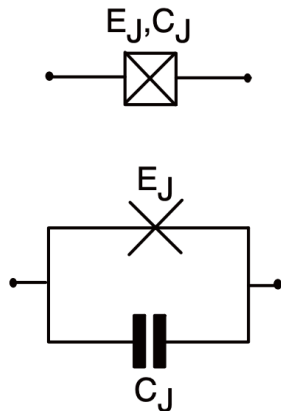
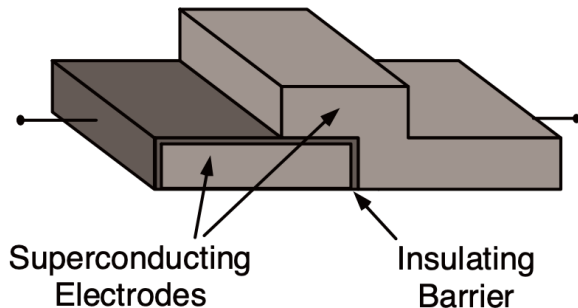


Quantum Electrodynamics and Quantum Optics: Lecture 6

Fall 2024

- Josephson Junctions
 - ▶ Current phase relationship
 - ▶ AC Josephson effect
- The cooper pair box (CPB)
 - ▶ Hamiltonian
 - ▶ Eigen-energies
 - ▶ Mathieu equation
 - ▶ Josephson Junctions Phase and Cooper pair box Number operator
- Cooper pair box and Transmon
 - ▶ Charge dispersion as a ratio of E_J and E_C
 - ▶ Anharmonicity
- Quantizing electrical circuits
 - ▶ Normal modes
 - ▶ Harmonic oscillator approximation

Josephson Junctions



Left: Sketch of a Josephson junction. Top Right: Electrical symbol.

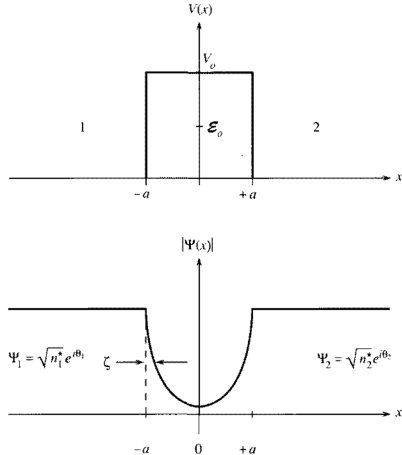


Figure 8.5 The model potential of the insulator $V(x)$ and the magnitude of the wavefunction $|\Psi|$. The two superconductors have densities of superconducting electron pairs n_1^* and n_2^* , respectively. Furthermore, the phase of the wavefunction θ can be different for the two superconductors at $x = \pm a$.

THE DISCOVERY OF TUNNELLING SUPERCURRENTS

Nobel Lecture, December 12, 1973

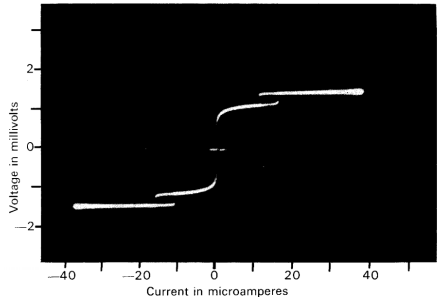


Fig. 3. The first published observation of tunnelling between two evaporated-film superconductors (Nicol, Shapiro and Smith, reference 6). A zero-voltage supercurrent is clearly visible. It was not until the experiments of Anderson and Rowell (reference 15) that such supercurrents could be definitely ascribed to the tunnelling process.

Macroscopic Quantum Model for Josephson junctions

Reminder: probability current

Schrödinger equation for a single particle (e.g. an electron) in a potential $V(\mathbf{r})$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{r})\psi.$$

According to Born's rule, the square of the magnitude of the wavefunction is interpreted as the probability density of the particle's position:

$$\wp(\mathbf{r}, t) \equiv |\psi(\mathbf{r}, t)|^2 = \psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t).$$

This probability density is shown to be conserved by defining a *probability current* $\mathbf{f}$, similar to the continuity equation for the case of charge density and electrical current:

$$\frac{\partial \wp}{\partial t} = -\nabla \cdot \mathbf{J}_\wp,$$

where the *probability current* is defined as

$$\mathbf{J}_\wp \equiv \frac{\hbar}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*) = \text{Re} \left\{ \psi^* \frac{\hbar}{im} \nabla \psi \right\}.$$

Note: The probability current is not a physical quantity and hence not measurable, in spite of resemblance with the electrical current and the continuity equation.

Macroscopic Quantum Model for superconductivity

Postulates:

- 1 Existence of carriers that do not scatter (superelectrons)
- 2 There exists a macroscopic quantum wavefunction, $\Psi(\mathbf{r}, t)$, that describes the behavior of the entire ensemble of charge carriers in the superconductor.

Superconductivity is a coherent phenomenon between the electrons. Similar to a coherent state of photons (laser) that describes a global state of a large number of photons.

Single particle:

$$\int dv \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) = 1 \rightarrow \begin{cases} \wp(\mathbf{r}, t) \equiv \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t), & \dots \text{probability density} \\ \mathbf{J}_\wp \equiv \text{Re} \left\{ \psi^* \frac{\hbar}{im} \nabla \psi \right\}. & \dots \text{probability current} \end{cases}$$

Entire ensemble of particles:

$$\int dv \Psi^*(\mathbf{r}, t) \Psi(\mathbf{r}, t) = N^* \rightarrow \begin{cases} \Psi^*(\mathbf{r}, t) \Psi(\mathbf{r}, t) = n^*(\mathbf{r}, t), & \dots \text{density of particles} \\ \mathbf{J}_s = \frac{q^*}{m^*} \text{Re} \left\{ \Psi^* \frac{\hbar}{i} \nabla \Psi \right\}, & \dots \begin{array}{l} \text{current of charge} \\ \text{(physical current)} \end{array} \end{cases}$$

where N^* is the number of particles.

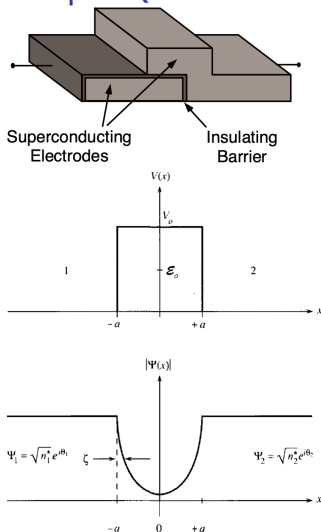


Figure 8.5 The model potential of the insulator $V(x)$ and the magnitude of the wavefunction $|\Psi|$. The two superconductors have densities of superconducting electron pairs n_1^* and n_2^* , respectively. Furthermore, the phase of the wavefunction θ can be different for the two superconductors at $x = \pm a$.

$$\mathbf{J}_s = \frac{q^*}{m^*} \text{Re} \left\{ \Psi^* \frac{\hbar}{i} \nabla \Psi \right\},$$

$$\Psi(+a) = \sqrt{n_2^*} e^{i\theta_2},$$

$$\Psi(-a) = \sqrt{n_1^*} e^{i\theta_1},$$

$$\zeta \equiv \sqrt{\frac{\hbar^2}{2m^*(V_o - \mathcal{E}_o)}},$$

$$\Psi(x) = C_1 \cosh \frac{x}{\zeta} + C_2 \sinh \frac{x}{\zeta}.$$

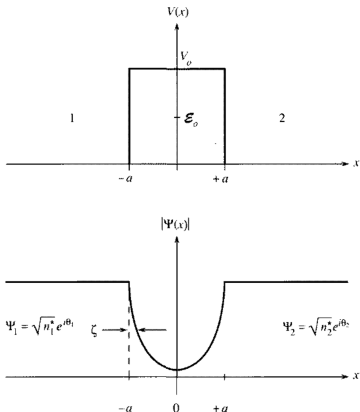


Figure 8.5 The model potential of the insulator $V(x)$ and the magnitude of the wavefunction $|\Psi|$. The two superconductors have densities of superconducting electron pairs n_1^* and n_2^* , respectively. Furthermore, the phase of the wavefunction θ can be different for the two superconductors at $x = \pm a$.

$$\mathbf{J}_S = \mathbf{J}_c \sin(\theta_1 - \theta_2),$$

$$J_c = -\frac{q^* \hbar}{m^* \zeta} \frac{\sqrt{n_1^* n_2^*}}{2 \sinh(a/\zeta) \cosh(a/\zeta)} = \frac{e \hbar \sqrt{n_1 n_2}}{2 m \zeta \sinh(2a/\zeta)},$$

$$\sinh 2a/\zeta \approx e^{2a/\zeta}/2.$$

Typical critical current J_C of a Josephson junction in a superconducting qubit is on the order of ~ 10 nA

$$\mathbf{J}_s(\mathbf{r}, t) = \mathbf{J}_c(y, z, t) \sin \varphi(y, z, t),$$

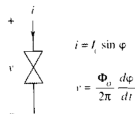
where φ is known as *the gauge-invariant phase difference* and is given by

$$\varphi(y, z, t) = \theta_1(y, z, t) - \theta_2(y, z, t) - \frac{2\pi}{\Phi_o} \int_1^2 \mathbf{A}(\mathbf{r}, t) \cdot d\mathbf{l}.$$

$$\frac{\partial \varphi(y, z, t)}{\partial t} = \frac{2\pi}{\Phi_o} \int_1^2 \mathbf{E}(\mathbf{r}, t) \cdot d\mathbf{l}$$

The latter equation is known as the Josephson *voltage-phase relation*. Consequently,

$$i = I_c \sin \varphi(t), \quad \frac{d\varphi}{dt} = \frac{2\pi}{\Phi_o} v$$



The basic Josephson junction as a lumped circuit parameter is denoted by the crossed symbol in the circuit diagram and is governed by the two equations shown.

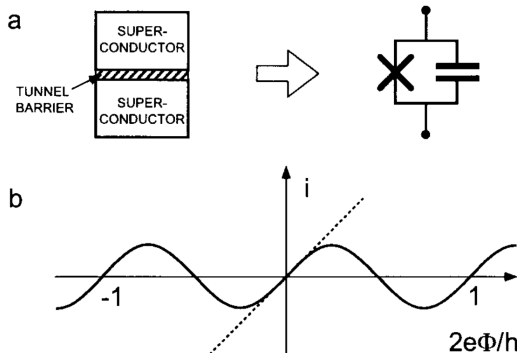


Fig. 2. (a) A Josephson tunnel junction can be modelled as a Josephson tunnel element (cross) in parallel with a capacitor. (b) Current-flux relation of the Josephson element. The dashed line is the current-flux relation of a linear inductance whose value is equal to the effective inductance of the junction.

Relation between phase and flux:

$$\delta = 2\pi\Phi/\Phi_0, \quad \Phi = \int_{-\infty}^t V_J(t') dt', \quad V_J = \frac{\Phi_0}{2\pi} \frac{d\delta}{dt}$$

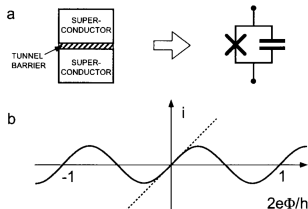


Fig. 2. (a) A Josephson tunnel junction can be modelled as a Josephson tunnel element (cross) in parallel with a capacitor. (b) Current-flux relation of the Josephson element. The dashed line is the current-flux relation of a linear inductance whose value is equal to the effective inductance of the junction.

$$W_J = \int_0^{t_o} i v dt$$

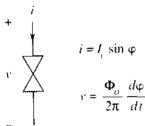
$$W_J = \int_0^{t_o} (I_c \sin \varphi') \left(\frac{\Phi_o}{2\pi} \frac{d\varphi'}{dt} \right) dt$$

$$W_J = \frac{\Phi_o I_c}{2\pi} \int_0^\varphi \sin \varphi' d\varphi'$$

$$W_{J_o} = \Phi_o I_c / 2\pi$$

$$W_J = W_{J_o} - \frac{\Phi_o I_c}{2\pi} \cos \varphi$$

Typical value for $W_{J_o} = 2.067 \cdot 10^{-15} \cdot 5 \cdot 10^{-8} / 2\pi = 1.65 \cdot 10^{-23}$ J.



$$\varphi(t) = \varphi(0) + \frac{2\pi}{\Phi_0} V_o t$$

The basic Josephson junction as a lumped circuit parameter is denoted by the crossed symbol in the circuit diagram and is governed by the two equations shown.

$$i = I_c \sin \left[\frac{2\pi}{\Phi_0} V_o t + \varphi(0) \right] = I_c \sin [2\pi f_J t + \varphi(0)]$$

develops across the junction. This effect is known as the *AC Josephson effect* and f_J is the *Josephson frequency*, given by

$$f_J = \frac{V_o}{\Phi_p} = \frac{2e}{h} V_o = 483.6 \times 10^{12} V_o \text{ (Hz)}.$$

For a constant driving voltage of $10 \mu\text{V}$, the current will oscillate at about 5 GHz. Because a few microvolts is typical of the lower voltage range applied

Current-voltage characteristics of a JJ

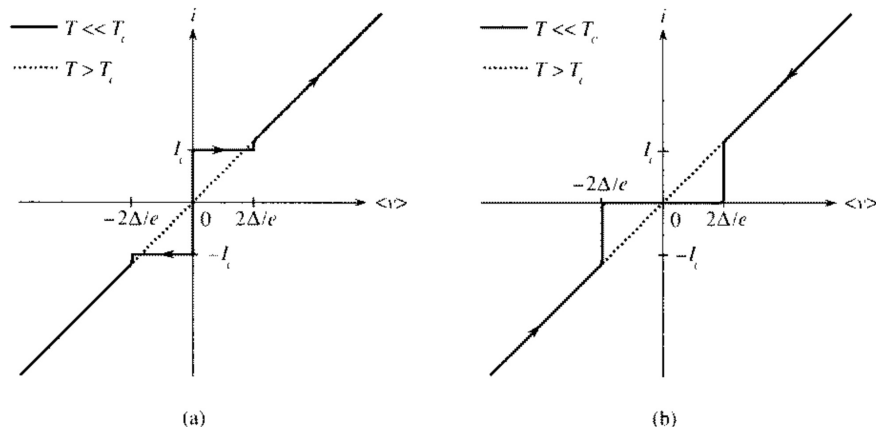
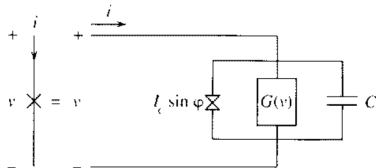


Figure 9.1 Current-voltage characteristic for a constant-current source driving the Josephson junction at a temperature of absolute zero. The voltage is time averaged. Curve (a) is for an increasing driving current and curve (b) is for a decreasing driving current. *Source: Courtesy of D. A. Rudman.*



A model of a generalized Josephson junction showing the three parallel channels.

$$i = I_c \sin \varphi + vG(v) + C \frac{dv}{dt}$$

$$i = I_c \sin \varphi + G(v) \frac{\Phi_o}{2\pi} \frac{d\varphi}{dt} + C \frac{\Phi_o}{2\pi} \frac{d^2\varphi}{dt^2}$$

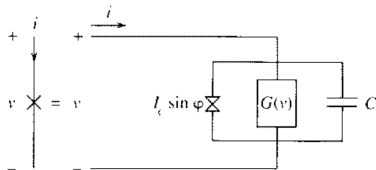
$$i = I_c \sin \varphi + \frac{1}{R} \frac{\Phi_o}{2\pi} \frac{d\varphi}{dt} + C \frac{\Phi_o}{2\pi} \frac{d^2\varphi}{dt^2}$$

$$\frac{i}{I_c} = \sin \varphi + \frac{d\varphi}{d\tilde{\tau}} + \beta_c \frac{d^2\varphi}{d\tilde{\tau}^2}$$

$$\tilde{\tau} = \frac{t}{\tau_J}$$

$$\tau_J = \frac{\Phi_o}{2\pi} \frac{1}{I_c R}$$

Nonlinear Josephson Inductance

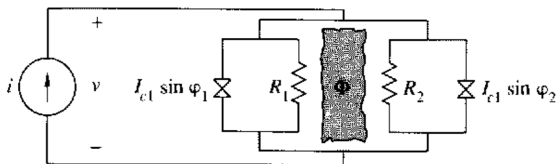


A model of a generalized Josephson junction showing the three parallel channels.

$$i(t) = I_c \sin \varphi(t) + \frac{v(t)}{R},$$
$$\frac{d}{dt} i(t) = \left[\frac{2\pi I_c}{\Phi_o} \cos \varphi(t) \right] v(t) + \frac{1}{R} \frac{d}{dt} v(t).$$

kinetic inductance $L(\varphi)$ and Josephson inductance L_J are given by

$$L(\varphi) = \frac{\Phi_0}{2\pi I_c \cos \varphi} = \frac{L_J}{\cos \varphi}.$$



$$\begin{aligned}
 i &= i_{J_1} + i_{J_2} + i_{R_1} + i_{R_2} \\
 &= I_{c1} \sin \varphi_1 + I_{c1} \sin \varphi_2 + \frac{v}{R_1} + \frac{v}{R_2}
 \end{aligned}$$

$$i = 2I_{c1} \cos\left(\frac{\pi\Phi_{\text{ext}}}{\Phi_o}\right) \sin\left(\varphi_1 + \frac{\pi\Phi_{\text{ext}}}{\Phi_o}\right) + \left(\frac{1}{R_1} + \frac{1}{R_2}\right) v(t)$$

$$I_c = 2I_{c1} \cos\left(\frac{\pi\Phi_{\text{ext}}}{\Phi_o}\right)$$

Quantizing electrical circuits – basics

Quantum LC oscillator

'lumped element' limit where the physical size of the LC oscillator is much smaller than the wavelength of electromagnetic waves at the frequency of the oscillator

$$\mathcal{L} = \frac{1}{2}LI^2 - \frac{1}{2}\frac{q^2}{C} = \frac{L}{2}\dot{q}^2 - \frac{1}{2C}q^2.$$

Euler-Lagrange equation of motion:

$$\ddot{q} = -\omega_0^2 q, \quad \omega_0 = 1/\sqrt{LC}.$$

Conjugate to charge – flux:

$$\Phi = \frac{\delta \mathcal{L}}{\delta \dot{q}} = L\dot{q} = LI.$$

Corresponding Hamiltonian

$$H = \Phi\dot{q} - \mathcal{L} = \frac{\Phi^2}{2L} + \frac{1}{2C}q^2.$$

Hamilton's equation of motion:

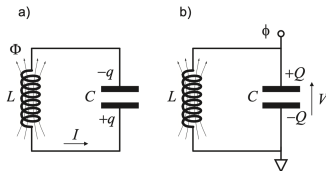
$$\dot{q} = \frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I,$$

$$\dot{\Phi} = -\frac{\partial H}{\partial q} = -\frac{q}{C} = V.$$

$$[\hat{\Phi}, \hat{q}] = -i\hbar$$

And we can write the Hamiltonian:

$$H = \frac{\hbar\omega_0}{2} \left(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \right) = \frac{\hbar\omega_0}{2} \left(2\hat{a}^\dagger \hat{a} + 1 \right)$$



Hamiltonian of an LC oscillator with charge q and flux Φ

$$H = \frac{q^2}{2C} + \frac{\Phi^2}{2L}$$

Operators of flux and charge

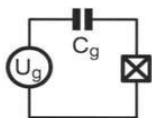
$$\hat{\Phi} = \sqrt{\frac{\hbar Z_0}{2}} (\hat{a} + \hat{a}^\dagger) \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{q} = \frac{1}{i} \sqrt{\frac{\hbar}{2Z_0}} (\hat{a} - \hat{a}^\dagger)$$

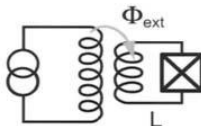
Hamiltonian

$$\hat{H} = \frac{\hbar\omega_0}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \frac{\hbar\omega_0}{2} (2\hat{a}^\dagger \hat{a} + 1)$$

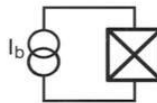
The Cooper Pair Box



a)

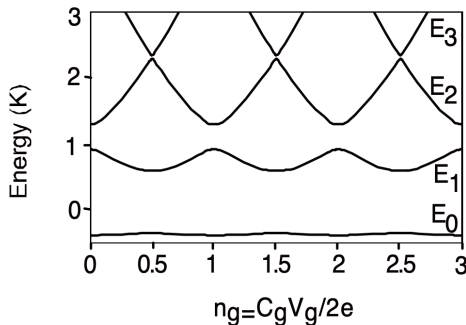
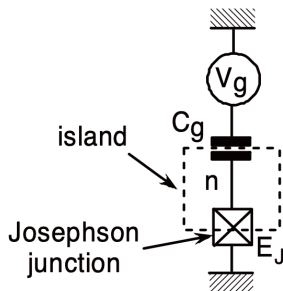


b)



c)

Here $E_C = \frac{(2e)^2}{2(C_J + C_g)}$ is the charging energy of the island of the box and $N_g = Q_r + C_g U / 2e$.



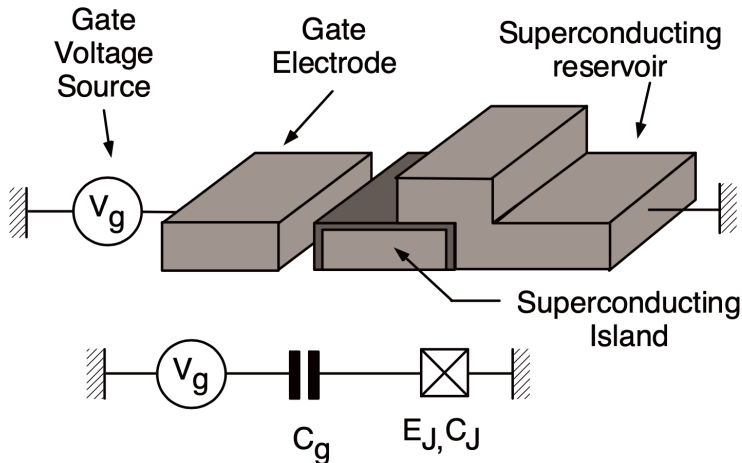
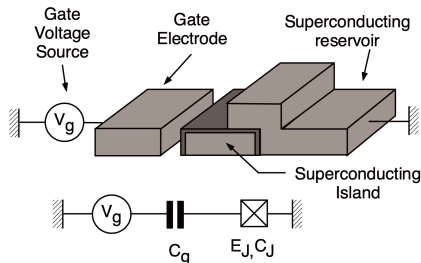


Figure 1.1: *The Cooper pair box. Top: Schematic representation of the Cooper pair box and of its biasing circuit, showing the Josephson junction with energy E_J and capacitance C_J , the superconducting island, the gate, and the gate voltage source V_g . Bottom: Corresponding electrical scheme.*



Pair Coulomb energy:

$$E_C = \frac{(2e)^2}{2C_\Sigma},$$

where

$$C_\Sigma = C_g + C_J$$

is the total capacitance of the island¹.

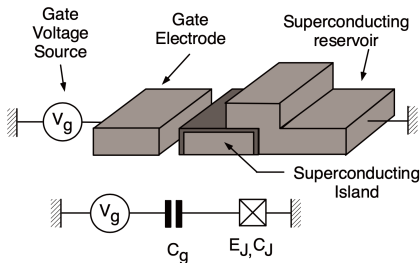
Charge representation

Let $\hat{n}$ be the operator associated to the number of Cooper pairs in excess from neutrality in the metallic island (more commonly called “the number of excess Cooper pairs”).

The eigenstates $|\mathbf{n}\rangle$ of $\hat{n}$ verify:

$$\hat{H}_{el} = E_C (\hat{n} - n_g)^2,$$

$$n_g = \frac{C_g V_g}{2e}$$



Charge representation

$$\hat{n}|\mathbf{n}\rangle = \mathbf{n}|\mathbf{n}\rangle, \mathbf{n} \in \mathbb{Z},$$

$$\hat{H}_J = -\frac{E_J}{2} \left(\sum_{\mathbf{n} \in \mathbb{Z}} |\mathbf{n}\rangle \langle \mathbf{n} + \mathbf{1}| + |\mathbf{n} + \mathbf{1}\rangle \langle \mathbf{n}| \right),$$

$$\hat{H}_{el} = E_C (\hat{n} - n_g)^2, \quad n_g = \frac{C_g V_g}{2e}$$

$$\hat{H}(n_g) = \sum_{\mathbf{n} \in \mathbb{Z}} \left[E_C (\mathbf{n} - n_g)^2 |\mathbf{n}\rangle \langle \mathbf{n}| - \frac{E_J}{2} (|\mathbf{n}\rangle \langle \mathbf{n} + \mathbf{1}| + |\mathbf{n} + \mathbf{1}\rangle \langle \mathbf{n}|) \right]$$

The spectrum associated to this hamiltonian is discrete and periodic in n_g . Let us call $|k\rangle$ the energy eigenstates and E_k their associated energies:

$$\hat{H}(n_g) |k\rangle = E_k |k\rangle.$$

$$\hat{H}(n_g) = \sum_{\mathbf{n} \in \mathbb{Z}} \left[E_C (\mathbf{n} - n_g)^2 |\mathbf{n}\rangle \langle \mathbf{n}| - \frac{E_J}{2} (|\mathbf{n}\rangle \langle \mathbf{n} + \mathbf{1}| + |\mathbf{n} + \mathbf{1}\rangle \langle \mathbf{n}|) \right]$$

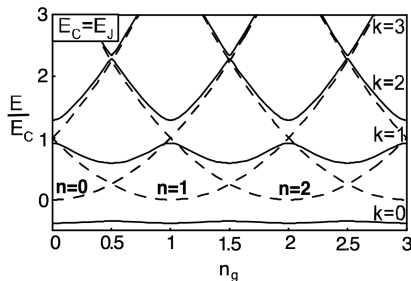


Figure 1.2: *Full lines: Eigenenergies of a Cooper pair box with $E_C = E_J$, calculated by diagonalizing the hamiltonian $\hat{H}$ in the charge representation, in a subspace of 10 charge states. Dotted lines: Electrostatic energies of the charge states. Note that the degeneracies between the energies of neighbouring charge states occurring at $n_g = \frac{1}{2} [\text{mod } 1]$ are lifted by the Josephson coupling.*

Phase representation

Conjugate of a 2e-quantized charge

According to the BCS theory, the conjugate of the number $\hat{N}$ of Cooper pairs in an electrode is the superconducting phase $\hat{\theta}$ of the electrode. The conjugate of the number $\hat{N}$ of Cooper pairs having passed through a Josephson junction, is the superconducting phase difference $\hat{\theta}$ through the junction. Let us note $|\theta\rangle$ the eigenstates of $\hat{\theta}$:

$$\hat{\theta}|\theta\rangle = \theta|\theta\rangle ,$$

with $\theta \in \mathbb{R}$. In the BCS theory, the discreteness of $\hat{N}$ causes that $|\theta\rangle$ and $|\theta + 2\pi\rangle$ have the same physical meaning. As a consequence, in order to describe the states of the node or branch, it is enough to work on the basis $\{|\theta\rangle, \theta \in [0, 2\pi[\}$.

Quantized charge case

In the case of a quantized number of charge $\hat{n}$, the relations (1.82) must include the charge $\hat{Q} = 2e\hat{n}$. The relationship between the superconducting phase $\hat{\theta}$ conjugated to $\hat{n}$ and the flux $\hat{\Phi}$ included in the Equations (1.83) and (1.84) is:

$$\hat{\theta} = \hat{\Phi}/\varphi_0 \text{ [mod } 2\pi] . \quad (1.85)$$

The superconducting tunnel junction can be modeled by a capacitor in parallel with a pure tunneling element, the Josephson junction. The Cooper pairs that tunnel through the junction can be expressed in the **charge basis**

$$Q_J(t) = -2eN(t), \quad \hat{N} = \sum_N N|N\rangle\langle N|.$$

Josephson junction Phase operator

$$|\delta\rangle = \sum_{N=-\infty}^{+\infty} e^{iN\delta}|N\rangle, \quad |N\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\delta e^{-iN\delta}|\delta\rangle, \quad [\hat{\delta}, \hat{N}] = i$$

Introducing the operator:

$$e^{i\hat{\delta}} = \frac{1}{2\pi} \int_0^{2\pi} d\delta e^{i\delta} |\delta\rangle\langle\delta|$$

Yields:

$$e^{i\hat{\delta}}|N\rangle = |N-1\rangle$$

Note, in contrast to quantum optics, where no phase operator can be defined, phase and number operator are here canonically conjugate operators as N can be negative.

$$\begin{aligned} e^{i\phi}|N\rangle &= \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi} |\phi\rangle\langle\phi|N\rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi} |\phi\rangle \sum_M e^{-iM\phi} \langle M|N\rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi} |\phi\rangle \sum_M e^{-iM\phi} \delta_{MN} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi} e^{-iN\phi} |\phi\rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-i(N-1)\phi} |\phi\rangle = |N-1\rangle \end{aligned}$$

Hamiltonian Description of the Josephson Junction

Hamiltonian of the Josephson tunnel Junction using the phase operator

$$\hat{H}_J = -E_J \cos \hat{\delta}, \quad E_J = \frac{1}{8} \frac{h}{e^2} G_t \Delta,$$

where Δ is the superconducting gap and G_t the tunnel conductance in the normal.

Re-expressing the Hamiltonian in the charge Basis of cooper pairs (N) that tunnel through the junction.

$$\hat{H}_J = -\frac{E_J}{2} \frac{1}{2\pi} \int_0^{2\pi} d\delta \left[e^{i\delta} + e^{-i\delta} \right] |\delta\rangle\langle\delta|,$$
$$e^{i\hat{\delta}}|N\rangle = |N-1\rangle, \quad e^{i\hat{\delta}} = \frac{1}{2\pi} \int_0^{2\pi} d\delta e^{i\delta} |\delta\rangle\langle\delta|,$$

$$\hat{H}_J = -\frac{E_J}{2} \sum_{N=-\infty}^{+\infty} [|N\rangle\langle N+1| + |N+1\rangle\langle N|].$$

Properties of conjugate variables

Because of the circular topology of the phase space associated to $\hat{\theta}$, physical expressions can only contain $\hat{\theta}$ through trigonometric expressions. The analog of formulas (1.72) are the Fourier transforms between the eigenstates $\{|n\rangle, n \in \mathbb{Z}\}$ of $\hat{n}$ and the eigenstates $\{|\theta\rangle, \theta \in [0, 2\pi[\}$ of $\hat{\theta}$ [51]:

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \exp(i n \theta) |n\rangle \Leftrightarrow |n\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta \exp(-i n \theta) |\theta\rangle . \quad (1.74)$$

This relation ensures the validity of some fundamental properties likes the translational relations:

$$\exp(i p \hat{\theta}) |n\rangle = |n+p\rangle \quad (1.75)$$

and

$$\exp(-i \theta_0 \hat{n}) |\theta\rangle = |\theta - \theta_0\rangle , \quad (1.76)$$

with $p \in \mathbb{N}$ and $\theta_0 \in \mathbb{R}$. It also infers that $\hat{n}$ can be expressed in the phase space as:

$$\hat{n} = \frac{1}{i} \frac{\partial}{\partial \theta} , \quad (1.77)$$

and that $\hat{\theta}$ can be expressed in the charge space as:

$$\hat{\theta} = i \frac{\partial}{\partial n} . \quad (1.78)$$

Eventually, for any observable $\hat{A}$,

$$[\hat{A}, \hat{H}] = i \hbar \frac{\partial \hat{A}}{\partial t} , \quad (1.79)$$

$$[\hat{A}, \hat{n}] = i \frac{\partial \hat{A}}{\partial \theta} , \quad (1.80)$$

and:

$$[\hat{A}, \hat{\theta}] = -i \frac{\partial \hat{A}}{\partial n} . \quad (1.81)$$

For a harmonic oscillator (in particular the electromagnetic field) with the Hamiltonian $\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$, it is possible to define a “phase operator” in analogy with the classical case:

$$\begin{aligned}\hat{a} &= e^{i\hat{\phi}} \sqrt{\hat{n}}, \\ \hat{a}^\dagger &= \sqrt{\hat{n}} e^{-i\hat{\phi}},\end{aligned}$$

with $\hat{n} = \hat{a}^\dagger \hat{a}$ and $\hat{\phi}$ being the hypothetical Hermitian phase operator.

This operator satisfies many identities similar to the previously defined superconducting phase operator $\hat{\theta}$, namely $[\hat{n}, \hat{\theta}] = i$. But careful calculation reveals that it is in fact not a well defined Hermitian operator. For example it is easy to show the commutator $[e^{i\hat{\phi}}, e^{-i\hat{\phi}}]$ is not equal to zero (see the exercise), which is a sign of inconsistency. This inconsistency in fact arises due to the fact that the Hilbert space for the harmonic oscillator is given by $n \in \mathbb{N}$. In the case of the superconducting phase the Hilbert space of the Cooper pairs also extends to negative values, allowing the well defined phase operator.

Phase representation

$$\begin{aligned}\hat{\theta}|\theta\rangle &= \theta|\theta\rangle & |\theta\rangle &\equiv |\theta + 2\pi\rangle \\ \exp(\pm i\hat{\theta})|\mathbf{n}\rangle &= |\mathbf{n} \pm 1\rangle & \hat{\mathbf{n}} &= \frac{1}{i} \frac{\partial}{\partial \theta}\end{aligned}$$

Hamiltonian in the phase representation

$$H(n_g) = E_C \left(\frac{1}{i} \frac{\partial}{\partial \theta} - n_g \right)^2 - E_J \cos(\theta)$$

$$\Psi_k(\theta) = \langle \theta | k \rangle$$

$$E_C \left(\frac{1}{i} \frac{\partial}{\partial \theta} - n_g \right)^2 \Psi_k(\theta) - E_J \cos(\theta) \Psi_k(\theta) = E_k \Psi_k(\theta)$$

$$\Psi_k(\theta) = \Psi_k(\theta + 2\pi)$$

the function

$$\varphi_k(\theta) = \exp(-in_g\theta)\Psi_k(\theta)$$

is a solution of the equation:

$$-E_C \frac{\partial^2}{\partial \theta^2} \varphi_k(\theta) - E_J \cos(\theta) \varphi_k(\theta) = E_k \varphi_k(\theta) . \quad (1.18)$$

The equation (1.18) takes the form of a Mathieu equation:

$$\frac{\partial^2 y(z)}{\partial z^2} - 2q \cos(2z)y(z) = -ay(z) , \quad z \in]-\infty, +\infty[, \quad (1.19)$$

with

$$\theta = 2z, \quad y(z) = \varphi_k(2z), \quad q = -2E_J/E_C, \quad a = 4E_k/E_C . \quad (1.20)$$

The solutions of this equation are analytically known [44]. The Mathieu functions³ $\mathcal{M}_C(a, q, z)$ and $\mathcal{M}_S(a, q, z)$, are the textbook solutions of (1.19) respectively even and odd in z . Note that when $q = 0$, these Mathieu functions are simply:

$$\begin{aligned} \mathcal{M}_C(a, 0, z) &= \cos(\sqrt{a}z) \\ \mathcal{M}_S(a, 0, z) &= \sin(\sqrt{a}z) . \end{aligned} \quad (1.21)$$

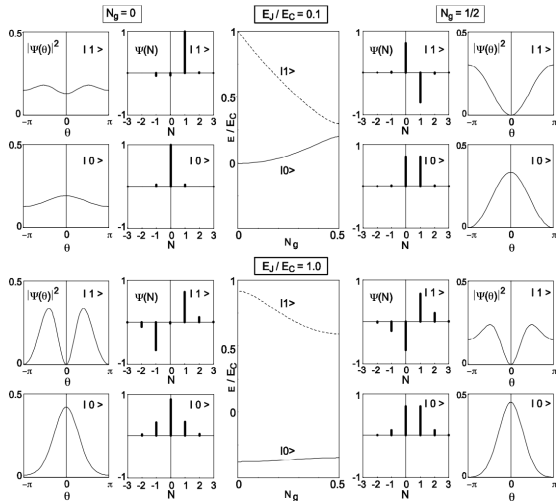
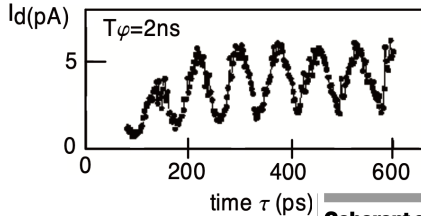
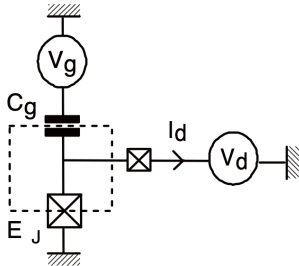


Fig. 3. Eigenenergies (middle panels) and wavefunctions of the $|0\rangle$ and $|1\rangle$ states in the charge and phase representations, for $N_g = 0$ (left panels) and for $N_g = 1/2$ (right panels), and for E_J/E_C ratios equal to 0.1 (top panels) and 1 (bottom panels). The $\Psi_k(N)$ eigenvectors are directly represented since they can be chosen real, whereas the $\Psi_k(\theta)$ wavefunctions are represented by their modulus squared.



Coherent control of macroscopic quantum states in a single-Cooper-pair box

Y. Nakamura*, Yu. A. Pashkin† & J. S. Tsai*

* NEC Fundamental Research Laboratories, Tsukuba, Ibaraki 305-8051, Japan

† CREST, Japan Science and Technology Corporation (JST), Kawaguchi, Saitama 332-0012, Japan

Figure 4: *Coherent manipulation of the quantum state of a Cooper pair box, in the experiment of Y. Nakamura et al. [19]. Left panel: Simplified electrical scheme of the experiment. A fast voltage pulse is applied to the gate to bring the system at the charge degeneracy point $n_g = 1/2$. The sudden change of the hamiltonian induces Rabi oscillations between the $|0\rangle$ and $|1\rangle$ states during the pulse duration τ . Right panel: Oscillations of the occupation probability of state $|1\rangle$ are revealed by oscillations of the current probe I_d with τ .*

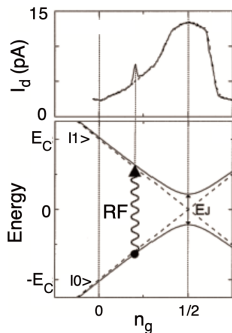


Figure 5: *Spectroscopy of the Cooper pair box. The energy difference between states $|0\rangle$ and $|1\rangle$ is measured by applying to the gate continuous radiofrequency signals at different frequencies and by sweeping the gate charge n_g . Top panel: A resonant increase of the current I_d through the probe junction (see text) is observed when n_g is such that the energy difference matches the applied frequency. Bottom panel: Energy diagram illustrating the excitation of the box by radiofrequency irradiation. Solid lines represent eigenenergies of the box whereas dashed lines show the electrostatic energy of charge states.*

Manipulating the Quantum State of an Electrical Circuit

D. Vion,* A. Aassime, A. Cottet, P. Joyez, H. Pothier,
C. Urbina,† D. Esteve, M. H. Devoret‡

We have designed and operated a superconducting tunnel junction circuit that behaves as a two-level atom: the "quantronium." An arbitrary evolution of its quantum state can be programmed with a series of microwave pulses, and a projective measurement of the state can be performed by a pulsed readout subcircuit. The measured quality factor of quantum coherence $Q_{\psi} \approx 25,000$ is sufficiently high that a solid-state quantum processor based on this type of circuit can be envisioned.

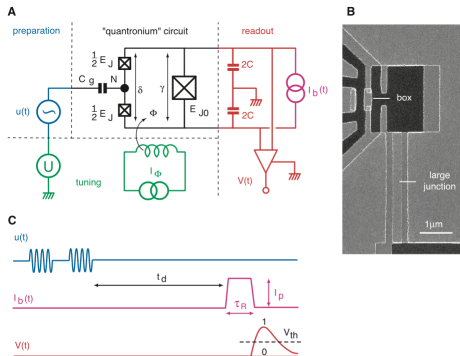
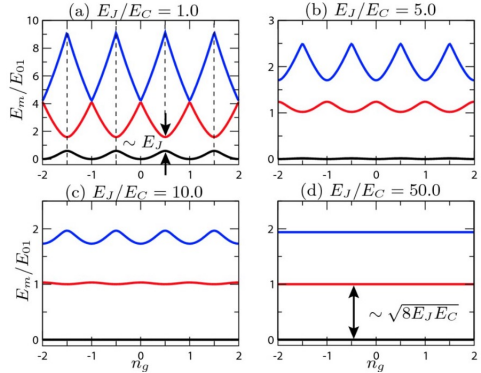
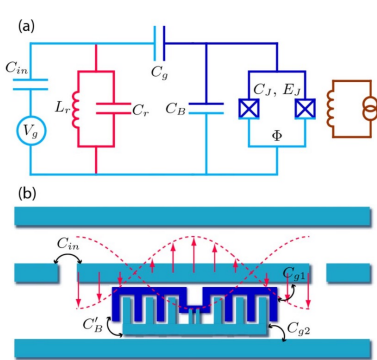


Fig. 1. (A) Idealized circuit diagram of the quantronium, a quantum-coherent circuit with its tuning, preparation, and readout blocks. The circuit consists of a Cooper pair box island (black node) delimited by two small Josephson junctions (crossed boxes) in a superconducting loop. The loop also includes a third, much larger Josephson junction shunted by a capacitance C . The Josephson energies of the box and the large junction are E_J and E_{J0} . The Cooper pair number N and the phases δ and γ are the degrees of freedom of the circuit. A dc voltage U applied to the gate capacitance C_g and a dc current I_d applied to a coil producing a flux Φ in the circuit loop tune the quantum energy levels. Microwave pulses $u(t)$ applied to the gate prepare arbitrary quantum states of the circuit. The states are read out by applying a current pulse $I_B(t)$ to the large junction and by monitoring the voltage $V(t)$ across it. (B) Scanning electron micrograph of a sample. (C) Signals involved in quantum state manipulation and measurement. Top: Microwave voltage pulses $u(t)$ are applied to the gate for state manipulation. Middle: A readout current pulse $I_B(t)$ with amplitude I_p is applied to the large junction t_d after the last microwave pulse. Bottom: Voltage $V(t)$ across the junction. The occurrence of a pulse depends on the occupation probabilities of the energy eigenstates. A discriminator with threshold V_{th} converts $V(t)$ into a boolean output for statistical analysis.

The transmon qubit

Transmon: a transmission-line shunted plasma oscillation qubit



- ① Large ratio of Josephson energy to charge energy E_J/E_C
- ② More insensitive to charge noise
- ③ Longer dephasing times

Circuit Quantization and anharmonicity

Hamiltonian of transmon qubit:

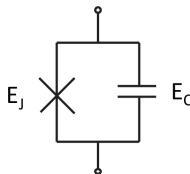
$$H = 4E_C n^2 - E_J \cos \delta, \quad \cos \delta \approx 1 - \frac{\delta^2}{2!} + \frac{\delta^4}{4!} + \dots$$

To lowest order, we have the harmonic oscillator Hamiltonian:

$$H \approx 4E_C n^2 + E_J \frac{\delta^2}{2} = \sqrt{8E_J E_C} \left(a^\dagger a + \frac{1}{2} \right)$$

Where n , δ is the conjugate pair of position and momentum:

$$\delta = \left(\frac{2E_C}{E_J} \right)^{1/4} (a + a^\dagger),$$
$$n = -i \left(\frac{E_J}{8E_C} \right)^{1/4} \frac{1}{\sqrt{2}} (a - a^\dagger).$$



So the Hamiltonian keeping up to the fourth order of δ is:

$$H \approx 4E_C n^2 + \frac{E_J \delta^2}{2} - \frac{\delta^4}{24} = \sqrt{8E_J E_C} \left(a^\dagger a + \frac{1}{2} \right) - \frac{E_C}{12E_J} (a + a^\dagger)^4$$

Circuit Quantization and anharmonicity

Correct the quartic term for each energy level:

$$\begin{aligned}\Delta E_m &= \langle m | -E_J \frac{\delta^4}{24} | m \rangle \\ &= -\frac{E_C}{12} \langle m | (a + a^\dagger)^4 | m \rangle \\ &= -\frac{E_C}{12} (6m^2 + 6m + 3).\end{aligned}$$

Corrected energy level for transmon qubit:

$$E_m = m\sqrt{8E_J E_C} - \frac{E_C}{12} (6m^2 + 6m + 3).$$

Lowest energy of transmon qubit,

$$E_{10} = E_1 - E_0 = \sqrt{8E_J E_C} - E_C,$$

Anharmonicity:

$$\eta = \omega_{21} - \omega_{10} = (E_{21} - E_{10}) / \hbar = -E_C / \hbar.$$

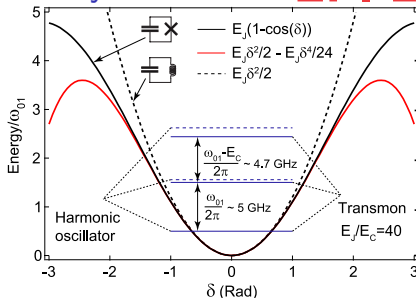
The relative anharmonicity $\eta_r \equiv \frac{\eta}{E_{10}} \approx -\left(\frac{8E_J}{E_C}\right)^{-\frac{1}{2}}$ decrease algebraically with E_J/E_C .

Charge dispersion decrease exponentially with E_J/E_C : (in large E_J/E_C approximation)

$$E_m(n_g) \simeq E_m(n_g = 1/4) - \frac{\epsilon_m}{2} \cos(2\pi n_g),$$

where $\epsilon_m \equiv E_m(n_g = 1/2) - E_m(n_g = 0)$ $\epsilon_m \simeq (-1)^m E_C \frac{2^{4m+5}}{m!} \sqrt{\frac{2}{\pi}} \left(\frac{E_J}{2E_C}\right)^{\frac{m}{2} + \frac{3}{4}} e^{-\sqrt{8E_J/E_C}},$

Exploiting these facts, transmon qubit operates on $\frac{E_J}{E_C} \gg 1$ (transmon regime).



In the absence of junction nonlinearity, one can solve the normal modes of the electrical circuit:

$$\sum_m \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m.$$

Introducing the Bosonic operators for the Josephson phase:

$$\begin{aligned}\hat{\varphi}_{m,j} &= \varphi_{\text{zpf},m,j} \left(\hat{a}_m + \hat{a}_m^\dagger \right), \\ \varphi_{\text{zpf},m,j} &= \frac{1}{\phi_0} \sqrt{\frac{\hbar}{2\omega_m C_m}}.\end{aligned}$$

Introducing the Bosonic operators for the Josephson phase, and Taylor expanding leads to the expression (note that the quadratic part of the cosine potential gives rise to the normal modes and is therefore removed from the expansion below).

$$\begin{aligned}\hat{H} &= \sum_m \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m + \sum_j \sum_{n \geq 2} E_j \frac{(-1)^{n+1}}{(2n)!} \hat{\varphi}_j^{2n}, \\ \hat{H} &= \sum_m \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m + E_j \left[1 - \cos \hat{\varphi}_j - \frac{\hat{\varphi}_j^2}{2} \right].\end{aligned}$$

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Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics

**A. Wallraff¹, D. I. Schuster¹, A. Blais¹, L. Frunzio¹, R.- S. Huang^{1,2},
J. Majer¹, S. Kumar¹, S. M. Girvin¹ & R. J. Schoelkopf¹**

¹*Departments of Applied Physics and Physics, Yale University, New Haven,
Connecticut 06520, USA*

²*Department of Physics, Indiana University, Bloomington, Indiana 47405, USA*

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- What is the Jaynes-Cummings (JC) model between a two-level system and quantized photons? What is the difference between JC and the semi-classical atom-light interaction?
- How does the physical system studied in the paper map to the JC model (what are the "atom" and "photon cavity")?
- How is the frequency of the "atom" tuned to match that of the "cavity"? How does the measured response change?
- What is the definition of the strong coupling regime?
- What is the Hamiltonian in the dispersive limit $\Delta \gg g$? Describe the transformation from the JC model to the dispersive limit?
- What is the physical meaning of the κ and γ parameters, how are they related to the measured cavity transmission and phase response in figure 2?
- Why is it important to achieve strong coupling regime?
- Which of the measurement results indicates the onset of strong coupling?