Quantum Electrodynamics and Quantum Optics: Lecture 3

Fall 2024

QuTiP[1]

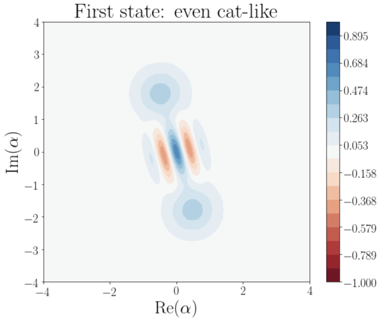
QuTiP

Quantum Toolbox in Python

QuTiP is an open-source software for simulating the dynamics of open quantum system. QuTiP aims to provide user-friendly and efficient numerical simulations of a wide variety of Hamiltonians, including those with arbitrary time-dependence, commonly found in a wide range of physics applications such as quantum optics, trapped ions, superconducting circuits, and quantum nanomechanical resonators.

In [8]:

```
W_even=np.around(W_even, decimals=2)
plt.figure(figsize=(10, 8))
plt.contourf(xvec,xvec, W_even, cmap='RdBu', levels=np.linspace(-1,
 1, 20))
plt.colorbar()
plt.xlabel(r'Re$(\alpha)$', fontsize=label_size), width="300"
plt.ylabel(r'Im$(\alpha)$', fontsize=label_size)
plt.title("First state: even cat-like", fontsize=title_font)
plt.show()
```



[1] Refer to http://qutip.org

QuTiP[2]

In a finite Hilbert space of Fock states one can

define operators and immediatly obtain their matrix form,

e.g. annihilation operator $\hat{a}$

```
a = destroy(5)
```

creation operator $\hat{a}^\dagger$

```
a.dag()
```

Quantum object: dims = [[5], [5]], shape = (5, 5), type = oper, isherm = False

$$\begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.414 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.732 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \end{pmatrix}$$

and compute the commutators $[\hat{a}, \hat{a}^\dagger]$

```
commutator(a, a.dag())
```

Quantum object: dims = [[5], [5]], shape = (5, 5), type = oper, isherm = True

$$\begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.000 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -4.0 \end{pmatrix}$$

A number operator $\hat{a}^\dagger \hat{a} = |n\rangle \langle n|$

in Hilbert space $|0\rangle, \dots, |99\rangle$

```
a = destroy(100)
n = a.dag()*a
n
```

Quantum object: dims = [[100], [100]], shape = (100, 100), type = oper, isherm = True

$$\begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 2.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.000 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 4.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 95.000 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 96.000 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 97.000 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 98.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 99.0 \end{pmatrix}$$

and its expectation value in a coherent state

```
alpha = coherent(100, 2 + 3 * 1j)
print(expect(n, alpha))
```

```
12.999999999999998
```

[2] Refer to http://qutip.org

Squeezed States Review[3]

Squeezing operator

$$\hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})}$$

We can also re-express $\xi = re^{i\theta}$ in terms of $\mu = \cosh r$ and $\nu = e^{i\theta} \sinh r$. Without giving full operator "disentangling" calculation, this is equal to the following normally ordered expression

$$\hat{S}(\xi) = e^{-\frac{\nu}{2\mu}\hat{a}^{\dagger 2}} \left(\frac{1}{\mu}\right)^{\hat{n}+\frac{1}{2}} e^{\frac{\nu^*}{2\mu}\hat{a}^2}.$$

Hence, the vacuum squeezed state can be expressed as

$$|\xi, 0\rangle = \frac{1}{\sqrt{\mu}} e^{-\frac{\nu}{2\mu}\hat{a}^{\dagger 2}} |0\rangle$$

which is also called two photon coherent state.

[3] Quantum Optics W. Vogel Chapter 3

Squeezed States Review

Squeezing operator : position / momentum basis

Starting from the transformation of the quadrature under the squeezing operator :

$$\hat{S}^\dagger(\xi)(\hat{Y}_1 + i\hat{Y}_2)\hat{S}(\xi) = \hat{Y}_1 e^{-r} + i\hat{Y}_2 e^{r},$$

we can derive the transformation of the $\hat{x}$ and $\hat{p}$ in the **Heisenberg picture**. Indeed

$$\hat{x}_{\mathrm{sq}} = \hat{S}^\dagger(\xi)\hat{x}\hat{S}(\xi) = e^{-r}\hat{x}$$

$$\hat{p}_{\mathrm{sq}} = \hat{S}^\dagger(\xi)\hat{p}\hat{S}(\xi) = e^{r}\hat{p}.$$

Now in the **Schrödinger picture**, this leads to rescaled coordinates

$$\psi_{\mathrm{sq}}(x) = \langle x|\hat{S}(\xi)|\psi\rangle = e^{r/2}\psi(e^{r}x)$$

$$\tilde{\psi}_{\mathrm{sq}}(p) = \langle p|\hat{S}(\xi)|\psi\rangle = e^{-r/2}\tilde{\psi}(e^{-r}p),$$

where $\tilde{\psi}$ is the Fourier Transform of ψ.

Squeezed States Review

Squeezing operator : position / momentum basis

This last step is straightforward using the resolution of identity $\mathbb{1} = \int \mathrm{d}x \, |x\rangle \langle x|$,

$$\langle \hat{x} \rangle_{\mathrm{sq}} = \langle \psi_{\mathrm{sq}} | \hat{x} | \psi_{\mathrm{sq}} \rangle = \int \mathrm{d}x \, x \left| \psi_{\mathrm{sq}}(x) \right|^2 = \langle \psi | \hat{S}^\dagger(\xi) \hat{x} \hat{S}(\xi) | \psi \rangle = e^{-r} \langle \hat{x} \rangle$$

$$= e^{-r} \langle \psi | \hat{x} | \psi \rangle = \int \mathrm{d}y \, y e^{-r} \left| \psi(y) \right|^2 = \int \mathrm{d}x \, x e^{r} \left| \psi(e^r x) \right|^2 = \int \mathrm{d}x \, x \left| e^{r/2} \psi(e^r x) \right|^2,$$

where we used $y = e^r x$.

Squeezing operator : Fock states

Since even Fock states are...even $\psi_{2n}(x) = \psi_{2n}(-x)$ and odd Fock states are odd $\psi_{2n}(x) = -\psi_{2n}(-x)$, and the squeezing operator is actually a **scaling operator for the coordinates** (does not modify the parity), odd component of a squeezed vacuum state vanish

Photon Number Distribution of a Squeezed State

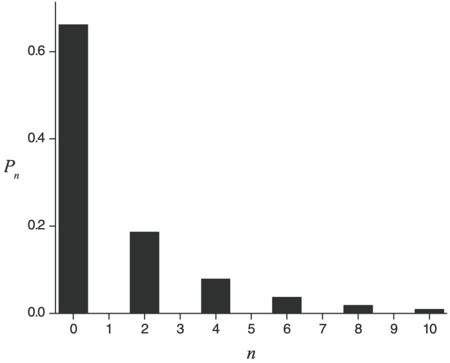
The photon number distribution

$$p_{\mathsf{n}} = \langle n | \hat{S}(\xi) | 0 \rangle \tag{1}$$

In position representation

$$p_{\mathsf{n}} = \int_{-\infty}^{\infty} \langle n | x \rangle \cdot \langle x | \hat{S}(\xi) | 0 \rangle \, dx = \tag{2}$$

$$= \int_{-\infty}^{\infty} \psi_{\mathsf{n}}(x) \cdot e^{r/2} \psi_0(e^r x) dx \tag{3}$$



Since $\psi_n(x)$ is even (i.e. symmetric $\psi(x) = \psi(-x)$) for even n (including $n = 0$), p_{n} vanishes for odd n.

Quadrature Representation

Optical quadratures

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \qquad [\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

or more generally

$$\hat{X}_\varphi = \hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi} \qquad [\hat{X}_\varphi, \hat{X}_{\varphi+\frac{\pi}{2}}] = 2i$$

The fluctuations of the quadrature[a] $\hat{X}_\varphi$ are $\left\langle \Delta\hat{X}_\varphi^2 \right\rangle \equiv \left\langle \hat{X}_\varphi^2 \right\rangle - \left\langle \hat{X}_\varphi \right\rangle^2$, and without proof:

$$\langle \beta, \xi | \, \Delta\hat{X}_\varphi^2 \, | \beta, \xi \rangle = |\mu e^{i\varphi} - \nu e^{-i\varphi}|^2 = |\mu - |\nu| e^{2i\varphi + \theta_\xi}|^2$$

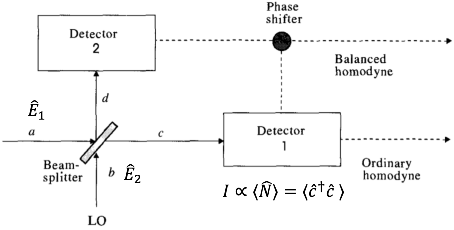
[a] note that different definitions exist e.g. $\frac{1}{2}$; $\frac{1}{\sqrt{2}}$; 1.

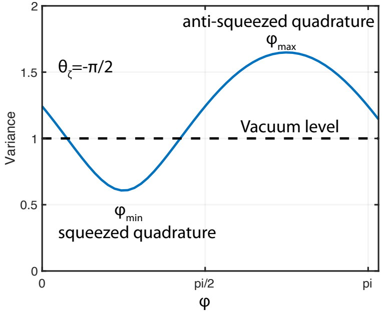
Quadrature Representation

$$\langle \beta, \xi | \, \Delta \hat{X}^2_{\varphi} \, | \beta, \xi \rangle = e^{2r}$$ enhanced fluctuations

$$\langle \beta, \xi | \, \Delta \hat{X}^2_{\varphi + \frac{\pi}{2}} \, | \beta, \xi \rangle = e^{-2r}$$ supressed fluctuations

Can be measured using homodyne detection





Homodyne Detection[4]

$$\hat{E}_1(r,t) = i\left(\frac{\hbar\omega}{2V\epsilon_0}\right)^{1/2}\left(\hat{a}e^{-i(kr+\omega t)} - \hat{a}^\dagger e^{i(kr+\omega t)}\right)$$

The beamsplitter output operator obeys $\hat{c} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{b}$. The photodetection current in the output mode $I_c \propto \langle\hat{N}\rangle = \langle\hat{c}^\dagger\hat{c}\rangle$ can then be calculated as:

$$I_c \propto \left\langle\hat{c}^\dagger\hat{c}\right\rangle \propto \eta\left\langle\hat{a}^\dagger\hat{a}\right\rangle + (1-\eta)\left\langle\hat{b}^\dagger\hat{b}\right\rangle + \sqrt{\eta(1-\eta)}(\langle\hat{a}\rangle\langle\hat{b}^\dagger\rangle - \langle\hat{a}^\dagger\rangle\langle b\rangle).$$

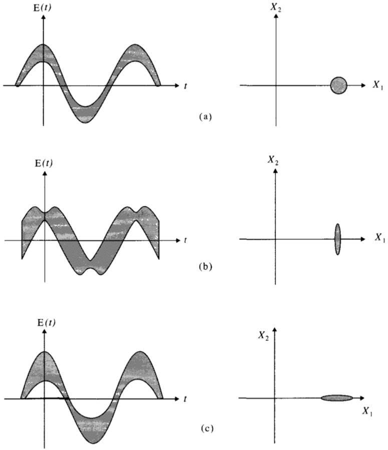
Notice that $\langle\hat{a}\hat{b}^\dagger\rangle = \langle\hat{a}\rangle\langle\hat{b}^\dagger\rangle$, assuming the states associated with $\hat{a}$ and $\hat{b}$ are uncorrelated. Let input mode $\hat{b}$ be in a relatively large coherent state $\left|\beta\right\rangle = \left||\beta|e^{i\varphi}\right\rangle$ compared to $\hat{E}_1$, one can measure arbitrary quadrature $\hat{X}_\varphi$:

$$\Rightarrow \left\langle\hat{c}^\dagger\hat{c}\right\rangle = (1-\eta)|\beta|^2 + |\beta|\sqrt{\eta(1-\eta)}\underbrace{\left\langle\hat{a}e^{-i\varphi} + \hat{a}^\dagger e^{i\varphi}\right\rangle}_{\propto\langle\hat{X}_\varphi\rangle}$$

[4]Scully, M.O., Zubairy, M.S. "Quantum optics" (1999). Chapter 4

Noise Properties of Squeezed State[5,6]

Fig. 2.7
Error contours and the corresponding graphs of electric field versus time for (a) a coherent state, (b) a squeezed state with reduced nosie in X_1, and (c) a squeezed state with reduced noise in X_2. (From C. Caves, *Phys. Rev. D* **23**, 1693 (1981).)



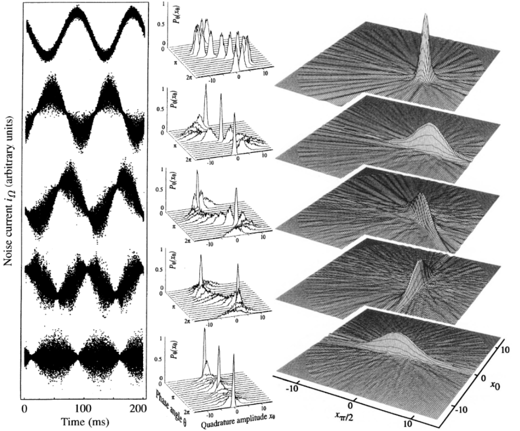


Figure 2 Noise traces in $i_\Omega(t)$ (left), quadrature distributions $P_\theta(x_\theta)$ (centre), and reconstructed Wigner functions (right) of generated quantum states. From the top: Coherent state, phase-squeezed state, state squeezed in the $\phi = 48°$-quadrature, amplitude-squeezed state, squeezed vacuum state. The noise traces as a function of time show the electric fields' oscillation in a 4π interval for the upper four states, whereas for the squeezed vacuum (belonging to a different set of measurements) a 3π interval is shown. The quadrature distributions (centre) can be interpreted as the time evolution of wave packets (position probability densities) during one oscillation period. For the reconstruction of the quantum states a π interval suffices.

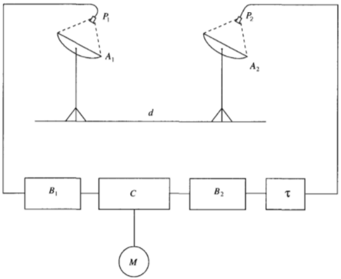
[5] Quantum Optics - Marlan O. Scully, M. Suhail Zubairy - Chapter 2

[6] Breitenbach, G., Schiller, S. & Mlynek, J. Measurement of the quantum states of squeezed light. Nature 387, 471–475 (1997).

Intensity autocorrelation[7, 8]

In photodectection one can also measure the intensity autocorrelation function
$$\left\langle : \hat{I}(t+\tau)\hat{I}(t) : \right\rangle = \left\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t) \right\rangle .$$
It is utilized in the Hanbury-Brown-Twiss interferometer to retrieve phase information by beating photocurrents.

Fig. 4.6 Schematic diagram of the Hanbury Brown-Twiss stellar intensity interferometer. Here P_1 and P_2 are the photodetectors, A_1 and A_2 are the mirrors, B_1 and B_2 are the amplifiers, τ is the delay time, C is a multiplier, and M is the integrator.



[7] Quantum Optics - Marlan O. Scully, M. Suhail Zubairy - Chapter 4

[8] Hanbury Brown, R.; Twiss, Dr R.Q. (1956). "A Test Of A New Type Of Stellar Interferometer On Sirius". Nature. 178: 1046–1048.

Review of Density Matrix[9]

Density matrix and probabilities

For a pure state $\left\langle \hat{M} \right\rangle = \langle \psi | \hat{M} | \psi \rangle$, $\quad \hat{\rho} \equiv \sum_i p(i) |\psi_i\rangle \langle \psi_i|$

For a mixed state $\left\langle \hat{M} \right\rangle = \sum_i p(i) \langle \psi | \hat{M} | \psi \rangle = \mathrm{Tr}(\hat{\rho}\hat{M})$.

$$
\begin{aligned}
\mathrm{Tr}(\hat{\rho}\hat{M}) &= \sum_n \sum_i p(i) \langle n|\psi_i\rangle \left\langle \psi_i|\hat{M}|n \right\rangle \\
&= \sum_n \sum_i p(i) \langle \psi_i | \hat{M} \underbrace{|n\rangle \langle n| \psi_i\rangle}_{\psi_i} = \sum p(i) \left\langle \psi_i|\hat{M}|\psi_i \right\rangle = \left\langle \hat{M} \right\rangle .
\end{aligned}
$$

[9] Stochastic Methods Gardiner Chapter 10

Review of Density Matrix

Density matrix and probabilities

Properties: $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}(\hat{C}\hat{A}\hat{B})$ cyclic
(i) $\mathrm{Tr}\hat{\rho} = 1$ for $\sum_i p(i) \langle \psi_i | \psi_i \rangle = \sum p(i) = 1$
(ii) Pure state $\hat{\rho}^2 = \hat{\rho}$
Time evolution:

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] \quad \text{von Neumann Equation}$$
$$\hat{\rho}(t) = e^{-i\hat{H}t/\hbar} \hat{\rho}(0) e^{i\hat{H}t/\hbar}$$

We can express the density matrix in Fock states basis

$$\hat{\rho} = \sum_{n,m} \rho_{n,m} |n\rangle \langle m| = \sum \langle n| \hat{\rho} |m\rangle |n\rangle \langle m| \,.$$

Phase Space[10]

Alternatively, one can express it in coherent state basis by inserting $\mathbb{1} = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha$, thus $\rho = \frac{1}{\pi^2}\iint d^2\alpha d^2\beta \rho(\alpha,\beta)|\alpha\rangle\langle\beta|$, where $\langle\alpha|\rho|\beta\rangle = \rho(\alpha,\beta)$ and for a normally ordered function $\hat{f}(\hat{a},\hat{a}^\dagger)$ we have

$$\left\langle \hat{f}(\hat{a},\hat{a}^\dagger)\right\rangle = \frac{1}{\pi^3}\iiint d^2\alpha d^2\beta d^2\gamma \langle\gamma \overbrace{|\alpha\rangle\langle\alpha|}^{\mathbb{1}} \rho|\beta\rangle\langle\beta|\hat{f}(\hat{a},\hat{a}^\dagger)|\gamma\rangle$$

$$\overset{\gamma\to\alpha}{=} \frac{1}{\pi^2}\iint d^2\alpha d^2\beta \rho(\alpha,\beta) f(\alpha,\beta^*)$$

R-representation

$R(\alpha^*,\beta) = \langle\alpha|\rho|\beta\rangle e^{\frac{1}{2}(|\alpha|^2+|\beta|^2)} = \sum_{n,m}\frac{\langle n|\rho|m\rangle}{\sqrt{n!m!}}\alpha^{*n}\beta^m$, thus:

$$\rho = \iint \frac{1}{\pi^2} d^2\alpha d^2\beta |\alpha\rangle\langle\beta| R(\alpha^*,\beta) e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}$$

$$\langle n|\rho|m\rangle = \frac{1}{\pi^2}\int R(\alpha^*,\beta)\sqrt{n!m!}^{-1}\alpha^n\beta^{*n}e^{-|\alpha|^2-|\beta|^2}d^2\alpha d^2\beta$$

[10] Glauber, Roy J. "Coherent and incoherent states of the radiation field". *Physical Review* 131.6 (1963): 2766.APA

Paper for next week

VOLUME 59, NUMBER 18 PHYSICAL REVIEW LETTERS 2 NOVEMBER 1987

Measurement of Subpicosecond Time Intervals between Two Photons by Interference

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(Received 10 July 1987)

A fourth-order interference technique has been used to measure the time intervals between two photons, and by implication the length of the photon wave packet, produced in the process of parametric down-conversion. The width of the time-interval distribution, which is largely determined by an interference filter, is found to be about 100 fs, with an accuracy that could, in principle, be less than 1 fs.

PACS numbers: 42.50.Bs, 42.65.Re

Questions to be addressed by the presenter

1. How are the correlated-two-photon state defined? Can they be separated into a product of two single photon states?
2. Which element enables precise tuning of delay between pulses? In this setting, does the detector's timing resolution impose any limitation?
3. How to define coincidence in detection? How to describe the propagation of optical field in Heisenberg picture? How displacement of the BS appears as a delay in the field expression at the detectors?
4. What's the spatial coherence extent of the optical pulses and how they are determined from IF filters? What is the width of the dip feature in coincidence measurement, is it consistent with the spatial coherence length?
5. What are the physical mechanisms that result in lower interference visibility? (Photon Flux rate? Detector resolution? Detector noise (dark count)?)