf{r})\right)}_{H_0}\phi(\mathbf{r},t),$$

which leads to a new form of total Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}},$$

where $H_{\text{int}} = q\hat{\mathbf{r}}\cdot\mathbf{E}(r_0,t)$ is the <u>rE</u>-interaction term

Light-matter interaction Hamiltonian

$\mathbf{p} \cdot \mathbf{A}$ Hamiltonian

Upon transformation $p \to p - q\mathbf{A}(r_0, t)$, one can obtain:

$$\hat{H} = \frac{1}{2m}\left[i\hbar\nabla - q\mathbf{A}(r_0, t)\right]^2 + V(\mathbf{r}) = \hat{H}_0 + \hat{H}_1 + \hat{H}_2,$$

where

$$\hat{H}_0 = -\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r}) \text{ is a free-electron Hamiltonian,}$$

$$\hat{H}_1 = \frac{i\hbar\nabla}{m} \cdot q\mathbf{A}(r_0, t) \propto \hat{\mathbf{p}} \cdot \mathbf{A} \text{ is a "pA"-interaction term, and}$$

$$\hat{H}_2 \propto [q\mathbf{A}(r_0, t)]^2/2m \text{ is a kinetic energy of electron induced by a field}$$

We obtain two terms in the Hamiltonian:

$$\hat{H}^{rE}_{\text{int}} = q\mathbf{r} \cdot \mathbf{E}(r_0, t) \text{ and } \hat{H}^{pA}_{\text{int}} = \mathbf{p} \cdot \mathbf{A}(r_0, t)/2m$$

Dipole approximation of two-level atomic system

Consider a two-level atom:

$$\hat{H}\,|1\rangle = \hbar\omega_1\,|1\rangle\,, \quad \hat{H}\,|2\rangle = \hbar\omega_2\,|2\rangle$$

We can calculate matrix elements on light-matter interaction Hamiltonian:

$$\begin{aligned}
\langle 1|\,\hat{H}_{\text{int}}\,|1\rangle &= \iint \langle 1|\,\mathbf{r}\rangle\,\langle \mathbf{r}|\,\hat{H}_{\text{int}}\,\left|\mathbf{r}'\right\rangle\,\langle \mathbf{r}'\,|1\rangle\,d^3r d^3r' && (1)\\
&= \int \underbrace{\phi_1^*(\mathbf{r})\phi_1(\mathbf{r})}_{\text{even}}\,q\,\underbrace{\mathbf{r}}_{\text{odd}}\cdot\mathbf{E}(r_0,t)d^3r \approx 0 && (2)\\
\langle 2|\,\hat{H}_{\text{int}}\,|2\rangle &= 0 && (3)\\
\langle 1|\,\hat{H}_{\text{int}}\,|2\rangle &= \underbrace{\int \phi_1^*(\mathbf{r})\;\phi_2(\mathbf{r})\;q\;\mathbf{r}\;d^3r}_{P_{12}-\text{ matrix element of dipole moment}}\cdot\mathbf{E}(r_0,t) && (4)
\end{aligned}$$

$\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ are wave functions with different spatial parity

Ladder operators for fermions

Two-level systems obey the fermionic anti-commutation relations:

$$\{\hat{a}, \hat{a}^\dagger\} = 1$$
$$\{\hat{a}, \hat{a}\} = \{\hat{a}^\dagger, \hat{a}^\dagger\} = 0$$

The ladder operator for the transitions between two fermionic levels $\hat{\sigma}^+ = \hat{a}_1 \hat{a}_2^\dagger$ and $\hat{\sigma}^- = \hat{a}_1^\dagger \hat{a}_2$ will thus obey the same anti-commutation relations

Two-level system:

$$\hat{\sigma}^+ |1,0\rangle = |0,1\rangle = |e\rangle$$
$$\hat{\sigma}^- |0,1\rangle = |1,0\rangle = |g\rangle$$

Ladder operators for fermions

Pseudo-spin operators

$$\hat{a}_1\hat{a}_2^\dagger = \hat{\sigma}^+ \text{ annihilates electron 1, creates electron 2} \equiv \textbf{excitation}$$

$$\hat{a}_1^\dagger\hat{a}_2 = \hat{\sigma}^- \text{ annihilates electron 2, creates electron 1} \equiv \textbf{de-excitation}$$

$$\frac{1}{2}\left(\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_1\right) = \hat{\sigma}^z \ \equiv \ \textbf{population inversion}$$

Pseudo-spin operators in $|1\rangle$,$|2\rangle$-basis

$$\hat{\sigma}^+ = |2\rangle\langle 1|$$

$$\hat{\sigma}^- = |1\rangle\langle 2|$$

$$\hat{\sigma}^z = \frac{1}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right)$$

Density matrix for two-level systems

Density matrix formalism

$$\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad \text{is a density matrix of two-level system}$$

$$\rho_{22} - \rho_{11} = 2\langle \hat{\sigma}^z \rangle \text{ is the population inversion}$$

$$\rho_{12} = \langle \hat{\sigma}^+ \rangle \text{ and } \rho_{21} = \langle \hat{\sigma}^- \rangle \text{ are } \textbf{coherences}$$

Light-matter interaction: Semiclassical dynamics

Schrödinger equation for two-level atom

State of a system (Schrödinger picture) $\left|\psi(t)\right\rangle = C_1(t)\left|1\right\rangle + C_2(t)\left|2\right\rangle$ evolves in time as follows:

$$i\hbar\partial_t \left|\psi(t)\right\rangle = \left(\hat{H}_0 + \hat{H}_{\text{int}}\right)\left|\psi(t)\right\rangle,$$

where $\hat{H}_0 = \hbar\omega_1 \left|1\right\rangle\left\langle 1\right| + \hbar\omega_2 \left|2\right\rangle\left\langle 2\right|$ and $\hat{H}_{\text{int}} = q\hat{\mathbf{r}}\cdot\mathbf{e}|\mathbf{E}|\cos\left(\omega_L t\right)$ ($\mathbf{e}$ and $|\mathbf{E}|$ are field polarization and amplitude).

Light-matter interaction: Semiclassical dynamics

Schrödinger equation for two-level atom

Multiplying both sides by $\langle 1|$, defining **Rabi frequency** $\Omega_R = P_{12}|\mathbf{E}|/\hbar$, and applying transform $C_{1,2} = \tilde{C}_{1,2}e^{-i\omega_{21}t}$, we get:

$$\begin{cases} \frac{d\tilde{C}_1}{dt} & = \tilde{C}_2 \frac{i\Omega_R}{2}\left[e^{+i\Delta t} + e^{+i(\omega_{21}+\omega_L)t}\right] \\ \frac{d\tilde{C}_2}{dt} & = \tilde{C}_1 \frac{i\Omega_R}{2}\left[e^{-i\Delta t} + e^{-i(\omega_{21}+\omega_L)t}\right], \end{cases}$$

where $\omega_{21} = \frac{E_2 - E_1}{\hbar}$, and $\Delta = \omega_{21} - \omega_L$ is the **detuning** of laser field from atomic resonance

Light-matter interaction: Semiclassical dynamics

Rotating wave approximation (RWA)

When the laser field **is close to atomic resonance**:

- $\Delta \approx 0$, which corresponds to $e^{\pm i\Delta t}$ being an almost **stationary** term in the equation for population dynamics
- $\omega_L + \omega_{21} \gg \Delta$, which makes $e^{\pm i(\omega_L+\omega_{21})t}$ a **fast-oscillating** term that does not affect averaged dynamics

Thus we can neglect fast-rotating terms in equations.

General solution of Rabi problem in terms of population inversion

$$|C_1|^2 - |C_2|^2 = \left(\frac{\Delta^2 - \Omega_R^2}{\Omega^2}\right)\sin^2\left(\frac{\Omega t}{2}\right) + \cos^2\left(\frac{\Omega t}{2}\right),$$

where $\Omega = \sqrt{\Omega_R^2 + \Delta^2}$ is the detuning-dependent "generalized Rabi frequency"

Light-matter interaction: Semiclassical dynamics

Rabi oscillations

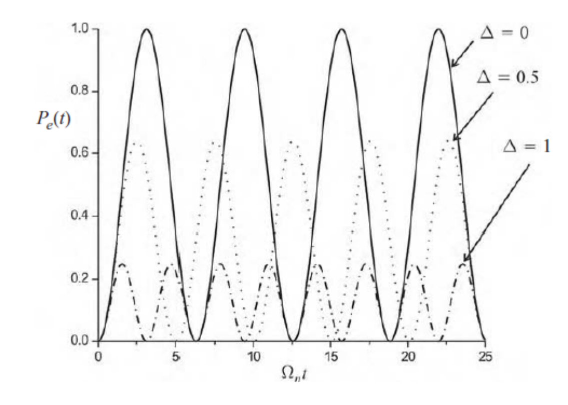

Figure: [1]Excited state population $P_e = |C_2(t)|^2$ dynamics for various detunings Δ

[1]Gerry, Christopher and Knight, Peter. *Introductory quantum optics*. Cambridge university press, 2005

Dipole moment

Considering atomic polarization $\hat{\mathbf{p}}$:

$$\langle\hat{\mathbf{p}}\rangle = e\left\langle\Psi(t)\right|\hat{\mathbf{r}}\left|\Psi(t)\right\rangle = ec_1c_2^*\underbrace{\langle 1|\,\hat{\mathbf{r}}\,|2\rangle}_{\mathbf{p}_{12}/e} + ec_1^*c_2\,\langle 2|\,\hat{\mathbf{r}}\,|1\rangle$$

Recall that in original frame (2 rotations) we have: $(\Delta = 0)$

$$c_1(t) = c_1(0)\cos\left(\frac{\Omega t}{2}\right)e^{-i\frac{E_1}{\hbar}t} + c_2(0)\sin\left(\frac{\Omega t}{2}\right)e^{i\frac{E_2}{\hbar}t}\sin\left(\frac{\Omega t}{2}\right)e^{i\frac{E_2}{\hbar}t}$$

Dipole moment

$$\langle\hat{p}\rangle = c_1(t)c_2^*(t)ep_{12} + c_1^*(t)c_2(t)ep_{21}$$

for $\Delta = 0$:

$$\langle\hat{p}\rangle = \mathrm{Re}(i\frac{p_{12}}{2}\sin(\Omega t)e^{i\omega t})$$

Paper for next week's presentation

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LETTERS

Synthesizing arbitrary quantum states in a superconducting resonator

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Questions for next week's presentation

- Explain the excitation swapping between qubit and the resonator shown in Fig.1c,d?
- Describe the set of operations (Q,S,Z) used on qubit and resonator based on Eq(1)?
- How is the sequence designed to synthesize an arbitrary state, how are complex state coefficients constructed?
- How to do Wigner tomography on the synthesized state, how is the parity measurement performed?
- What's the difference between $|0\rangle + |3\rangle$ and $|0\rangle + e^{i\phi}|3\rangle$ in terms of Wigner function, why does the relative phase of Fock states correspond to a global rotation of the Wigner function?
- How are phases of Fock state superposition (density matrix) measured?