Quantum Electrodynamics and Quantum Optics: Lecture 12

Fall 2024

Quantum measurements and measurement back-action

Free-particle Hamiltonian: $\hat{H}_0 = \frac{\hat{p}^2}{2m}$.
First measurement at time $t = 0$, yields $\hat{x}(0)$ and $\hat{p}(0)$. Variances of these quantities $\Delta\hat{x}^2 = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2$ follow Heisenberg uncertainty:

$$\Delta\hat{p}(0)\Delta\hat{x}(0) \geq \frac{\hbar}{2}$$

Time-evolution for small time τ:

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = -\frac{i}{\hbar}[\hat{x}, \hat{H}] = \hat{p}/m \implies \hat{x}(\tau) = \hat{x}(0) + \frac{\hat{p}(0)\tau}{m}$$

Measurement at time τ will have the uncertainty:

$$\Delta\hat{x}(\tau)^2 = \Delta\hat{x}(0)^2 + \frac{\Delta\hat{p}(0)^2\tau^2}{m^2} + \left(\langle\hat{x}(0)\hat{p}(0) + \hat{p}(0)\hat{x}(0)\rangle - 2\langle\hat{p}(0)\rangle\langle\hat{x}(0)\rangle\right)\frac{\tau}{m}$$

Quantum measurements and measurement back-action

Assume that position and momentum are not correlated. Thus we have for the uncertainty at time τ:

$$\Delta\hat{x}(\tau)^2 = \Delta\hat{x}(0)^2 + \frac{\hbar^2\tau^2}{4m^2\Delta\hat{x}(0)^2}.$$

First measurement at $t = 0$ introduces an uncertainty for the second measurement at $t = \tau$. This is referred to as the *measurement back-action*. The minimum possible uncertainty due to measurement back-action is called the *standard quantum limit* (SQL)

Standard Quantum Limit (SQL)

$$\Delta\hat{x}(\tau)^2 = \frac{\hbar\tau}{m}$$

Consider the case of gravitational wave detection[1], where $\tau \sim 1$ ms and $m = 4$ kg. These parameters give $\Delta\hat{x}(\tau)_{SQL} \sim 10^{-18}$ m.

[1] Ref. Ch. 14 *Quantum Optics* GJ Milburn, DF Walls

Harmonic oscillator quantum limit

Consider harmonic oscillator Hamiltonian: $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\Omega_\mathrm{m}^2\hat{x}^2}{2}$, the dynamics follows

$$\hat{x}(\tau) = \hat{x}(0)\cos(\Omega_m\tau) + \frac{\hat{p}(0)}{m\Omega_m}\sin(\Omega_m\tau)$$

$$[\hat{x}(0), \hat{x}(\tau)] = \frac{i\hbar}{m\Omega_\mathrm{m}}\sin(\Omega_\mathrm{m}\tau), \quad \Delta\hat{x}(0)^2\Delta\hat{p}(0)^2 \geq \frac{\hbar^2}{4}.$$

When measuring at a later time, at half-period (when $\Omega_\mathrm{m}\tau = \pi/2$):

Limit for periodic measurement

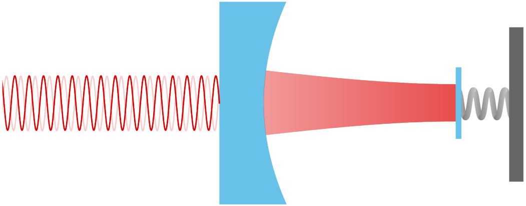
$$\sqrt{\Delta\hat{x}(\tau)^2} \geq \sqrt{\frac{\hbar}{2m\Omega_\mathrm{m}}} \equiv \text{zero-point motion}$$

Velocity and position cannot be determined simultaneously due to the uncertainty principle.

Optomechanics

Dispersive coupling of the mass to the cavity field:

$$\begin{aligned}\omega_{c,m} &= m\frac{c\pi}{L}\\ \omega_{c,m}(x) &= m\frac{c\pi}{L+x}\\ &= \omega_{c,m}\left(1-\frac{x}{L}\right).\end{aligned}$$



We consider the fundamental mode that $m=1$. Hamiltonian of the mass-cavity system is thus modified:

$$\hat{H} = \hbar\omega_c\left(1-\frac{\hat{x}}{L}\right)\hat{a}^\dagger\hat{a} + \hbar\Omega_\mathrm{m}\hat{b}^\dagger\hat{b}.$$

The interaction part of the Hamiltonian arises from the changing cavity frequency:

$$\hat{H}_\mathrm{int} = -\hbar\frac{\omega_c}{L}x_\mathrm{zpf}\hat{a}^\dagger\hat{a}\left(\hat{b}+\hat{b}^\dagger\right).$$

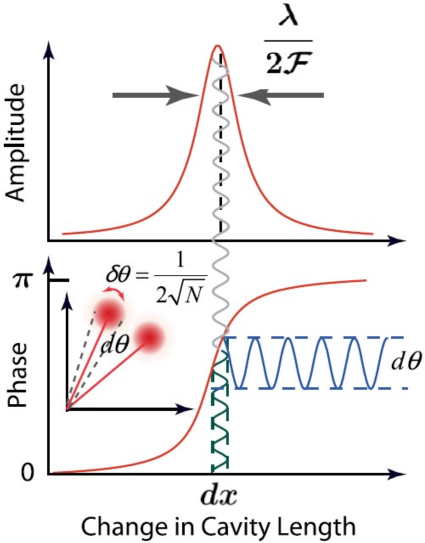
Optomechanics

Reflection from the cavity at resonance ($\omega = \omega_c$):

$$r[\omega] = \frac{\omega - \omega_c(x) - i\kappa/2}{\omega - \omega_c(x) + i\kappa/2} = \frac{\omega_c x/L - i\kappa/2}{\omega_c x/L + i\kappa/2}.$$

Phase shift around resonance of the reflected field

$$\theta \sim 2\frac{x}{L}\frac{\omega_c}{\hbar} \propto x$$



Quantum Langevin equations

Recall: Coupling between system and bath

$$\hat{H}_B = \int d\omega\, \hbar\omega \hat{b}^\dagger[\omega]\hat{b}[\omega] \quad \text{and} \quad \hat{H}_{SB} = \int d\omega\, \hbar g[\omega] \left(\hat{a}\hat{b}^\dagger[\omega] + \hat{a}^\dagger\hat{b}[\omega]\right).$$

Time-evolution of the operators in the Heisenberg picture includes a dissipation term and a fluctuation term:

$$\frac{d\hat{a}}{dt} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{\text{sys}}] - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\text{in}}.$$

For the optomechanical Hamiltonian, $\hat{H} = \hbar\omega_c \left(1 - \frac{\hat{x}}{L}\right)\hat{a}^\dagger\hat{a} + \hbar\Omega_{\text{m}}\hat{b}^\dagger\hat{b}$, we get the following equations of motion with drive:

$$\frac{d\hat{a}}{dt} = -i\omega_c\hat{a} + i\frac{\omega_c}{L}x_{\text{zpf}}\hat{a}\left(\hat{b} + \hat{b}^\dagger\right) - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}(\hat{a}_{\text{in}} + \bar{a}_{in}e^{i\omega_L t}),$$

$$\frac{d\hat{b}}{dt} = -i\Omega_{\text{m}}\hat{b} + i\frac{\omega_c}{L}x_{\text{zpf}}\hat{a}^\dagger\hat{a} - \frac{\Gamma_{\text{m}}}{2}\hat{b} + \sqrt{\Gamma_{\text{m}}}\hat{b}_{\text{in}}.$$

Quantum Langevin equations

We transfer to a rotating frame, $\hat{a} \to \hat{a}e^{i\omega_L t}$. We consider the case when the cavity is resonantly driven, i.e. $\omega_L = \omega_c$. Next we assume that the fields are strong, so they can be represented as a sum of a mean value and small fluctuations:

$$\hat{a} \to \bar{\alpha} + \delta\hat{a} \quad \text{and} \quad \hat{b} \to \bar{\beta} + \delta\hat{b}.$$

The interaction Hamiltonian $\hat{H}_{\text{int}} = \hbar\frac{\omega_c}{L}x_{\text{zpf}}\hat{a}^\dagger\hat{a}\left(\hat{b}^\dagger + \hat{b}\right)$ is thus *linearized*:

$$\hat{a}^\dagger\hat{a} = (\bar{\alpha}^* + \delta\hat{a}^\dagger)(\bar{\alpha} + \delta\hat{a}) \to \bar{\alpha}\left(\delta\hat{a} + \delta\hat{a}^\dagger\right)$$

Redefining $\delta\hat{a}$ as $\hat{a}$, we get *linearized quantum Langevin equations*:

$$\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = i\frac{\omega_c}{L}x_{\text{zpf}}\bar{\alpha}\left(\hat{b} + \hat{b}^\dagger\right) - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}_{\text{in}},$$

$$\frac{\mathrm{d}\hat{b}}{\mathrm{d}t} = -i\Omega_{\text{m}}\hat{b} + i\frac{\omega_c}{L}\bar{\alpha}(\hat{a} + \hat{a}^\dagger) - \frac{\Gamma_{\text{m}}}{2}\hat{b} + \sqrt{\Gamma_{\text{m}}}\hat{b}_{\text{in}}.$$

Quadratures

Next we consider the fluctuations in amplitude and phase quadratures:

$$\hat{X} \equiv \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right) \quad \text{and} \quad \hat{Y} \equiv \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right).$$

The Langevin equations for the optical field can be expressed as

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} = -\frac{\kappa}{2}\hat{X} + \sqrt{\kappa}\hat{X}_{\mathrm{in}}$$
$$\frac{\mathrm{d}\hat{Y}}{\mathrm{d}t} = i\sqrt{2}x_{\mathrm{zpf}}\frac{\omega_c}{L}\bar{\alpha}(\hat{b} + \hat{b}^\dagger) - \frac{\kappa}{2}\hat{Y} + \sqrt{\kappa}\hat{Y}_{\mathrm{in}}$$

Position of the mechanical oscillator is only imprinted on the phase of the optical field
$\implies$ We can infer position using *homodyne detection*.

Input-output relations

The input-output relation for fields also applies to the quadratures:

$$\hat{a}_{\text{out}} = -\hat{a}_{\text{in}} + \sqrt{\kappa}\hat{a} \quad \Longrightarrow \quad \hat{Y}_{\text{out}} = -\hat{Y}_{\text{in}} + \sqrt{\kappa}\hat{Y}.$$

Taking the Fourier transform $\hat{Y}[\omega] = \int_{-\infty}^{\infty} e^{i\omega t}\hat{Y}(t)dt$, and redefining $\hat{q} = \hat{x} = x_{\text{zpf}}\left(\hat{b} + \hat{b}^{\dagger}\right)$ for readability yields

$$-i\omega\hat{Y}[\omega] = i\sqrt{2}\frac{\omega_c}{L}\bar{\alpha}\hat{q}[\omega] - \frac{\kappa}{2}\hat{Y}[\omega] + \sqrt{\kappa}\hat{Y}_{\text{in}}[\omega].$$

After substitution, we assume so-called bad-cavity limit $\kappa \gg \Omega_{\text{m}}$ and derive the output phase quadrature:

$$\hat{Y}_{\text{out}}[\omega] = \hat{Y}_{\text{in}}[\omega] + i\frac{\bar{\alpha}\omega_c}{L}\sqrt{\frac{8}{\kappa}}\hat{q}[\omega].$$

Spectral densities

We can find spectral densities by Wiener-Khinchin theorem:

$$S_{\hat{Y}\hat{Y}}[\omega] = \lim_{\tau\to\infty} \frac{1}{\tau} \langle \hat{Y}_\tau^\dagger[\omega] \hat{Y}_\tau[\omega] \rangle$$
$$= \int \mathrm{d}\tau \, e^{-i\omega\tau} \langle \hat{Y}^\dagger(\tau) \hat{Y}(0) \rangle = \int_\infty^\infty \mathrm{d}\omega' \langle \hat{Y}^\dagger[-\omega] \hat{Y}[\omega'] \rangle$$

The input noise and mechanical motion are not correlated: $\langle \hat{Y}_{in}(\omega)\hat{q}(\omega)\rangle = 0$, and using the relations:

$$\langle \hat{Y}_{in}(t)\hat{Y}_{in}^\dagger(t')\rangle = (\bar{n}+1)\delta(t-t'), \quad \langle \hat{Y}_{in}^\dagger(t)\hat{Y}_{in}(t')\rangle = \bar{n}\delta(t-t')$$

Spectral density of the output noise is given by

$$S_{\hat{Y}_{\mathrm{out}}\hat{Y}_{\mathrm{out}}}[\omega] = \underbrace{1}_{\text{Shot noise } S_{\hat{Y}_{\mathrm{in}}\hat{Y}_{\mathrm{in}}}} + \underbrace{\frac{8\omega_c^2\bar{\alpha}^2}{\kappa L^2} S_{\hat{q}\hat{q}}[\omega]}_{\text{signal}}$$

SQL and Heisenberg uncertainty

The weakest signal $S^{\text{imp}}_{\hat{q}\hat{q}}$ that can be measured is when the signal-to-noise ratio is equal to 1:

$$S^{\text{imp}}_{\hat{q}\hat{q}} = \left(\frac{\kappa L^2}{8\omega_c^2\bar{\alpha}^2}\right) S_{\hat{Y}_{\text{in}}\hat{Y}_{\text{in}}}$$

Force acting on the mechanical oscillator is $\hat{F} = -\partial\hat{H}/\partial\hat{q}$. Assuming $\dot{\hat{X}} = 0$,

$$\hat{F} = \sqrt{2}\hbar\frac{\omega_c}{L}\hat{X} \implies \hat{F} = \sqrt{\frac{8}{\kappa}}\hbar\frac{\omega_c}{L}\hat{X}_{\text{in}}$$

$$\implies S_{\hat{F}\hat{F}}[\omega] = \frac{8}{\kappa}\left(\hbar\frac{\omega_c}{L}\bar{\alpha}\right)^2 S_{\hat{X}_{\text{in}}\hat{X}_{\text{in}}}[\omega]$$

From these two expressions, it can be seen that

$$S_{\hat{F}\hat{F}}[\omega]S^{\text{imp}}_{\hat{q}\hat{q}} = \hbar^2 S_{\hat{Y}_{\text{in}}Y_{\text{in}}} S_{\hat{X}_{\text{in}}\hat{X}_{\text{in}}} = \frac{\hbar^2}{4}.$$

Spectrum of position fluctuations

We write a second order differential equation for position:

$$\ddot{\hat{q}} = -\Omega_{\mathrm{m}}^2 \hat{q} - 2i\frac{\omega_c}{L}\bar{\alpha} x_{\mathrm{zpf}} \hat{X} - \Gamma_{\mathrm{m}} \dot{\hat{q}} + \sqrt{\Gamma_{\mathrm{m}}}\hat{q}_{\mathrm{in}}.$$

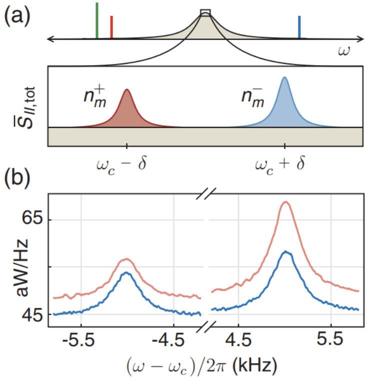
Taking the Fourier transform as $\hat{q}[\omega] = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} \hat{q}(t)$, we get:

$$\hat{q}[\omega] = \chi[\omega] \left[-2i\frac{\omega_c}{L}\bar{\alpha} x_{\mathrm{zpf}} \hat{X}[\omega] + \sqrt{\Gamma_{\mathrm{m}}}\hat{q}_{\mathrm{in}} \right],$$

where $\chi[\omega] = \left(\Omega_{\mathrm{m}}^2 - \omega^2 - i\omega\Gamma_{\mathrm{m}}\right)^{-1}$.

Noise spectral density

$$S_{\hat{q}\hat{q}}[\omega] = 2\Gamma_{\mathrm{m}}|\chi[\omega]|^2 \left[S_{\hat{q}_{\mathrm{in}}\hat{q}_{\mathrm{in}}} + \underbrace{4\frac{\left(x_{\mathrm{zpf}}\bar{\alpha}\frac{\omega_c}{L}\right)^2}{\Gamma_{\mathrm{m}}}}_{C_{\mathrm{eff}}} S_{\hat{X}\hat{X}} \right]$$



$$S_{\hat{q}\hat{q}}[\omega] = 2\Gamma_{\mathrm{m}}|\chi[\omega]|^2(n_{\mathrm{th}} + C_{\mathrm{eff}} + 1)$$
$$S_{\hat{q}\hat{q}}[-\omega] = 2\Gamma_{\mathrm{m}}|\chi[\omega]|^2(n_{\mathrm{th}} + C_{\mathrm{eff}})$$

Asymmetric noise spectral density $\rightarrow$ In contrast to the classical results!

Paper for this week

Squeezed light from a silicon micromechanical resonator

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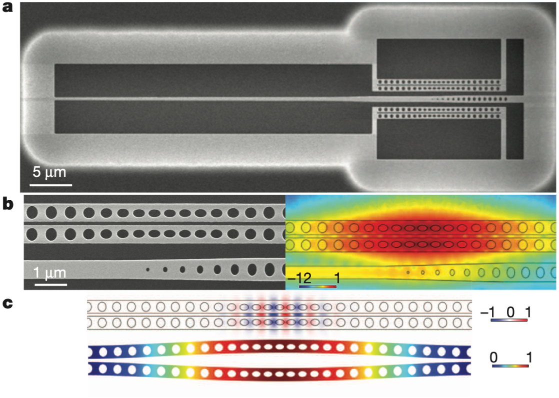

Questions

- What is optomechanical Hamiltonian? How does linearization work and how to justify it?
- What is the quantum backaction force on the mechanical oscillator?
- What is the physical mechanism that enables optical squeezing?
- What are the Fourier domain operators? What are their commutation relations and statistical properties?

Questions

- What is the physically the optical cavity and the mechanical oscillator?
- What is the technical difficulties in observing large optical squeezing?
- How is different optical quadrature spectrum measured?
- What is the typical bandwidth in observing this type of optical squeezing?