Quantum Electrodynamics and Quantum Optics: Lecture 4

Fall 2024

P-Function[1]

P-function

We introduce the P-function or the Glauber-Sudarshan phase space representation $P(\alpha)$ as

$$\rho = \int P(\alpha) \left|\alpha\right\rangle \left\langle\alpha\right| \mathrm{d}^2\alpha.$$

The P-function has the following properties:

$$\int \mathrm{d}^2\alpha P(\alpha) = 1 \quad P^*(\alpha) = P(\alpha).$$

It is often used to compute the expectations of a normally ordered function:

$$\left\langle \hat{a}^{\dagger m}\hat{a}^n \right\rangle = \mathrm{Tr}(\hat{a}^n \rho \hat{a}^{\dagger m}) = \int \mathrm{d}^2\alpha P(\alpha) \alpha^{*m} \alpha^n$$

where $P(\alpha)$ is a quasi-probability function which diagonalizes the density operator in the coherent state basis.

[1] Glauber, R. J. (1963). Coherent and incoherent states of the radiation field. Physical Review, 131(6), 2766.

P-Function

- Coherent states $\rho = |\alpha_0\rangle\langle\alpha_0|$: $P(\alpha) = \delta(\alpha - \alpha_0)$

$$\text{Variance:} \quad \left\langle \Delta\hat{n}^2 \right\rangle = \left\langle (\hat{a}^\dagger\hat{a})^2 - \left\langle \hat{a}^\dagger\hat{a} \right\rangle^2 \right\rangle = \left\langle \hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}^\dagger\hat{a} - \left\langle \hat{a}^\dagger\hat{a} \right\rangle^2 \right\rangle \quad \text{normal order}$$

$$= \int \mathrm{d}^2\alpha \underbrace{P(\alpha)}_{\delta(\alpha-\alpha_0)} (|\alpha|^4 + |\alpha|^2) - \left[\int \mathrm{d}^2\beta \underbrace{P(\beta)}_{\delta(\beta-\alpha_0)} |\beta|^2 \right]^2$$

$$= |\alpha_0|^2$$

- Fock states $\rho = |n\rangle\langle n|$: $P(\alpha) = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)$, where $\delta^{(2)}(\alpha) = \delta(\alpha)\delta(\alpha^*)$ and δ the complex Dirac function.
- Also worth noting the operator correspondences:[2] $\hat{a}^\dagger |\alpha\rangle\langle\alpha| = (\alpha^* + \frac{\partial}{\partial\alpha})|\alpha\rangle\langle\alpha|$ and $|\alpha\rangle\langle\alpha|\hat{a} = (\alpha + \frac{\partial}{\partial\alpha^*})|\alpha\rangle\langle\alpha|$.

[2] Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Page 79

Can a $P(\alpha, \alpha^*)$ be found for every density matrix?

The Characteristic Function of the P representation[3]

$$\chi_N(z, z^*) \equiv \mathbf{Tr}[\rho e^{iz^* a^\dagger} e^{iza}] = \int P(\alpha, \alpha^*) e^{iz^*\alpha^*} e^{iz\alpha} \mathrm{d}^2\alpha$$

P function expressed in the Fock state basis

$$P(\alpha, \alpha^*) = \int \mathrm{d}^2 z \sum_{n,m,k} \rho_{n+k,m+k} \sqrt{(n+k)!(m+k)!} (iz^*)^m (iz)^n e^{-iz^*\alpha^*} e^{iz\alpha} / k!m!n!$$

Normal ordered operator products

$$\langle (\hat{a}^\dagger)^p (\hat{a})^q \rangle = \frac{\partial^{p+q}}{\partial (iz^*)^p \partial (iz)^q} \chi_N(z, z^*)|_{z=z^*=0}$$

[3] Scully, M. O., & Zubairy, M. S. (1999). Quantum optics., page 81

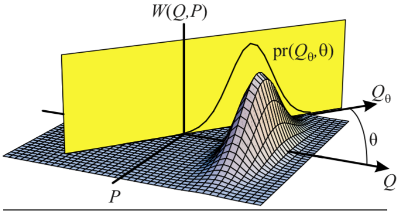
Wigner Function[4]

Wigner function, the phase-space quasi-probability density

$$W_{\hat{\rho}}(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle q+\frac{1}{2}q'\right|\hat{\rho}\left|q-\frac{1}{2}q'\right\rangle e^{-ipq'}dq'$$

This function uniquely defines the state and directly relates to the quadrature histograms measured experimentally via

$$\Pr(q_\theta,\theta) = \int_{-\infty}^{\infty} W_{\det}(q_\theta\cos\theta - p_\theta\sin\theta, q_\theta\sin\theta + p_\theta\cos\theta)dp_\theta.$$



The experimentally measured probability density $\Pr(q_\theta,\theta)$ is the integral projection of the Wigner function $W_{\hat{\rho}}(q,p)$ onto a vertical plane defined by the phase of the local oscillator.

[4] Lvovsky, Alexander I., and Michael G. Raymer. "Continuous-variable optical quantum-state tomography." *Reviews of Modern Physics* 81.1 (2009): 299.

Wigner function

Definition

$$W(p,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ipx) \left\langle q - \frac{x}{2}\right| \hat{\rho} \left| q + \frac{x}{2}\right\rangle dx$$

Marginal distributions

$$\langle p|\,\rho\,|p\rangle = |\psi(p)|^2 = \int_{-\infty}^{\infty} W(p,q)dq \qquad \langle q|\,\rho\,|q\rangle = |\psi(q)|^2 = \int_{-\infty}^{\infty} W(p,q)dp$$

Basic Properties

- Real $W^*(q,p) = W(q,p)$
- $\mathbf{Tr}[\hat{\rho}_1\hat{\rho}_2] = 2\pi \iint W_1(q,p)W_2(q,p)dqdp$ or $\langle \Psi_1\rangle\, \Psi_2 = 2\pi \iint W_1(q,p)W_2(q,p)dqdp$
- $\mathbf{Tr}[\hat{\rho}^2] = \int 2\pi W(q,p)^2 dqdp \leq 1$

Wigner function and the time-frequency distribution[5]

Time-frequency

$P(t, \omega)$: intensity at time t and frequency ω

$$S(\omega) = \frac{1}{\sqrt{2\pi}} \int s(t) e^{-i\omega t} dt$$

$$\int P(t, \omega) dt = |S(\omega)|^2$$

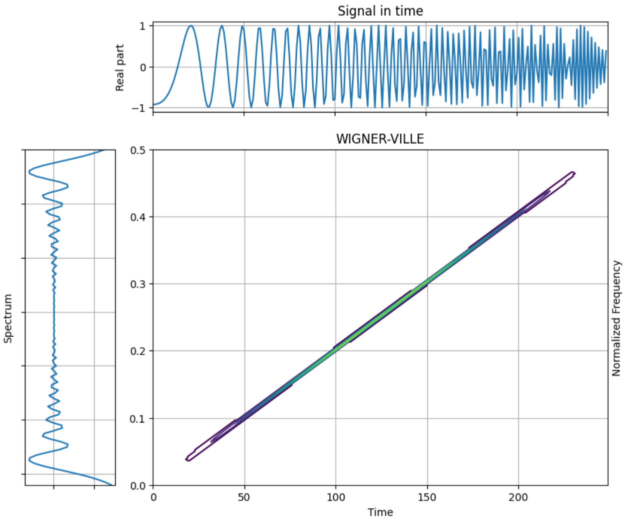
$$\int P(t, \omega) d\omega = |s(t)|^2$$

Wigner-Ville distribution

$$P(t, \omega) = \frac{1}{2\pi} \int s^*(t - \frac{1}{2}\tau) e^{-i\tau\omega} s(t + \frac{1}{2}\tau) d\tau$$

[5] Ville, J. "Theorie et application dela notion de signal analytique." Cables et transmissions 2.1 (1948): 61-74.

Wigner-Ville distribution for a chirped pulse[6]



[6] PyTFTB, https://tftb.readthedocs.io

Wigner function

Coherent state

$|\alpha\rangle = \left|\frac{1}{2}X_1 + i\frac{1}{2}X_2\right\rangle$

$x_i' = x_i - X_i$ and $\langle \Delta \hat{x}_1^2 \rangle = \langle \Delta \hat{x}_2^2 \rangle = 1$

$W(x_1', x_2') = \frac{2}{\pi} e^{-\frac{1}{2}(x_1'^2 + x_2'^2)}$

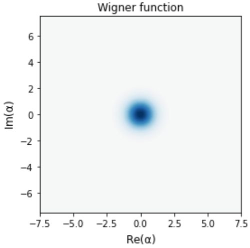
Squeezed state

$W(x_1', x_2') = \frac{2}{\pi} e^{-\frac{1}{2}(x_1'^2 e^{-2r} + x_2'^2 e^{+2r})}$

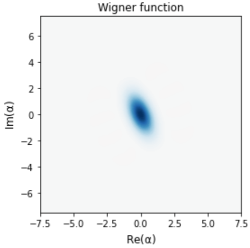
Fock state

For the state $|1\rangle$: $W(\alpha, \alpha^*) = \frac{2}{\pi} e^{-2|\alpha|^2}(4|\alpha|^2 - 1)$
is negative at origin

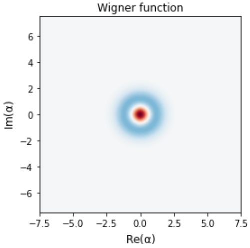
Visualizations of Wigner function



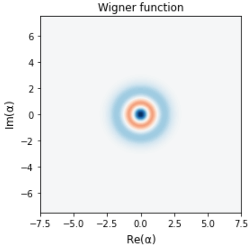
Vaccum state



Squeezed state

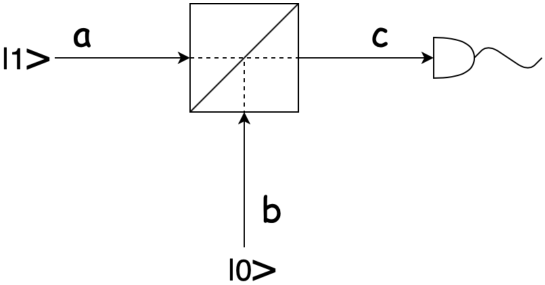


Fock state $|1\rangle$



Fock state $|2\rangle$

Influence of a beamsplitter splitting ratio on Wigner function measurement



$$\hat{c} = \sqrt{\eta}\hat{a} + i\sqrt{1-\eta}\hat{b}$$

If we measure the Wigner function at port c, with η larger than 0.5, the negativity of the Wigner function is observable.

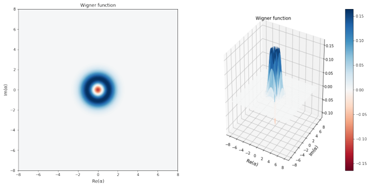


Figure: $\eta = 0.75$

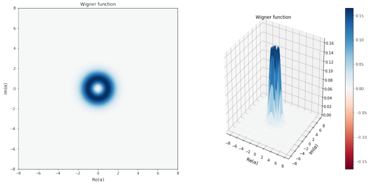


Figure: $\eta = 0.5$

Quasi probability function

Husimi-Q function

$$\chi_A \equiv \mathbf{Tr}[\rho e^{iza} e^{iz^* a^\dagger}]$$

$$\langle \hat{a}^q \hat{a}^{\dagger p} \rangle = \frac{\partial^{p+q}}{\partial (iz^*)^p \partial (iz)^q} \chi_A(z, z^*)$$

Wigner function

$$\chi_s \equiv \mathbf{Tr}[\rho e^{iz^* a^\dagger + iza}]$$

$$W(\alpha, \alpha^*) \equiv \int \chi_s(z, z^*) e^{-iz^* \alpha^*} e^{iz\alpha} \mathrm{d}^2 z$$

$$\left\langle \frac{\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger}{2} \right\rangle = \int W(\alpha, \alpha^*) \alpha^* \alpha \partial^2 \alpha$$

Quantum state tomography

Motivation

To reconstruct a quantum state of light, we cannot directly measure ρ_{nn} with a photo-detector but we can measure $\mathrm{Pr}(X_\theta)$ and reconstruct the full Wigner function.

$$X_\theta = \langle X_\theta | \rho | X_\theta \rangle = \langle X | U_\theta^\dagger \rho U_\theta | X \rangle$$

$$\mathrm{Pr}(X_\theta) = \int_{-\infty}^{\infty} W(p_\theta, q_\theta) dp_\theta$$

$$W_\theta(r) = -\frac{1}{\pi} \int_r^\infty \mathrm{Pr}(X_\theta)(X_\theta^2 - r^2)^{-1/2} dX_\theta$$

It is only posible to obtain such $W_\theta(r)$ when the Wigner function is rotationally symmetric[7]

[7] Vogel, W., Welsch, D.G. "Quantum Optics" (2001). Chapter 7

State Reconstruction

Inverse Radon transformation

$$W_{\text{det}}(q,p) = \frac{1}{2\pi^2} \int_0^{\pi} \int_{-\infty}^{\infty} \Pr(q_\theta,\theta) \times K(q\cos\theta + p\sin\theta - q_\theta) dq_\theta d\theta$$

with the integration kernel $K(x) = \frac{1}{2}\int_{-\infty}^{\infty} |\xi| e^{i\xi x} d\xi$. The density matrix can then be reconstructed using the pattern function method.

Maximum likelihood reconstruction

$$L = \Pi_i \Pr_{\hat{\rho}}(q_i, \theta_i)$$

is the likelihood function given the measured data set $\{(q_i, \theta_i)\}$ where $\hat{\rho}$ is the density matrix to be optimized.

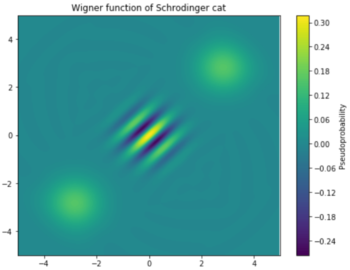
Quantum Toolbox

QuTiP
Quantum Toolbox in Python

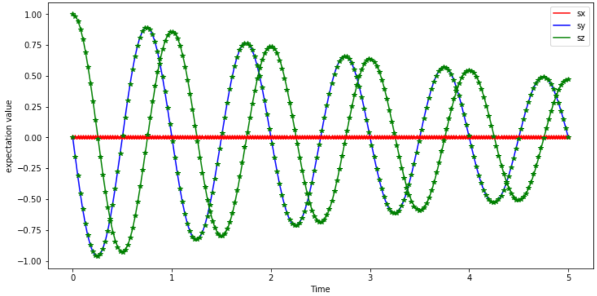
Papers Using QuTiP

56. **Molony** et al., "Creation of Ultracold ^{87}RbCs Molecules in the Rovibrational Ground State",	Phys. Rev. Lett. **113**, 255301 (2014).
55. **Lecocq** et al., "Resolving the vacuum fluctuations of an optomechanical system using an artificial atom",	Nat. Phys. **11**, 635 (2015).
54. **Bassereh** et al., "Effect of Noise on the Efficiency of Quantum Excitation Energy Transfer in a Toy Model of a Linear Protein Structure",	arXiv:1408.6256
53. **Müller** et al., "Coherent Generation of Nonclassical Light on Chip via Detuned Photon Blockade",	Phys. Rev. Lett. **114**, 233601 (2015).
52. **Reimann** et al., "Cavity-Modified Collective Rayleigh Scattering of Two Atoms",	Phys. Rev. Lett. **114**, 023601 (2015).
51. **Ostermann** et al., "Protected subspace Ramsey metrology",	Phys. Rev. A **90**, 053823 (2014).
50. **Mari** et al., "Quantum optomechanical piston engines powered by heat",	J. Phys. B **48**, 175501 (2015).
49. **Lin** et al., "Josephson parametric phase-locked oscillator and its application to dispersive readout of superconducting qubits",	Nat. Commun. **5**, 4480 (2014).
48. **Lagoudakis** et al., "Hole Spin Pumping and Re-pumping in a p-type δ-doped InAs Quantum Dot",	Phys. Rev. B **90**, 121402(R) (2014).
47. **Figueiredo Roque** et al., "Dissipation-driven squeezed and sub-Poissonian mechanical states in quadratic optomechanical systems",	arXiv:1406.1987

Quantum Toolbox



Wigner function of cat states



Time evolution of a qubit

Paper for next week

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Quantum State Reconstruction of the Single-Photon Fock State

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(Received 14 March 2001; published 11 July 2001)

We have reconstructed the quantum state of optical pulses containing single photons using the method of phase-randomized pulsed optical homodyne tomography. The single-photon Fock state $|1\rangle$ was prepared using conditional measurements on photon pairs born in the process of parametric down-conversion. A probability distribution of the phase-averaged electric field amplitudes with a strongly non-Gaussian shape is obtained with the total detection efficiency of $(55 \pm 1)\%$. The angle-averaged Wigner function reconstructed from this distribution shows a strong dip reaching classically impossible negative values around the origin of the phase space.

DOI: 10.1103/PhysRevLett.87.050402 PACS numbers: 03.65.Wj, 42.50.Dv

Questions for next week

- What's the expression of Wigner function they used
- What measurement data is taken and how is it used to reconstruct the Wigner function
- How does measurement efficiency impact the result
- How does signal-LO mode-matching influence efficiency in homodyne detection
- Why a single laser is used for local-oscillator and signal
- Why do they use a doubler and down-converter for single photon sources
- How does the spatial-temporal pulse shape of single photon match LO
- How is the density matrix reconstructed? What are the values of the off-diagonal terms.