CHAPTER 1

Quantum theory of radiation

Light occupies a special position in our attempts to understand nature both classically and quantum mechanically. We recall that Newton, who made so many fundamental contributions to optics, championed a particle description of light and was not favorably disposed to the wave picture of light. However, the beautiful unification of electricity and magnetism achieved by Maxwell clearly showed that light was properly understood as the wave-like undulations of electric and magnetic fields propagating through space.

The central role of light in marking the frontiers of physics continues on into the twentieth century with the ultraviolet catastrophe associated with black-body radiation on the one hand and the photoelectric effect on the other. Indeed, it was here that the era of quantum mechanics was initiated with Planck's introduction of the quantum of action that was necessary to explain the black-body radiation spectrum. The extension of these ideas led Einstein to explain the photoelectric effect, and to introduce the photon concept.

It was, however, left to Dirac* to combine the wave- and particle-like aspects of light so that the radiation field is capable of explaining all interference phenomena and yet shows the excitation of a specific atom located along a wave front absorbing one photon of energy. In this chapter, following Dirac, we associate each mode of the radiation field with a quantized simple harmonic oscillator, this is the essence of the quantum theory of radiation. An interesting consequence of the quantization of radiation is the fluctuations associated with the zero-

* The pioneering papers on the quantum theory of radiation by Dirac [1927] and Fermi [1932] should be read by every student of the subject. Excellent modern treatments are to be found in the textbooks by: Loudon, *The Quantum Theory of Light* [1973], Cohen-Tannoudji, Dupont-Roc, and Grynberg, *Atom–Photon Interactions* [1992], Weinberg, *Theory of Quantum Fields* [1995], and Pike and Sarkar, *Quantum Theory of Radiation* [1995].

point energy or the so-called vacuum fluctuations. These fluctuations have no classical analog and are responsible for many interesting phenomena in quantum optics. As is discussed at length in Chapters 5 and 7, a semiclassical theory of atom–field interaction in which only the atom is quantized and the field is treated classically, can explain many of the phenomena which we observe in modern optics. The quantization of the radiation field is, however, needed to explain effects such as spontaneous emission, the Lamb shift, the laser linewidth, the Casimir effect, and the full photon statistics of the laser. In fact, each of these physical effects can be understood from the point of view of vacuum fluctuations perturbing the atoms, e.g., spontaneous emission is often said to be the result of 'stimulating' the atom by vacuum fluctuations. However, as compelling as these reasons are for quantizing the radiation field, there are other strong reasons and logical arguments for quantizing the radiation field.

For example, the problem of quantum beat phenomena provides us with a simple example in which the results of self-consistent fully quantized calculation differ qualitatively from those obtained via a semiclassical theory with or without vacuum fluctuations. Another experiment wherein a quantized theory of radiation is required for the proper interpretation of the observed results is two-photon interferometry and the production of entangled states associated with such a configuration. This is discussed in detail in Chapter 21. Further support that the electromagnetic field is quantized is provided by the experimental observations of nonclassical states of the radiation field, e.g., squeezed states, sub-Poissonian photon statistics, and photon antibunching.

Following this brief motivation for the quantum theory of radiation, we now turn to the quantization of the free electromagnetic field.

1.1 Quantization of the free electromagnetic field

With the objective of quantizing the electromagnetic field in free space, it is convenient to begin with the classical description of the field based on Maxwell's equations. These equations relate the electric and magnetic field vectors **E** and **H**, respectively, together with the displacement and inductive vectors **D** and **B**, respectively, and have the form (in mks units):

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{1.1.1a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.1.1b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.1.1c}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1.1.1d}$$

with the constitutive relations

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{1.1.2}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}. \tag{1.1.3}$$

Here ϵ_0 and μ_0 are the free space permittivity and permeability, respectively, and $\mu_0 \epsilon_0 = c^{-2}$ where c is the speed of light in vacuum.

It follows, on taking the curl of Eq. (1.1.1b) and using Eqs. (1.1.1a), (1.1.1d), (1.1.2), and (1.1.3), that $\mathbf{E}(\mathbf{r}, t)$ satisfies the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{1.1.4}$$

In deriving Eq. (1.1.4) we also used $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$.

1.1.1 Mode expansion of the field

We first consider the electric field to have the spatial dependence appropriate for a cavity resonator of length L (Fig. 1.1). We take the electric field to be linearly polarized in the x-direction and expand in the normal modes of the cavity

$$E_x(z, t) = \sum_j A_j q_j(t) \sin(k_j z), \tag{1.1.5}$$

where q_j is the normal mode amplitude with the dimension of a length, $k_j = j\pi/L$, with $j = 1, 2, 3, \ldots$, and

$$A_j = \left(\frac{2\nu_j^2 m_j}{V \epsilon_0} \right)^{1/2}, \tag{1.1.6}$$

with $\nu_j = j\pi c/L$ being the cavity eigenfrequency, $V = LA$ (A is the transverse area of the optical resonator) is the volume of the resonator and m_j is a constant with the dimension of mass. The constant m_j has been included only to establish the analogy between the dynamical problem of a single mode of the electromagnetic field and that of the simple harmonic oscillator. The equivalent mechanical oscillator will have a mass m_j, and a Cartesian coordinate q_j. The nonvanishing component of the magnetic field H_y in the cavity* is obtained from Eq. (1.1.5):

* In the present treatment of field quantization in vacuum we are focussing on the electric $\mathbf{E}(\mathbf{r}, t)$ and magnetic $\mathbf{H}(\mathbf{r}, t)$ fields. In a material medium it is preferable to work with $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$; see Bialynicki-Birula and Bialynicka-Birula [1976].

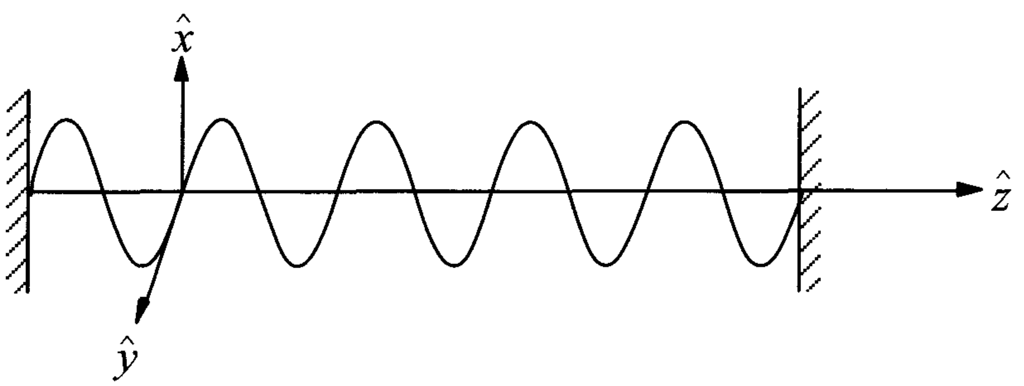


Fig. 1.1 Electromagnetic field of frequency ν inside a cavity. The field is assumed to be transverse with the electric field polarized in the x-direction.

$$H_y = \sum_j A_j \left(\frac{\dot{q}_j \epsilon_0}{k_j}\right) \cos(k_j z). \tag{1.1.7}$$

The classical Hamiltonian for the field is

$$\mathscr{H} = \frac{1}{2} \int_V d\tau (\epsilon_0 E_x^2 + \mu_0 H_y^2), \tag{1.1.8}$$

where the integration is over the volume of the cavity. It follows, on substituting from Eqs. (1.1.5) and (1.1.7) for E_x and H_y, respectively, in Eq. (1.1.8), that

$$\begin{aligned} \mathscr{H} &= \frac{1}{2} \sum_j (m_j \nu_j^2 q_j^2 + m_j \dot{q}_j^2) \\ &= \frac{1}{2} \sum_j \left(m_j \nu_j^2 q_j^2 + \frac{p_j^2}{m_j}\right), \end{aligned} \tag{1.1.9}$$

where $p_j = m_j \dot{q}_j$ is the canonical momentum of the jth mode. Equation (1.1.9) expresses the Hamiltonian of the radiation field as a sum of independent oscillator energies. Each mode of the field is therefore dynamically equivalent to a mechanical harmonic oscillator.

1.1.2 Quantization

The present dynamical problem can be quantized by identifying q_j and p_j as operators which obey the commutation relations

$$[q_j, p_{j'}] = i\hbar \delta_{jj'}, \tag{1.1.10a}$$

$$[q_j, q_{j'}] = [p_j, p_{j'}] = 0. \tag{1.1.10b}$$

It is convenient to make a canonical transformation to operators a_j and $a_j^\dagger$:

$$a_j e^{-i\nu_j t} = \frac{1}{\sqrt{2 m_j \hbar \nu_j}} (m_j \nu_j q_j + i p_j), \tag{1.1.11a}$$

$$a_j^\dagger e^{i\nu_j t} = \frac{1}{\sqrt{2m_j\hbar\nu_j}}(m_j\nu_j q_j - ip_j). \tag{1.1.11b}$$

In terms of a_j and $a_j^\dagger$, the Hamiltonian (1.1.9) becomes

$$\mathscr{H} = \hbar\sum_j \nu_j\left(a_j^\dagger a_j + \frac{1}{2}\right). \tag{1.1.12}$$

The commutation relations between a_j and $a_j^\dagger$ follow from those between q_j and p_j:

$$[a_j, a_{j'}^\dagger] = \delta_{jj'}, \tag{1.1.13}$$

$$[a_j, a_{j'}] = [a_j^\dagger, a_{j'}^\dagger] = 0. \tag{1.1.14}$$

The operators a_j and $a_j^\dagger$ are referred to as the annihilation and the creation operators, respectively. The reason for these names will become clear in the next section. In terms of a_j and $a_j^\dagger$, the electric and magnetic fields (Eqs. (1.1.5) and (1.1.7)) take the form

$$E_x(z,t) = \sum_j \mathscr{E}_j(a_j e^{-i\nu_j t} + a_j^\dagger e^{i\nu_j t})\sin k_j z, \tag{1.1.15}$$

$$H_y(z,t) = -i\epsilon_0 c\sum_j \mathscr{E}_j(a_j e^{-i\nu_j t} - a_j^\dagger e^{i\nu_j t})\cos k_j z, \tag{1.1.16}$$

where the quantity

$$\mathscr{E}_j = \left(\frac{\hbar\nu_j}{\epsilon_0 V}\right)^{1/2} \tag{1.1.17}$$

has the dimensions of an electric field.

So far we have considered the quantization of the radiation field in a finite one-dimensional cavity. We can now quantize the field in unbounded free space as follows.

We consider the field in a large but finite cubic cavity of side L. Here we regard the *cavity* merely as a region of space with no specific boundaries. We consider the running-wave solutions instead of the standing-wave solutions considered above and impose periodic boundary conditions.

The classical electric and magnetic fields can be expanded in terms of the plane waves

$$\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}}\mathscr{E}_{\mathbf{k}}\alpha_{\mathbf{k}}e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{c.c.}, \tag{1.1.18}$$

$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\mu_0}\sum_{\mathbf{k}} \frac{\mathbf{k}\times\hat{\epsilon}_{\mathbf{k}}}{\nu_k}\mathscr{E}_{\mathbf{k}}\alpha_{\mathbf{k}}e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{c.c.}, \tag{1.1.19}$$

where the summation is taken over an infinite discrete set of values of the wave vector $\mathbf{k} \equiv (k_x, k_y, k_z)$, $\hat{\epsilon}_{\mathbf{k}}$ is a unit polarization vector, $\alpha_{\mathbf{k}}$ is a dimensionless amplitude and

$$\mathscr{E}_{\mathbf{k}} = \left(\frac{\hbar\nu_k}{2\epsilon_0 V}\right)^{1/2}. \tag{1.1.20}$$

In Eqs. (1.1.18) and (1.1.19) c.c. stands for complex conjugate. The periodic boundary conditions require that

$$k_x = \frac{2\pi n_x}{L}, \quad k_y = \frac{2\pi n_y}{L}, \quad k_z = \frac{2\pi n_z}{L}, \tag{1.1.21}$$

where n_x, n_y, n_z are integers $(0, \pm 1, \pm 2, \ldots)$. A set of numbers (n_x, n_y, n_z) defines a mode of the electromagnetic field. Equation (1.1.1d) requires that

$$\mathbf{k} \cdot \hat{\epsilon}_{\mathbf{k}} = 0, \tag{1.1.22}$$

i.e., the fields are purely transverse. There are, therefore, two independent polarization directions of $\hat{\epsilon}_{\mathbf{k}}$ for each $\mathbf{k}$.

The change from a discrete distribution of modes to a continuous distribution can be made by replacing the sum in Eqs. (1.1.18) and (1.1.19) by an integral:

$$\sum_{\mathbf{k}} \rightarrow 2\left(\frac{L}{2\pi}\right)^3 \int d^3k, \tag{1.1.23}$$

where the factor 2 accounts for two possible states of polarization.

In many problems, we shall be interested in the density of modes between the frequencies ν and $\nu + d\nu$. This can be obtained by transforming from the rectangular components $(k_x,\ k_y,\ k_z)$ to the polar coordinates $(k \sin\theta \cos\phi,\ k \sin\theta \sin\phi,\ k \cos\theta)$, so that the volume element in $\mathbf{k}$ space is

$$d^3k = k^2 dk \sin\theta d\theta d\phi = \frac{\nu^2}{c^3} d\nu \sin\theta d\theta d\phi. \tag{1.1.24}$$

The total number of modes in volume L^3 in the range between ν and $\nu + d\nu$ is given by

$$d\mathscr{N} = 2\left(\frac{L}{2\pi}\right)^3 \frac{\nu^2 d\nu}{c^3} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi = \frac{L^3 \nu^2}{\pi^2 c^3} d\nu. \tag{1.1.25}$$

Therefore the number of modes with frequencies in the range ν to $\nu + d\nu$ is

$$D(\nu) d\nu = \frac{L^3 \nu^2}{\pi^2 c^3} d\nu, \tag{1.1.26}$$

where $D(\nu)$ is called the mode density.

As before, the radiation field is quantized by identifying $\alpha_{\mathbf{k}}$ and $\alpha_{\mathbf{k}}^*$ with the harmonic oscillator operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$, respectively, which satisfy the commutation relation $[a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger] = 1$. The quantized electric and magnetic fields take the form

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}} a_{\mathbf{k}} e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}, \tag{1.1.27}$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{\mu_0} \sum_{\mathbf{k}} \frac{\mathbf{k} \times \hat{\epsilon}_{\mathbf{k}}}{\nu_k} \mathscr{E}_{\mathbf{k}} a_{\mathbf{k}} e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}, \tag{1.1.28}$$

where H.c. stands for Hermitian conjugate. Usually the positive and negative frequency parts of these field operators are written separately. For example, the electric field operator $\mathbf{E}(\mathbf{r}, t)$ is written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}^{(+)}(\mathbf{r}, t) + \mathbf{E}^{(-)}(\mathbf{r}, t), \tag{1.1.29}$$

where

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}} a_{\mathbf{k}} e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}}, \tag{1.1.30}$$

$$\mathbf{E}^{(-)}(\mathbf{r}, t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}} a_{\mathbf{k}}^\dagger e^{i\nu_k t - i\mathbf{k}\cdot\mathbf{r}}. \tag{1.1.31}$$

Here $\mathbf{E}^{(+)}(\mathbf{r}, t)$ contains only the annihilation operators and its adjoint $\mathbf{E}^{(-)}(\mathbf{r}, t)$ contains only the creation operators.

1.1.3 Commutation relations between electric and magnetic field components

An important consequence of imposing the quantum conditions (1.1.13) and (1.1.14) is that as the electric and magnetic field strengths do not commute they are thus not measurable simultaneously. In order to show this we rewrite the quantized mode expansions (1.1.27) and (1.1.28) for $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$, respectively by including explicitly the two states of polarization denoted by the symbol λ:

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{k},\lambda} \hat{\epsilon}_{\mathbf{k}}^{(\lambda)} \mathscr{E}_{\mathbf{k}} a_{\mathbf{k},\lambda} e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}, \tag{1.1.32}$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{\mu_0} \sum_{\mathbf{k},\lambda} \frac{\mathbf{k} \times \hat{\epsilon}_{\mathbf{k}}^{(\lambda)}}{\nu_k} \mathscr{E}_{\mathbf{k}} a_{\mathbf{k},\lambda} e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.} \tag{1.1.33}$$

The corresponding commutation relations between the operators $a_{\mathbf{k},\lambda}$ and $a_{\mathbf{k},\lambda}^\dagger$ are

$$\begin{aligned} &[a_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}] = [a_{\mathbf{k},\lambda}^\dagger, a_{\mathbf{k}',\lambda'}^\dagger] = 0, \\ &[a_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}. \end{aligned} \tag{1.1.34}$$

It then follows that the equal time commutator between the field components is given by

$$[E_x(\mathbf{r},t), H_y(\mathbf{r}',t)] = \frac{\hbar c^2}{2V}\sum_{\mathbf{k},\lambda} \epsilon_{\mathbf{k}x}^{(\lambda)} \left[\epsilon_{\mathbf{k}x}^{(\lambda)} k_z - \epsilon_{\mathbf{k}z}^{(\lambda)} k_x\right] \times \left[e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right], \tag{1.1.35}$$

where $\epsilon_{\mathbf{k}i}^{(\lambda)}$ $(i = x, y, z)$ is the ith component of $\hat{\epsilon}_{\mathbf{k}}^{(\lambda)}$. We proceed by using the operator identity of Problem 1.9 to write

$$\hat{\epsilon}_{\mathbf{k}}^{(1)}\hat{\epsilon}_{\mathbf{k}}^{(1)} + \hat{\epsilon}_{\mathbf{k}}^{(2)}\hat{\epsilon}_{\mathbf{k}}^{(2)} + \frac{\mathbf{kk}}{k^2} = \mathbf{1}, \tag{1.1.36}$$

where $\hat{\epsilon}_{\mathbf{k}}^{(1)}\hat{\epsilon}_{\mathbf{k}}^{(1)}$, $\hat{\epsilon}_{\mathbf{k}}^{(2)}\hat{\epsilon}_{\mathbf{k}}^{(2)}$, and $\mathbf{kk}$ denote dyadic products. One can verify that taking the inner product of (1.1.36) with the Cartesian unit vector $\hat{\mathbf{e}}_i$ from the left and $\hat{\mathbf{e}}_j$ from the right yields

$$\epsilon_{\mathbf{k}i}^{(1)}\epsilon_{\mathbf{k}j}^{(1)} + \epsilon_{\mathbf{k}i}^{(2)}\epsilon_{\mathbf{k}j}^{(2)} = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{1.1.37}$$

The summation over the polarization states in Eq. (1.1.35) can now be carried out using (1.1.37). The resulting expression for the commutator is

$$[E_x(\mathbf{r},t), H_y(\mathbf{r}',t)] = \frac{\hbar c^2}{2V}\sum_{\mathbf{k}} k_z \left[e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right]. \tag{1.1.38}$$

We now replace the summation by an integral via

$$\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3}\int d^3k. \tag{1.1.39}$$

The factor of 2 has not been included as was done in Eq. (1.1.23) because, in the present case, we have summed over two polarization states explicitly. We obtain

$$[E_x(\mathbf{r},t), H_y(\mathbf{r}',t)] = -i\hbar c^2 \frac{\partial}{\partial z}\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{1.1.40}$$

In general

$$[E_j(\mathbf{r},t), H_j(\mathbf{r}',t)] = 0 \quad (j = x, y, z), \tag{1.1.41}$$

$$[E_j(\mathbf{r},t), H_k(\mathbf{r}',t)] = -i\hbar c^2 \frac{\partial}{\partial \ell}\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{1.1.42}$$

where j, k, and ℓ form a cyclic permutation of x, y, and z.

We, therefore, conclude that the parallel components of $\mathbf{E}$ and $\mathbf{H}$ may be measured simultaneously whereas the perpendicular components cannot.

1.2 Fock or number states

In this section we first restrict ourselves to a single mode of the field of frequency ν having creation and annihilation operators $a^\dagger$ and a, respectively. Let $|n\rangle$ be the energy eigenstate corresponding to the energy eigenvalue E_n, i.e.,

$$\mathscr{H}|n\rangle = \hbar\nu\left(a^\dagger a + \frac{1}{2}\right)|n\rangle = E_n|n\rangle. \tag{1.2.1}$$

If we apply the operator a from the left, we obtain after using the commutation relation $[a, a^\dagger] = 1$ and some rearrangement

$$\mathscr{H}a|n\rangle = (E_n - \hbar\nu)a|n\rangle. \tag{1.2.2}$$

This means that the state

$$|n-1\rangle = \frac{a}{\alpha_n}|n\rangle, \tag{1.2.3}$$

is also an energy eigenstate but with the reduced eigenvalue

$$E_{n-1} = E_n - \hbar\nu. \tag{1.2.4}$$

In Eq. (1.2.3), α_n is a constant which will be determined from the normalization condition

$$\langle n-1|n-1\rangle = 1. \tag{1.2.5}$$

If we repeat this procedure n times we move down the energy ladder in steps of $\hbar\nu$ until we obtain

$$\mathscr{H}a|0\rangle = (E_0 - \hbar\nu)a|0\rangle. \tag{1.2.6}$$

Here E_0 is the ground state energy such that $(E_0 - \hbar\nu)$ would correspond to an energy eigenvalue smaller than E_0. Since we do not allow energies lower than E_0 for the oscillator, we must conclude

$$a|0\rangle = 0. \tag{1.2.7}$$

The state $|0\rangle$ is referred to as the vacuum state. Using this relation we can find the value of E_0 from the eigenvalue equation

$$\mathscr{H}|0\rangle = \frac{1}{2}\hbar\nu|0\rangle = E_0|0\rangle. \tag{1.2.8}$$

This gives

$$E_0 = \frac{1}{2}\hbar\nu. \tag{1.2.9}$$

It then follows from Eq. (1.2.4) that

$$E_n = \left(n + \frac{1}{2}\right) \hbar\nu. \tag{1.2.10}$$

From Eq. (1.2.1), we obtain

$$a^\dagger a|n\rangle = n|n\rangle, \tag{1.2.11}$$

i.e., the energy eigenstate $|n\rangle$ is also an eigenstate of the 'number' operator

$$n = a^\dagger a. \tag{1.2.12}$$

The normalization constant α_n in Eq. (1.2.3) can now be determined.

$$\langle n-1|n-1\rangle = \frac{1}{|\alpha_n|^2}\langle n|a^\dagger a|n\rangle = \frac{n}{|\alpha_n|^2}\langle n|n\rangle = \frac{n}{|\alpha_n|^2} = 1. \tag{1.2.13}$$

If we take the phase of the normalization constant α_n to be zero then $\alpha_n = \sqrt{n}$. Equation (1.2.3) then becomes

$$a|n\rangle = \sqrt{n}|n-1\rangle. \tag{1.2.14}$$

We can proceed along the same lines with the operator $a^\dagger$. The resulting equation is

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle. \tag{1.2.15}$$

A repeated use of this equation gives

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{1.2.16}$$

It is useful to interpret the energy eigenvalues (1.2.10) as corresponding to the presence of n quanta or *photons* of energy $\hbar\nu$. The eigenstates $|n\rangle$ are called Fock states or photon number states. They form a complete set of states, i.e.,

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1. \tag{1.2.17}$$

The energy eigenvalues are discrete, in contrast to classical electromagnetic theory where energy can have any value. The energy expectation value can however take on any value, for the state vector is, in general, an arbitrary superposition of energy eigenstates, i.e.,

$$|\psi\rangle = \sum_n c_n|n\rangle \tag{1.2.18}$$

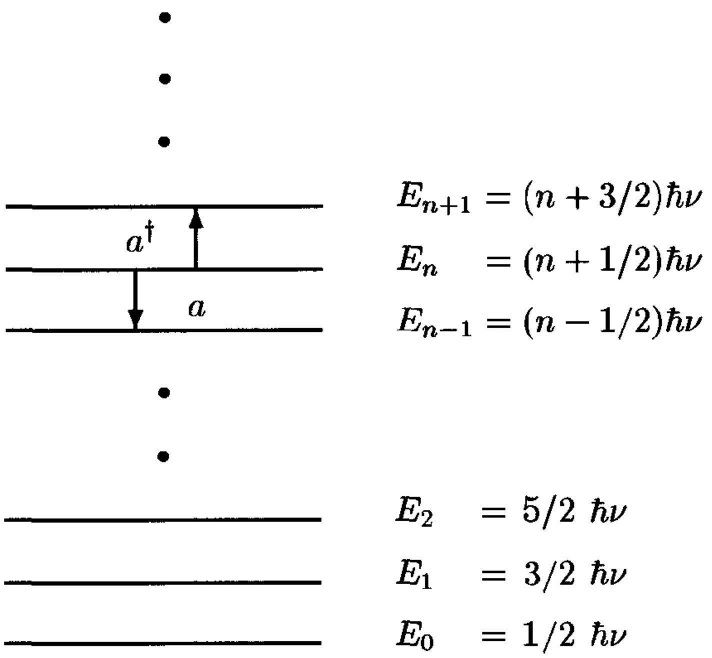


Fig. 1.2
Energy levels for the quantum mechanical harmonic oscillators associated with the electromagnetic field. The creation operator $a^\dagger$ adds a quantum of energy $\hbar\nu$, whereas the destruction operator a subtracts the same amount of energy.

where c_n are complex coefficients. The residual energy $\hbar\nu/2$ corresponding to E_0 is called the zero-point energy. In Fig. 1.2, the energy levels for the quantum mechanical oscillations associated with the electromagnetic field are given.

An important property of the number state $|n\rangle$ is that the corresponding expectation value of the single-mode linearly polarized field operator

$$E(\mathbf{r}, t) = \mathscr{E} a e^{-i\nu t + i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.} \tag{1.2.19}$$

vanishes, i.e.,

$$\langle n|E|n\rangle = 0. \tag{1.2.20}$$

However, the expectation value of the intensity operator E^2 is given by

$$\langle n|E^2|n\rangle = 2|\mathscr{E}|^2 \left(n + \frac{1}{2}\right), \tag{1.2.21}$$

i.e., there are fluctuations in the field about its zero ensemble average. It is interesting to note that there are nonzero fluctuations even for a *vacuum* state $|0\rangle$. These vacuum fluctuations are responsible for many interesting phenomena in quantum optics as discussed earlier. For example, it may be considered that they *stimulate* an excited atom to emit spontaneously. They also account for the Lamb shift of $2P_{1/2} \to 2S_{1/2}$ energy levels of atomic hydrogen. In particular in Section 1.3, we shall see how the vacuum fluctuations of the electromagnetic field are responsible for the Lamb shift.

The operators a and $a^\dagger$ annihilate and create photons, respectively, for, as seen in Eqs. (1.2.14) and (1.2.15), they change a state with n photons into one with $n-1$ or $n+1$ photons. The operators a and $a^\dagger$ are therefore referred to as annihilation (or destruction) and creation operators, respectively. These operators are not themselves Hermitian ($a \neq a^\dagger$) and do not represent observable quantities such as the electric and magnetic field amplitudes. However, some combinations of the operators are Hermitian such as $a_1 = (a + a^\dagger)/2$ and $a_2 = (a - a^\dagger)/2i$.

So far we have considered a single-mode field and have found that, in general, the wave function can be written as a linear superposition of photon number states $|n\rangle$. We now extend this formalism to deal with multi-mode fields.

We can rewrite the Hamiltonian $\mathscr{H}$ in Eq. (1.1.12) as

$$\mathscr{H} = \sum_{\mathbf{k}} \mathscr{H}_{\mathbf{k}} \tag{1.2.22}$$

where

$$\mathscr{H}_{\mathbf{k}} = \hbar\nu_k \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \right). \tag{1.2.23}$$

The energy eigenstate $|n_{\mathbf{k}}\rangle$ of $\mathscr{H}_{\mathbf{k}}$ is defined in a manner similar to the single-mode field via the energy eigenvalue equation

$$\mathscr{H}_{\mathbf{k}}|n_{\mathbf{k}}\rangle = \hbar\nu_k \left(n_{\mathbf{k}} + \frac{1}{2} \right) |n_{\mathbf{k}}\rangle. \tag{1.2.24}$$

The general eigenstate of $\mathscr{H}$ can therefore have $n_{\mathbf{k}_1}$ photons in the first mode, $n_{\mathbf{k}_2}$ in the second, $n_{\mathbf{k}_\ell}$ in the ℓth and so forth, and can be written as $|n_{\mathbf{k}_1}\rangle|n_{\mathbf{k}_2}\rangle \dots |n_{\mathbf{k}_\ell}\rangle \dots$ or more conveniently

$$|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots, n_{\mathbf{k}_\ell}, \dots\rangle \equiv |\{n_{\mathbf{k}}\}\rangle. \tag{1.2.25}$$

The annihilation and creation operators $a_{\mathbf{k}_\ell}$ and $a_{\mathbf{k}_\ell}^\dagger$ lower and raise the ℓth entry alone, i.e.,

$$a_{\mathbf{k}_\ell}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots, n_{\mathbf{k}_\ell}, \dots\rangle = \sqrt{n_{\mathbf{k}_\ell}}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots, n_{\mathbf{k}_\ell} - 1, \dots\rangle, \tag{1.2.26}$$

$$a_{\mathbf{k}_\ell}^\dagger|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots, n_{\mathbf{k}_\ell}, \dots\rangle = \sqrt{n_{\mathbf{k}_\ell} + 1}|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots, n_{\mathbf{k}_\ell} + 1, \dots\rangle. \tag{1.2.27}$$

The general state vector for the field is a linear superposition of these eigenstates:

$$\begin{aligned} |\psi\rangle &= \sum_{n_{\mathbf{k}_1}} \sum_{n_{\mathbf{k}_2}} \cdots \sum_{n_{\mathbf{k}_\ell}} \cdots c_{n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots, n_{\mathbf{k}_\ell}, \dots} |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots, n_{\mathbf{k}_\ell}, \dots\rangle \\ &\equiv \sum_{\{n_{\mathbf{k}}\}} c_{\{n_{\mathbf{k}}\}} |\{n_{\mathbf{k}}\}\rangle. \end{aligned} \tag{1.2.28}$$

This is a more general superposition than

$$|\psi\rangle = |\psi_{\mathbf{k}_1}\rangle|\psi_{\mathbf{k}_2}\rangle \ldots |\psi_{\mathbf{k}_\ell}\rangle \ldots, \tag{1.2.29}$$

where $|\psi_{\mathbf{k}_\ell}\rangle$ are state vectors for individual modes. Equation (1.2.28) includes state vectors of the type (1.2.29) as well as more general states having correlations between the field modes which can result from interaction of the various field modes with a common system.

1.3 Lamb shift

The precision observation of the Lamb shift, between the $2S_{1/2}$ and $2P_{1/2}$ levels in hydrogen, was in a real sense the stimulus for modern quantum electrodynamics (QED). According to Dirac theory, the $2S_{1/2}$ and $2P_{1/2}$ levels should have equal energies. However, radiative corrections due to the interaction between the atomic electron and the vacuum, shift the $2S_{1/2}$ level higher in energy by around 1057 MHz relative to the $2P_{1/2}$ level.

Early attempts to calculate such 'vacuum induced' radiative corrections were frustrated in that they predicted infinite level shifts. However, the beautiful measurement of Lamb and Retherford provided the stimulus for renormalization theory which has been so successful in handling these divergences.

On the occasion of Lamb's sixty-fifth birthday, Freeman Dyson* wrote:

> Your work on the hydrogen fine structure led directly to the wave of progress in quantum electrodynamics on which I took a ride to fame and fortune. You did the hard, tedious, exploratory work. Once you had started the wave rolling, the ride for us theorists was easy. And after we had zoomed ashore with our fine, fancy formalisms, you still stayed with your stubborn experiment. For many years thereafter you were at work, carefully coaxing the hydrogen atom to give us the accurate numbers which provided the solid foundations for all our speculations...
>
> Those years, when the Lamb shift was the central theme of physics, were golden years for all the physicists of my generation. You were the first to see that that tiny shift, so elusive and hard to measure, would clarify in a fundamental way our thinking about particles and fields.

* Dyson [1978].

Shortly after the experimental results were announced, Bethe produced a simple nonrelativistic calculation which was in good qualitative agreement with theory, by using the suggestion of Kramers, Schwinger, and Weisskopf for 'subtracting off' infinities. This was extended to a full relativistic theory in quantitative agreement with experiments by Kroll and Lamb, and French and Weisskopf; and was the harbinger of modern QED as developed by Schwinger, Feynman, and Dyson.

The excellent agreement between the full quantum theory of radiation and matter, and experiment, e.g., the Lamb shift, provides strong support for the quantization of the radiation field. However, a detailed calculation of the Lamb shift would take us too far from mainstream quantum optics. Therefore, we will present here a heuristic derivation of the electromagnetic level shift following Welton.

The effect of the fluctuations in the electric and magnetic fields associated with the vacuum is a perturbation of the electron in a hydrogen atom from the standard orbits of the Coulomb potential $-e^2/4\pi\epsilon_0 r$ due to the proton; so the electron radius $r \rightarrow r + \delta r$, where δr is the fluctuation in the position of the electron due to the fluctuating fields. The change in potential energy, and thus the associated level shift, is given by

$$\begin{aligned}\Delta V &= V(\mathbf{r}+\delta\mathbf{r}) - V(\mathbf{r}) \\ &= \delta\mathbf{r}\cdot\nabla V + \frac{1}{2}(\delta\mathbf{r}\cdot\nabla)^2 V(\mathbf{r}) + \ldots\end{aligned} \tag{1.3.1}$$

Since the fluctuations are isotropic, $\langle\delta\mathbf{r}\rangle_{\text{vac}} = 0$, the first term can be neglected. Moreover,

$$\langle(\delta\mathbf{r}\cdot\nabla)^2\rangle_{\text{vac}} = \frac{1}{3}\langle(\delta\mathbf{r})^2\rangle_{\text{vac}}\nabla^2, \tag{1.3.2}$$

again due to the isotropy of the fluctuations. We therefore obtain

$$\langle\Delta V\rangle = \frac{1}{6}\langle(\delta\mathbf{r})^2\rangle_{\text{vac}}\left\langle\nabla^2\left(\frac{-e^2}{4\pi\epsilon_0 r}\right)\right\rangle_{\text{at}}, \tag{1.3.3}$$

where $\langle\ldots\rangle_{\text{at}}$ represents the quantum average with respect to the atomic states.

For the $2S$ state of hydrogen

$$\begin{aligned}\left\langle\nabla^2\left(\frac{-e^2}{4\pi\epsilon_0 r}\right)\right\rangle_{\text{at}} &= \frac{-e^2}{4\pi\epsilon_0}\int d\mathbf{r}\,\psi_{2S}^*(\mathbf{r})\nabla^2\left(\frac{1}{r}\right)\psi_{2S}(\mathbf{r}) \\ &= \frac{e^2}{\epsilon_0}|\psi_{2S}(0)|^2 \\ &= \frac{e^2}{8\pi\epsilon_0 a_0^3},\end{aligned} \tag{1.3.4}$$

where $a_0 = 4\pi\epsilon_0\hbar^2/me^2$ (m is the mass of the electron) is the Bohr radius and we use

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r}), \tag{1.3.5}$$

and

$$\psi_{2S}(0) = \frac{1}{(8\pi a_0^3)^{1/2}}. \tag{1.3.6}$$

For P-states, the nonrelativistic wave function vanishes at the origin and hence so does the energy shift.

Next we consider the contribution $\langle(\delta\mathbf{r})^2\rangle_{\text{vac}}$ due to the vacuum fluctuations in Eq. (1.3.3). The classical equation of motion for the electron displacement $(\delta r)_{\mathbf{k}}$ induced by a single mode of the field of wave vector $\mathbf{k}$ and frequency ν is

$$m\frac{d^2}{dt^2}(\delta r)_{\mathbf{k}} = -eE_{\mathbf{k}}. \tag{1.3.7}$$

This is valid if the field frequency ν is greater than the frequency ν_0 in the Bohr orbit, i.e., if $\nu > \pi c/a_0$. For the field oscillating at frequency ν,

$$\delta r(t) \cong \delta r(0)e^{-i\nu t} + \text{c.c.} \tag{1.3.8}$$

We thus have

$$(\delta r)_{\mathbf{k}} \cong \frac{e}{mc^2k^2}E_{\mathbf{k}}, \tag{1.3.9}$$

where, from Eq. (1.1.27),

$$E_{\mathbf{k}} = \mathscr{E}_{\mathbf{k}}(a_{\mathbf{k}}e^{-i\nu t+i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.}). \tag{1.3.10}$$

After summing over all modes, we obtain

$$\begin{aligned}\langle(\delta\mathbf{r})^2\rangle_{\text{vac}} &= \sum_{\mathbf{k}}\left(\frac{e}{mc^2k^2}\right)^2\langle 0|(E_{\mathbf{k}})^2|0\rangle \\ &= \sum_{\mathbf{k}}\left(\frac{e}{mc^2k^2}\right)^2\left(\frac{\hbar ck}{2\epsilon_0 V}\right),\end{aligned} \tag{1.3.11}$$

where we have made the substitution $\mathscr{E}_{\mathbf{k}} = (\hbar ck/2\epsilon_0 V)^{1/2}$. For the continuous mode distribution, the summation in Eq. (1.3.11) is changed to an integral (Eq. (1.1.23)). We then obtain after carrying out the angular integrations

$$\begin{aligned}\langle(\delta\mathbf{r})^2\rangle_{\text{vac}} &= 2\frac{V}{(2\pi)^3}4\pi\int dk k^2\left(\frac{e}{mc^2k^2}\right)^2\left(\frac{\hbar ck}{2\epsilon_0 V}\right) \\ &= \frac{1}{2\epsilon_0\pi^2}\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar}{mc}\right)^2\int\frac{dk}{k}.\end{aligned} \tag{1.3.12}$$

This gives a divergent result. However as noted before, the present method is only valid for $v > \pi c/a_0$, or equivalently $k > \pi/a_0$. It is also valid only for wavelengths longer than the Compton wavelength, i.e., $k < mc/\hbar$, because of magnetic effects on the motion which begin when $v/c = p/mc = \hbar k/mc \lesssim 1$. The present method is invalid if the electron is relativistic. We can therefore choose the lower and upper limits for the integral in Eq. (1.3.12) to be π/a_0 and $mc/\hbar$, respectively. We then obtain

$$\langle(\delta \mathbf{r})^2\rangle_{\text{vac}} \cong \frac{1}{2\epsilon_0 \pi^2}\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar}{mc}\right)^2 \ln\left(\frac{4\epsilon_0 \hbar c}{e^2}\right). \tag{1.3.13}$$

On substituting Eqs. (1.3.4) and (1.3.13) into Eq. (1.3.3), we obtain the following expression for the Lamb shift

$$\langle \Delta V\rangle = \frac{4}{3}\frac{e^2}{4\pi\epsilon_0}\frac{e^2}{4\pi\epsilon_0 \hbar c}\left(\frac{\hbar}{mc}\right)^2 \frac{1}{8\pi a_0^3}\ln\left(\frac{4\epsilon_0 \hbar c}{e^2}\right). \tag{1.3.14}$$

This shift is about 1 GHz in good agreement with the observed shift, considering the crude approximations made in the calculation.

Finally, we note the exciting developments in Lamb shift physics made possible by modern quantum optical techniques, namely the measurement of the radiation shift of the $1S$ state via precise measurements of the two-photon $1S$–$2S$ transition first performed by Hänsch and co-workers.

1.4 Quantum beats

Over the past decades several alternative theories to quantum electrodynamics (QED) have been proposed. One such theory is based on stochastic electrodynamics. In this theory, matter is treated quantum mechanically while radiation is described according to Maxwell's equations, to which one adds *vacuum fluctuations*. In this picture, it would seem that almost all quantum phenomena, such as spontaneous emission, Lamb shift, and the laser linewidth, can be understood in a semiquantitative fashion.

Quantum beat* phenomena however provide us with a simple example of a case in which the results of a self-consistent fully quantized calculation differ substantially from those obtained via a semiclassical theory (SCT) even when augmented by the notion of vacuum fluctuations. This is a good example of a problem which cannot be explained, let alone calculated, by semiclassical-type arguments.

* Svanberg [1991].

In later chapters, we shall present elaborate theories of atom–field interaction based on semiclassical and fully quantum mechanical treatments. In this section, however, we discuss quantum beats via QED and SCT in three-level atomic systems using simple arguments. We consider two different types of three-level atoms in the so-called V and Λ type configurations which are prepared in a coherent superposition of all three states. Both systems are first treated semiclassically and then by QED methods in order to compare the results of both approaches.

As depicted in Fig. 1.3, an ensemble of atoms prepared in a coherent superposition of states is described by a state vector,

$$
\begin{aligned}
|\psi(t)\rangle = {} & c_a \exp(-i\omega_a t)|a\rangle + c_b \exp(-i\omega_b t)|b\rangle \\
& + c_c \exp(-i\omega_c t)|c\rangle,
\end{aligned}
\tag{1.4.1}
$$

where c_a, c_b, and c_c are probability amplitudes for the atom to be in levels $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. Furthermore, if the nonvanishing dipole matrix elements are denoted by

$$
\begin{array}{ll}
V \text{ type atoms} & \Lambda \text{ type atoms} \\[6pt]
\mathscr{P}_{ac} = e\langle a|r|c\rangle & \mathscr{P}_{ac} = e\langle a|r|c\rangle \\
\mathscr{P}_{bc} = e\langle b|r|c\rangle & \mathscr{P}_{ab} = e\langle a|r|b\rangle,
\end{array}
\tag{1.4.2}
$$

where the designations V and Λ are explained in Fig. 1.3, then the state (1.4.1) implies that each atom contains two microscopic oscillating dipoles, that is,

$$
\begin{array}{ll}
V \text{ type atoms} & \Lambda \text{ type atoms} \\[6pt]
P(t) = \mathscr{P}_{ac}(c_a^* c_c)\exp(i\nu_1 t) & P(t) = \mathscr{P}_{ac}(c_a^* c_c)\exp(i\nu_1 t) \\
\quad + \mathscr{P}_{bc}(c_b^* c_c)\exp(i\nu_2 t) & \quad + \mathscr{P}_{ab}(c_a^* c_b)\exp(i\nu_2 t) \\
\quad + \text{c.c.} & \quad + \text{c.c.},
\end{array}
\tag{1.4.3}
$$

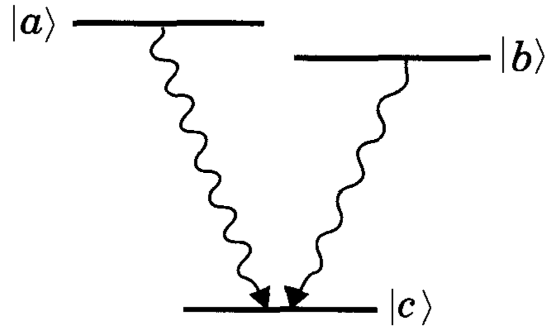
where $\nu_1 = \omega_a - \omega_c$, $\nu_2 = \omega_b - \omega_c$ for V type atoms and $\nu_1 = \omega_a - \omega_b$, $\nu_2 = \omega_a - \omega_c$ for Λ type atoms. From a semiclassical perspective, the field radiated will then be a sum of two terms

$$
E^{(+)} = \mathscr{E}_1 \exp(-i\nu_1 t) + \mathscr{E}_2 \exp(-i\nu_2 t),
\tag{1.4.4}
$$

in an obvious notation. Hence it is clear that a square law detector contains an interference or *beat note* term

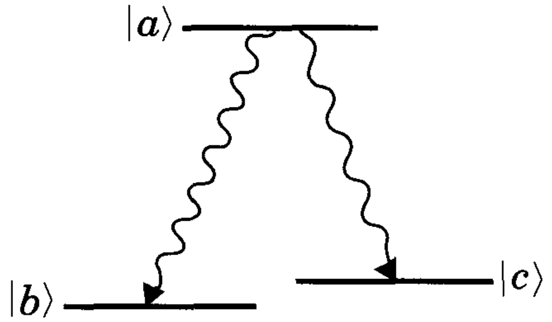
$$
|E^{(+)}|^2 = |\mathscr{E}_1|^2 + |\mathscr{E}_2|^2 + \{\mathscr{E}_1^* \mathscr{E}_2 \exp[i(\nu_1 - \nu_2)t] + \text{c.c.}\}.
\tag{1.4.5}
$$

Fig. 1.3
Three-level atomic structures for (a) V type and (b) Λ type quantum beats.



(a) V-type atom

$$|\psi_V(t)\rangle = \sum_{i=a,b,c} c_i\,|i,0\rangle + c_1\,|c,1_{\nu_1}\rangle + c_2\,|c,1_{\nu_2}\rangle$$



(b) Λ-type atom

$$|\psi_\Lambda(t)\rangle = \sum_{i=a,b,c} c_i'\,|i,0\rangle + c_1'\,|b,1_{\nu_1}\rangle + c_2'\,|c,1_{\nu_2}\rangle$$

Such a beat note is frequently observed in beam–foil spectroscopy experiments.

Finally we note, and this is the central point, that such an interference term is predicted by SCT for atoms of both types V and Λ.

Let us now consider the same problem as viewed from a QED perspective. For an atom of the V type we now calculate a beat note

$$\langle\psi_V(t)|E_1^{(-)}(t)E_2^{(+)}(t)|\psi_V(t)\rangle, \tag{1.4.6}$$

where $E_1^{(-)}(t)$ and $E_2^{(+)}(t)$ are proportional to the creation and annihilation operator expressions $a_1^\dagger \exp(i\nu_1 t)$ and $a_2 \exp(-i\nu_2 t)$, respectively. In view of $|\psi_V(t)\rangle$, as given in Fig. 1.3(a), Eq. (1.4.6) reduces to

$$\kappa\langle 1_{\nu_1}0_{\nu_2}|a_1^\dagger a_2|0_{\nu_1}1_{\nu_2}\rangle \exp[i(\nu_1-\nu_2)t]\langle c|c\rangle, \tag{1.4.7}$$

where κ is a constant. Hence, the beat note calculated via QED is given by

$$\kappa \exp[i(\nu_1 - \nu_2)t] \underbrace{\langle c|c\rangle}_{=1} . \tag{1.4.8}$$

On the other hand for Λ type atoms we have

$$\langle \psi_\Lambda(t)|E_1^{(-)}(t)E_2^{(+)}(t)|\psi_\Lambda(t)\rangle, \tag{1.4.9}$$

and taking $|\psi_\Lambda(t)\rangle$ from Fig. 1.3(b) this becomes

$$\begin{aligned} &\kappa' \langle 1_{\nu_1} 0_{\nu_2}|a_1^\dagger a_2|0_{\nu_1} 1_{\nu_2}\rangle \exp[i(\nu_1 - \nu_2)t]\langle c|b\rangle \\ &= \kappa' \exp[i(\nu_1 - \nu_2)t] \underbrace{\langle c|b\rangle}_{=0} . \end{aligned} \tag{1.4.10}$$

Summarizing these QED considerations,

$$\begin{aligned} &V \text{ type atoms} : \langle \psi_V(t)|E_1^{(-)}(t)E_2^{(+)}(t)|\psi_V(t)\rangle = \kappa \exp[i(\nu_1 - \nu_2)t], \\ &\Lambda \text{ type atoms} : \langle \psi_\Lambda(t)|E_1^{(-)}(t)E_2^{(+)}(t)|\psi_\Lambda(t)\rangle = 0 \end{aligned} \tag{1.4.11}$$

whereas in the SCT calculations one finds the beat note amplitude to be nonvanishing for both V type and Λ type atoms.

The following argument based on the *quantum theory of measurement* provides some physical insight concerning the *missing* beats. A V type atom when coherently excited will decay via the emission of a photon with frequency ν_1 or ν_2. Since both transitions lead to the same final atomic state, one cannot determine along *which path*, ν_1 or ν_2, the atom decayed. Analogous to Young's double-slit problem, this uncertainty in atomic trajectory leads to an interference between photons with frequencies ν_1 and ν_2, giving rise to quantum beats. The complementary nature of *which-path* information and the appearance of quantum beats will be discussed in detail in Chapter 19. A coherently excited Λ type atom will also decay via the emission of a photon with frequency ν_1 or ν_2. However, after the emission is long past, an observation of the atom would now tell us which decay channel (1 or 2) was taken (atom in $|c\rangle$ or $|b\rangle$). Consequently, we expect no beats in this case.

The clear conclusion is that a QED calculation is consistent with our most fundamental notions of quantum theory, while SCT applied to this problem is not.

1.5 What is light? – The photon concept

The quantum theory of radiation provides a complete description of radiation–matter interactions (when supplemented by certain renormalization presumptions). It is however tempting to argue that the conceptual underpinnings of the quantum theory of radiation and the concept of a photon can be best thought of as involving a classical electromagnetic field plus the fluctuations associated with vacuum. However, advances in quantum optics have brought forward new arguments for quantizing the electromagnetic field, and with them, deeper insight into the conceptual nature of photons. With such examples as quantum beat phenomena, the quantum eraser, and certain two-photon interference phenomena, as discussed later in this book, it becomes necessary to think of the photon as a quantum mechanical entity whose basic physics is much deeper than the semiclassical theory plus vacuum fluctuation logic. We also note that there are deep questions associated with the question of metric in a quantized field theory, and that, in one of his last papers, Feynman* makes interesting comments connecting the possibility of a deeper understanding of renormalization theory by combining negative probability concepts with indefinite-metric physics. Some of these ideas and the extensions of our conceptual understanding of the photon are the subject of this concluding section of this chapter.

1.5.1 *Vacuum fluctuations and the photon concept*

While the formal quantum theory of radiation and quantum electrodynamics has had amazing success in explaining the interaction of electromagnetic radiation with matter, there are certain conceptual problems. For example, the various infinities associated with the calculations of quantities, such as the Lamb shift, the anomalous magnetic moment.

On the other hand, as we shall see in later chapters of this book, there are many processes associated with the radiation–matter interaction which can be well explained by a semiclassical theory in which the field is treated classically and the matter is treated quantum mechanically. Examples of physical phenomena which can be explained either totally or largely by semiclassical theory include the photoelectric effect which was first explained semiclassically by Wentzel in 1927. Stimulated emission, resonance fluorescence, and many other effects

* In: Negative Probabilities in Quantum Mechanics, ed. B. Hiley and F. Peat (Routledge, London, 1978).

do not require the full machinery of the quantum theory of radiation for their explanation; they can rather be explained by a semiclassical analysis.

In the same spirit, it is interesting to note that the two clouds on the horizon of physics at the beginning of the twentieth century both involved electromagnetic radiation. As the reader will no doubt recall, it was stated that the only two issues that were not completely understood in physics at that time were the null result of the Michelson–Morley experiment and the Rayleigh–Jeans catastrophe associated with black-body radiation. The Michelson–Morley experiment, of course, led to special relativity, which was the logical capstone of classical mechanics and electrodynamics, and the Planck solution to the Rayleigh–Jeans catastrophe was the beginning of quantum mechanics.

It is, however, interesting and important to realize that neither of these phenomena involved the concept of a photon. In the first instance, Einstein was thinking essentially of transformations involving Maxwell's equations and in the second instance, Planck was thinking in terms of quantizing the energies of the oscillators in the walls of his cavity, not quantizing the radiation field. Up to this point, neither the quantum theory of radiation nor the ideal concept of the photon had been conceived.

The first introduction of the photon concept was Einstein's utilization of the idea to explain the photoelectric effect. It is again interesting to note, as we alluded to earlier, that most of the photoelectric effect can be understood semiclassically. We recall for the reader that there are three issues associated with the photoelectric effect that any theory needs to explain. First, when light of frequency ν falls on a photoemissive surface, the energy of the ejected electrons T_e obeys the expression

$$\hbar\nu = \Phi + T_e, \tag{1.5.1}$$

where Φ is a work function and is a parameter characterizing the particular material under discussion. Second, the rate of electron ejection is proportional to the square of the electric field of the incident light. Third, there is no time delay between the time in which the field begins falling on the photoactive surface and the instance of photoelectron emission. The first two of these phenomena can, in contrast to what we read in most textbooks, be explained fully by simply quantizing the atoms associated with the photodetector. However, the third point, namely, the lack of a delay is a bit more subtle. It may be reasonably argued that quantum mechanics teaches us that the rate of ejection is finite even for small times, i.e., times involving a few optical cycles of the radiation field. Nevertheless, one may argue that the concept of the photon is really explicit here in the sense that conservation of energy is at stake. That is, if we have only a short period of time τ elapsing between the instants that the radiation field begins to interact with the photoemitting atoms and the emission of the photoelectron, the amount of energy which has fallen on the surface would be governed by $\epsilon_0 E^2 A\tau$, where A is the cross-section of the incident beam. For sufficiently short times, the energy which has fallen on the photodetector may not exceed Φ. This clearly shows that we are not able to conserve energy if we take a semiclassical point of view. However, the photon concept in which the ejection of the photoelectron implies that a photon is annihilated gets around this problem completely. This is one of the triumphs of the quantum field theory.

In any case, it is a tribute to Einstein's deep understanding of physics that he was able to introduce the photon concept from such limited, and in some ways, misleading information. Having listed some of the virtues of the semiclassical theory, we now turn to the question of where it breaks down. In many arguments of this type, one hears the statement that it is the lack of the back-action of the field on the atom that is missing in semiclassical theory. This is, of course, not the case, as this back-action is contained by forcing the theory to be self-consistent as shown in Fig. 1.4. There we see that the existence of a field enters into the Schrödinger equation in such a way as to induce a dipole in an otherwise unperturbed atom. This dipole then radiates and is the source of absorption, stimulated emission, resonance fluorescence, etc. Now, the radiation which is emitted by the dipole is itself a source of perturbation of the atomic wave function (i.e., back-action) in a self-consistent analysis, as indicated in Fig. 1.4. However, the success of the semiclassical theory can only go so far and we now turn to the problems in which it breaks down and indicate how these examples can be understood by supplementing semiclassical theory with fluctuations due to the vacuum.

1.5.2 Vacuum fluctuations

Perhaps the most important example of a situation which is not covered by the semiclassical theory of Fig. 1.4 is the spontaneous emission of light. We note that an atom, which is initially in the excited state, will remain in the excited state since there is no dipole associated with an atom in any pure quantum state and therefore the atom never starts radiating. The situation is that of unstable

Fig. 1.4
Self-consistent equations demonstrating that an assumed field $\mathbf{E}'(\mathbf{r}_0, t)$ perturbs the ith atom according to the laws of quantum mechanics and induces an electric dipole expectation value. Values for atoms localized at $\mathbf{z}_0$ are added to yield the macroscopic polarization, $\mathbf{P}(\mathbf{r}_0, t)$. This polarization acts as a source in Maxwell's equations for a field $\mathbf{E}(\mathbf{r}_0, t)$. The loop is completed by the self-consistency requirement that the field assumed, $\mathbf{E}'$, is equal to the field produced, $\mathbf{E}$.

Quantum Mechanics

$$i\hbar \frac{\partial}{\partial t} \left|\psi^i\right\rangle = \left[\mathcal{H}_0 - e\mathbf{r} \cdot \mathbf{E}'(\mathbf{r}_0, t)\right] \left|\psi^i\right\rangle$$

Statistical summation

$$\mathbf{P}(\mathbf{r}_0, t) = e \sum_i \left\langle \psi^i \left| \mathbf{r} \right| \psi^i \right\rangle \delta(\mathbf{r}_{0i} - \mathbf{r}_0)$$

Electrodynamics

$$\nabla \times \left[\nabla \times \mathbf{E}(\mathbf{r}_0, t)\right] + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}_0, t) = -\mu_0 \frac{\partial^2}{\partial t^2} \mathbf{P}(\mathbf{r}_0, t)$$

Self-consistent field: $\mathbf{E}' = \mathbf{E}$

equilibrium and the atom remains in the excited state for a long, potentially infinite, time if there are no fluctuations to get things started. Furthermore, the Lamb shift is a good example of a physical situation which is only understood with the introduction of the vacuum into the problem. As we recall, the Dirac solution of the hydrogen atom shows a complete degeneracy between the $2^2S_{1/2}$ and the $2^2P_{1/2}$ levels of the hydrogen atom. However, when vacuum fluctuations are included, as in Section 1.3, we see that the Lamb shift is qualitatively accounted for and conceptually understood. Other phenomena, such as the Planck distribution of black-body radiation and the linewidth of the laser, can be understood by such semiclassical plus vacuum fluctuation arguments.

The general feeling in the early 1970s then, was that vacuum fluctuations play a very important role in our understanding of the photon concept and that perhaps the best paradigm to apply to such problems was the notion of a classical field plus a vacuum fluctuation noise or uncertainty. The discussions of squeezing as a redistribution of this uncertainty (as discussed in Chapter 2) and other related physical arguments tend to support this perspective. However, we soon realize that this concept of a 'photon", while useful, is incomplete and we now turn to a deeper and more compelling argument for quantizing the radiation field.

1.5.3 *Quantum beats, the quantum eraser, Bell's theorem, and more*

As we discussed in Section 1.4, the existence of quantum beats in an upper-state V type doublet ensemble in contrast to the absence of quantum beats associated with a lower-doublet in a Λ type atomic configuration forms the basis for an alternative argument for quantizing the radiation field which has nothing to do with the previous vacuum fluctuations. The quantum beat argument provides an example of the insufficiency of semiclassical theory plus vacuum fluctuations to understand the physics of the phenomenon. From this early example sprang concepts such as the quantum eraser and the two-photon correlation interference phenomena. This eventually showed that the early arguments and statements to the effect 'photons interfere only with themselves' were to be understood only within the context of Young's double-slit type experiments, and should not be pushed beyond that limit. We here have a great example of the importance of photon entangled states. Such entangled states are used in optical tests of Bell's inequalities and it could therefore be argued that they provide a deeper insight into the photon concept and indeed all quantum mechanics. As discussed in the last chapter of this book, we have a deeper appreciation of the nature of the quantum theory of light as a result of recent quantum optical studies.

1.5.4 *'Wave function for photons'*

The heading of this section is put in quotes for two reasons. First, it is the heading of a section in Power's classic book on QED. Second, the quotes serve to alert the reader to the fact that there is, strictly speaking, no such a thing as a 'photon wave function'.

For example, Power and also Kramers make the point that one may not think* of the 'photon' in the same sense as a massive (nonrelativistic) particle. On the other hand, some physicists argue that a single photon in free space is analogous to the meson if we let the meson mass go to zero. It is therefore interesting to consider the evidence and arguments for and against the concept of a 'photon wave function'.

The 'wave–particle duality' of light was the philosophical notion which led De Broglie to suggest that electrons might display wave-like behavior. However from the perspective of modern quantum optics, the wave mechanical, Maxwell–Schrödinger, treatment makes a clear distinction between light and matter waves. The interference

* See also Bialynicki-Birula [1994].

and diffraction of matter waves are the essence of *quantum* mechanics. However the corresponding behavior in light is described by the *classical* Maxwell equations.

But the question naturally rises: can we think of the electric field of light as a kind of 'wave function for the photon'? Specifically in his book on quantum mechanics Kramers asks in the section entitled 'The Photon Wave Function: Motivation and Definition',

> How far and how exactly can one consistently compare the radiation field with an ensemble of independent particles?
>
> When in 1924 De Broglie suggested that material particles should show wave phenomena ... such a comparison was of great heuristic importance. Now that wave mechanics has become a consistent formalism one could ask whether it is possible to consider the Maxwell equations to be a kind of Schrödinger equation for light particles, instead of considering them, as we have done up to now, to be classical equations of motion which formally look like a wave equation, and which are quantized only later on; or are both ideas equivalent?

At the end of the section Kramers answers the question as follows:

> The answer to the question put at the beginning of this section is thus that *one can not speak of particles in a radiation field in the same sense as in the (non-relativistic) quantum mechanics of systems of point particles.*

Kramers' reason for this conclusion is the same as that clearly stated by Power who says (in Section 5.1 entitled 'Wave Function For Photons')

> Thus it is natural to ask what are the ϕ's for photons? Strictly speaking there are no such wave functions! One may not speak of particles in a radiation field in the same sense as in the elementary quantum mechanics of systems of particles as used in the last chapter. The reason is that the wave equation ... solutions of Schrödinger's time-dependent wave function corresponding to an energy E_λ have a circular frequency $\omega_\lambda = +E_\lambda/\hbar$, while the monochromatic solutions of the wave equation have both $\pm\omega_\lambda$. The E and B fields satisfying the Maxwell equations in free space, and therefore satisfying the wave equation too, are real and are not eigenfunctions of $i\hbar\partial/\partial t$. A Schrödinger wave of given energy must be complex.

That is, the *real* electric wave (Eq. (1.1.27)) has both $\exp(-i\nu_k t)$ and $\exp(i\nu_k t)$ parts while the matter wave has only $\exp(-i\nu_p t)$ type terms. We shall return to this point later, but let us first recall the arguments of Bohm in his classic *Quantum Theory* book on the subject. On page 98 he notes that

> The probability that an electron can be found with position between x and $x + dx$ is
> $$P(x) = \psi^*(x)\psi(x)dx.$$

He then compares this with the situation for light and goes on to say:

> There is, strictly speaking, no function that represents the probability of finding a light quantum at a given point. If we choose a region large compared with a wavelength, we obtain approximately
> $$P(x) \cong \frac{\mathscr{E}^2(x) + \mathscr{H}^2(x)}{8\pi h\nu(x)},$$
> but if this region is defined too well, $\nu(x)$ has no meaning.

Later on Bohm makes the statement that for matter

> There is a probability current
> $$S = \frac{\hbar}{2mi}(\psi^*\Delta\psi - \psi\Delta\psi^*)$$
> which satisfies the relation
> $$\frac{\partial P}{\partial t} + \mathrm{div} S = 0,$$

but he notes that

> There is no corresponding quantity for light.

We agree with the conclusion of Kramers and Bohm, namely that the concept of a photon wave function must be used with care and can be very misleading. However as we shall see, each of the above objections to the concept can be overcome.

We begin by noting that, from the perspective of a semiclassical theory, we are dealing with a wave description of the (classical) radiation, and (quantum) matter systems. Only when we proceed to quantize the radiation field are the radiation–matter equations treated on the same footing. In this fully quantized theory, it is instructive to consider matter from a second quantized vantage. Recall the quantization procedure of Section 1.1 in which we replaced the Fourier amplitudes of the field by operators. Consider the classical complex field $E(\mathbf{r}, t)$ for polarized light. Since the light is polarized, we can ignore the vector character of the field. In passing from the classical to the quantum description of the field we replace the coefficients of the field eigenfunctions, $U_{\mathbf{k}}(\mathbf{r})$, by operators, i.e.,

$$E^{(+)}(\mathbf{r},t)=\sum_{\mathbf{k}}\mathscr{E}_{\mathbf{k}}\alpha_{\mathbf{k}}e^{-i\nu_k t}U_{\mathbf{k}}(\mathbf{r}), \tag{1.5.2}$$

where $\alpha_{\mathbf{k}}$ are classical field amplitudes, is replaced by

$$E^{(+)}(\mathbf{r},t)=\sum_{\mathbf{k}}\mathscr{E}_{\mathbf{k}}a_{\mathbf{k}}e^{-i\nu_k t}U_{\mathbf{k}}(\mathbf{r}), \tag{1.5.3}$$

where $a_{\mathbf{k}}$ are quantum field operators.

Now, a corresponding quantization procedure can be, and is, applied to matter. For example, the wave function of a massive system (atom, electron, meson, etc.) is described by the superposition of states

$$\psi(\mathbf{r},t)=\sum_{\mathbf{p}}c_{\mathbf{p}}e^{-i\nu_p t}\phi_{\mathbf{p}}(\mathbf{r}), \tag{1.5.4}$$

where $\nu_p = E_p/\hbar$ and $c_{\mathbf{p}}$ is the probability amplitude for a particle being in state $\phi_{\mathbf{p}}(\mathbf{r})$, e.g., for a particle of momentum $\mathbf{p}$ we have

$$\phi_{\mathbf{p}}(\mathbf{r})=\frac{1}{\sqrt{V}}e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{1.5.5}$$

The (*second*) quantization procedure now is to turn each probability amplitude $c_{\mathbf{p}}$ into an annihilation operator $\hat{c}_{\mathbf{p}}$ obeying Fermi–Dirac or Bose–Einstein commutation relations, etc. In such a case, the wave function becomes an operator

$$\hat{\psi}(\mathbf{r},t)=\sum_{\mathbf{p}}\hat{c}_{\mathbf{p}}e^{-i\nu_p t}\phi_{\mathbf{p}}(\mathbf{r}), \tag{1.5.6}$$

which annihilates a particle at $\mathbf{r}$ and the state of the system is described by a state vector $|\psi\rangle$. At this level both the matter and photons are described by quantized fields and the state of the photon and/or meson *field* is described by a state vector $|\psi\rangle$. The logic of semiclassical and fully second quantized treatments of the radiation–matter system is summarized in Fig. 1.5.

Notice that the terminology 'second' quantization is appropriate for the matter field, since we are introducing operators for the second time; i.e., we first set $p_x \to (\hbar/i)\partial/\partial x$, etc., and second we replace probability amplitudes by operators $c_{\mathbf{p}}(t) \to \hat{c}_{\mathbf{p}}(t)$. However, this does not appear to be the case for the photon since $\hbar$ appears only once. In this sense, the quantization of the radiation field can be argued to be a 'first' quantization procedure.

We now turn the picture around and pretend that we first learn of photons and mesons, etc., from a fully quantized field perspective. The particle wave function is obtained from the state vector by taking the inner product between the position eigenstate $|\mathbf{r}\rangle$ and the state vector $|\psi(t)\rangle$

	Light	Matter
Semiclassical	$\mathbf{E}(\mathbf{r},t)$ $\Box^2\mathbf{E} = -\mu_0\ddot{\mathbf{P}}$ Maxwell	$\psi(\mathbf{r},t)$ $\dot{\psi}(\mathbf{r},t) = -\frac{\mathrm{i}}{\hbar}\mathcal{H}\psi(\mathbf{r},t)$ Schrödinger
Quantum field	$\|\dot{\psi}_f\rangle = -\frac{\mathrm{i}}{\hbar}\mathcal{H}_f\,\|\psi_f\rangle$ $\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}(t)U_{\mathbf{k}}(\mathbf{r})$ Dirac	$\|\dot{\psi}_m\rangle = -\frac{\mathrm{i}}{\hbar}\mathcal{H}_m\,\|\psi_m\rangle$ $\hat{\psi}(\mathbf{r},t) = \sum_{\mathbf{p}} \hat{c}_{\mathbf{p}}(t)\phi_{\mathbf{p}}(\mathbf{r})$ Schwinger

Fig. 1.5
The semiclassical theory of the radiation and matter 'fields' are treated according to the Maxwell and Schrödinger equations. Both fields display wave-like behavior but $\hbar$ appears only in the matter equation. Applying the full quantum field theory of, e.g., Dirac and Schwinger, the radiation and matter are treated on the same footing.

$$\Psi(\mathbf{r},t) = \langle \mathbf{r}|\psi(t)\rangle. \tag{1.5.7}$$

We recall that the state $|\mathbf{r}\rangle$ can be written as

$$|\mathbf{r}\rangle = \hat{\psi}^\dagger(\mathbf{r})|0\rangle, \tag{1.5.8}$$

that is, the creation operator

$$\hat{\psi}^\dagger(\mathbf{r}) = \sum_{\mathbf{p}} \hat{c}_{\mathbf{p}}^\dagger e^{i\nu_p t}\phi_{\mathbf{p}}^*(\mathbf{r}), \tag{1.5.9}$$

acting on the vacuum creates a particle at $\mathbf{r}$. So from Eqs. (1.5.7) and (1.5.8), we have the usual result for the matter wave function

$$\Psi(\mathbf{r},t) = \langle 0|\hat{\psi}(\mathbf{r})|\psi(t)\rangle. \tag{1.5.10}$$

Now it is natural to ask: can we write something like Eq. (1.5.7) for the photon? The answer is, strictly speaking, 'no'; because there is no $|\mathbf{r}\rangle$ state for the photon.

With that in mind, let us push on and ask the operational question: what is the probability that a single-photon state of the radiation field, that is

$$|\psi\rangle = \sum_{\{n\}} c_{\{n\}}(t)|\{n\}\rangle, \tag{1.5.11}$$

where $\{n\}$ stands for the set of states with one (and only one) photon in each mode $\mathbf{k}$, will lead to the ejection of a photoelectron by a detector (atom) placed at point $\mathbf{r}$?

For example, the state Eq. (1.5.11) might be produced by an excited atom decaying to a ground state, an example we will return to later. In any case, we have in mind a wave packet representing a single photon propagating through space and the probability amplitudes $c_{\{n\}}$ contain the information normally associated with the Fourier coefficients for the single-photon pulse.

Now, as we will discuss in Section 4.2, the probability of exciting an atom (a detector atom) at $\mathbf{r}$ is governed by

$$P_\psi(\mathbf{r},t) \propto \langle\psi|E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)|\psi\rangle, \tag{1.5.12}$$

where the annihilation operator $E^{(+)}(\mathbf{r},t)$ is given by

$$E^{(+)}(\mathbf{r},t) = \sum_{\mathbf{k}} \mathscr{E}_{\mathbf{k}} a_{\mathbf{k}} e^{-i\nu_k t} U_{\mathbf{k}}(\mathbf{r}), \tag{1.5.13}$$

and the creation operator $E^{(-)}(\mathbf{r},t)$ is just the adjoint of Eq. (1.5.13). We insert a sum over a complete set of states, $\sum_{\{n'\}} |\{n'\}\rangle\langle\{n'\}| = 1$ in Eq. (1.5.12) and write

$$P_\psi(\mathbf{r},t) \propto \sum_{\{n'\}} \langle\psi|E^{(-)}(\mathbf{r},t)|\{n'\}\rangle\langle\{n'\}|E^{(+)}(\mathbf{r},t)|\psi\rangle. \tag{1.5.14}$$

But since there is only one photon in ψ and $E^{(+)}(\mathbf{r},t)$ annihilates it, only the vacuum term $|0\rangle\langle 0|$ will contribute to Eq. (1.5.14). Hence we have

$$P_\psi(\mathbf{r},t) \propto \langle\psi|E^{(-)}(\mathbf{r},t)|0\rangle\langle 0|E^{(+)}(\mathbf{r},t)|\psi\rangle, \tag{1.5.15}$$

and we are therefore led to define the 'electric field' associated with the single photon state $|\psi_\gamma\rangle$ as

$$\Psi_{\mathscr{E}}(\mathbf{r},t) = \langle 0|E^{(+)}(\mathbf{r},t)|\psi_\gamma\rangle. \tag{1.5.16}$$

Now for the state $|\psi_\gamma\rangle$ prepared by atomic decay, Eq. (6.3.24), we find

$$\Psi_{\mathscr{E}}(\mathbf{r},t) = \frac{\mathscr{E}_0}{r}\Theta\left(t-\frac{r}{c}\right) e^{-i(t-r/c)(\omega-i\Gamma/2)}, \tag{1.5.17}$$

where $\mathscr{E}_0$ is a constant, r is the distance from the atom to the detector, $\Theta(x)$ is the usual step function and Γ is the atomic decay rate. We note that the wave packet (1.5.17) is sharply peaked about the atomic transition frequency ω. This will be the case in all the packets we consider in this section.

Let us write Eq. (1.5.16) more explicitly using the positive frequency part in Eq. (1.1.32) for the electric field annihilation operator, that is

$$\begin{aligned}\boldsymbol{\Psi}_{\mathscr{E}}(\mathbf{r},t) &= \langle 0|\mathbf{E}^{(+)}(\mathbf{r},t)|\psi_\gamma\rangle \\ &= \langle 0|\sum_{\mathbf{k},\lambda} \hat{\epsilon}_{\mathbf{k}}^{(\lambda)} \sqrt{\frac{\hbar\nu_k}{2\epsilon_0 V}}\, a_{\mathbf{k},\lambda} e^{-i\nu_k t+i\mathbf{k}\cdot\mathbf{r}}|\psi_\gamma\rangle.\end{aligned} \tag{1.5.18}$$

As discussed in the previous paragraph, the field is sharply peaked about the frequency ω so that we may replace the slowly varying frequency ν_k as it appears in the square-root factor by ω and write

$$\mathbf{\Psi}_{\mathscr{E}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\langle 0|\sum_{\mathbf{k},\lambda}\hat{\epsilon}_{\mathbf{k}}^{(\lambda)} a_{\mathbf{k},\lambda} e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}}|\psi_\gamma\rangle. \tag{1.5.19}$$

Comparing (1.5.19) with the wave function (1.5.4) we are led to define the photodetection probability amplitude as

$$\varphi_\gamma(\mathbf{r},t) = \sum_{\mathbf{k},\lambda}\hat{\epsilon}_{\mathbf{k}}^{(\lambda)}\langle 0|a_{\mathbf{k},\lambda}\frac{e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}|\psi_\gamma\rangle, \tag{1.5.20}$$

which is to say

$$\mathbf{\Psi}_{\mathscr{E}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega}{2\epsilon_0}}\varphi_\gamma(\mathbf{r},t). \tag{1.5.21}$$

We may write an equation of motion for $\varphi_\gamma(\mathbf{r},t)$ by using Maxwell's equations, which couple together the electric field (1.5.16) with the magnetic field

$$\mathbf{\Psi}_{\mathscr{H}}(\mathbf{r},t) = \langle 0|\mathbf{H}^{(+)}(\mathbf{r},t)|\psi_\gamma\rangle, \tag{1.5.22}$$

where $\mathbf{H}^{(+)}(\mathbf{r},t)$ is the positive frequency part of the magnetic field operator (1.1.33), which we here write in the form

$$\mathbf{H}^{(+)}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda}\frac{\mathbf{k}}{k}\times\hat{\epsilon}_{\mathbf{k}}^{(\lambda)}\sqrt{\frac{\hbar\nu_k}{2\mu_0}}a_{\mathbf{k},\lambda}\frac{e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}. \tag{1.5.23}$$

Using Eqs. (1.5.22) and (1.5.23) and proceeding as in the case of $\Psi_{\mathscr{E}}(\mathbf{r},t)$ we find

$$\begin{aligned}\mathbf{\Psi}_{\mathscr{H}}(\mathbf{r},t) &= \sqrt{\frac{\hbar\omega}{2\mu_0}}\langle 0|\sum_{\mathbf{k},\lambda}\frac{\mathbf{k}}{k}\times\hat{\epsilon}_{\mathbf{k}}^{(\lambda)}a_{\mathbf{k},\lambda}\frac{e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}|\psi_\gamma\rangle \\ &= \sqrt{\frac{\hbar\omega}{2\mu_0}}\boldsymbol{\chi}_\gamma(\mathbf{r},t). \end{aligned} \tag{1.5.24}$$

Now we may write Maxwell's equations (1.1.1) in terms of $\boldsymbol{\varphi}_\gamma$ (Eq. (1.5.21)) and $\boldsymbol{\chi}_\gamma$ (Eq. (1.5.24)) as

$$\nabla\times\boldsymbol{\chi}_\gamma = \frac{1}{c}\frac{\partial\boldsymbol{\varphi}_\gamma}{\partial t}, \tag{1.5.25a}$$

$$\nabla\times\boldsymbol{\varphi}_\gamma = -\frac{1}{c}\frac{\partial\boldsymbol{\chi}_\gamma}{\partial t}, \tag{1.5.25b}$$

$$\nabla\cdot\boldsymbol{\chi}_\gamma = 0, \tag{1.5.25c}$$

$$\nabla\cdot\boldsymbol{\varphi}_\gamma = 0. \tag{1.5.25d}$$

We proceed to express Eqs. (1.5.25a-1.5.25d) in an aesthetically pleasing matrix form by writing $\boldsymbol{\varphi}$ and $\boldsymbol{\chi}$ as 1×3 column matrices

$$\boldsymbol{\varphi}_\gamma = \begin{bmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{bmatrix}; \qquad \boldsymbol{\chi}_\gamma = \begin{bmatrix} \chi_x \\ \chi_y \\ \chi_z \end{bmatrix} \tag{1.5.26}$$

in terms of which, see Problem 1.7, Maxwell's equations (1.5.25a-1.5.25d) may be written as

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{\varphi}_\gamma \\ \boldsymbol{\chi}_\gamma \end{bmatrix} = \begin{bmatrix} 0 & -c\mathbf{s}\cdot\mathbf{p} \\ c\mathbf{s}\cdot\mathbf{p} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\varphi}_\gamma \\ \boldsymbol{\chi}_\gamma \end{bmatrix}, \tag{1.5.27a}$$

and

$$\nabla\cdot\begin{bmatrix} \boldsymbol{\varphi}_\gamma \\ \boldsymbol{\chi}_\gamma \end{bmatrix} = 0, \tag{1.5.27b}$$

where s_x, s_y, and s_z are the 3×3 matrices given in Problem 1.7, and $\mathbf{p}$ is the usual momentum operator $(\hbar/i)\boldsymbol{\nabla}$.

It is interesting to compare Maxwell's equations in the form (1.5.27a, 1.5.27b) to the Dirac equations* for the neutrino

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{\varphi}_\eta \\ \boldsymbol{\chi}_\eta \end{bmatrix} = \begin{bmatrix} 0 & c\boldsymbol{\sigma}\cdot\mathbf{p} \\ c\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{\varphi}_\eta \\ \boldsymbol{\chi}_\eta \end{bmatrix}, \tag{1.5.28}$$

where the two-component spinors $\boldsymbol{\varphi}_\eta$ and $\boldsymbol{\chi}_\eta$ make up the Dirac wave function for the neutrino

$$\boldsymbol{\Psi}_\eta = \begin{bmatrix} \boldsymbol{\varphi}_\eta \\ \boldsymbol{\chi}_\eta \end{bmatrix}. \tag{1.5.29}$$

With equations of motion (1.5.27a,1.5.27b) in hand we easily derive the equation of continuity

$$\frac{\partial}{\partial t}\boldsymbol{\Psi}_\gamma^\dagger\boldsymbol{\Psi}_\gamma = -\nabla\cdot\mathbf{j}, \tag{1.5.30}$$

where

$$\boldsymbol{\Psi}_\gamma = \begin{bmatrix} \boldsymbol{\varphi}_\gamma \\ \boldsymbol{\chi}_\gamma \end{bmatrix}, \tag{1.5.31}$$

and the current density $\mathbf{j}$ is found, see Problem 1.8, to be

$$\mathbf{j} = \boldsymbol{\Psi}_\gamma^\dagger\mathbf{v}\boldsymbol{\Psi}_\gamma \tag{1.5.32}$$

* The correspondence between the Maxwell and Dirac equations is well known. See, for example, Bialynicki-Birula, [1994].

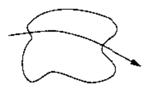
	Photon	Neutrino
Quantum field theory	$\vert\dot{\psi}_\gamma\rangle = -\frac{\mathrm{i}}{\hbar}\mathcal{H}_\gamma \vert\psi_\gamma\rangle$	$\vert\dot{\psi}_\eta\rangle = -\frac{\mathrm{i}}{\hbar}\mathcal{H}_\eta \vert\psi_\eta\rangle$
"Wave" mechanics	$\Psi_\gamma = \begin{bmatrix}\phi_\gamma \\ \chi_\gamma\end{bmatrix}$ $\dot{\Psi}_\gamma = -\frac{\mathrm{i}}{\hbar}\begin{bmatrix}0 & -c\mathbf{s}\cdot\mathbf{p} \\ c\mathbf{s}\cdot\mathbf{p} & 0\end{bmatrix}\Psi_\gamma$	$\Psi_\eta = \begin{bmatrix}\phi_\eta \\ \chi_\eta\end{bmatrix}$ $\dot{\Psi}_\eta = -\frac{\mathrm{i}}{\hbar}\begin{bmatrix}0 & -c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & 0\end{bmatrix}\Psi_\eta$
Classical limit: Eikonal physics	 $n(\mathbf{r})$ Ray optics $\delta\int n\,\mathrm{d}s = 0$ Fermat's principle	 $V(\mathbf{r})$ Classical mechanics $\delta\int L\,\mathrm{d}t = 0$ Hamilton's principle

Fig. 1.6
A symmetric description for a photon and a neutrino. In the classical limit (Last row), light is described by ray optics whereas matter is described by the analogous classical Hamilton's principle. The quantum field theoretical description of a photon and a neutrino in the first row is also quite symmetric. The 'wave' mechanics row indicates the equations of motion for $\Psi_\gamma(\mathbf{r}, t)$ and $\Psi_\eta(\mathbf{r}, t)$.

with the 'velocity' operator given by

$$\mathbf{v} = c\begin{bmatrix}0 & -\mathbf{s} \\ \mathbf{s} & 0\end{bmatrix}. \tag{1.5.33}$$

The comparison between $\mathbf{\Psi}_\gamma$ and $\mathbf{\Psi}_\eta$ is summarized in Fig. 1.6.

While it is amusing to note the similarities between the photon and the neutrino equations of motion, important and basic differences must be noted. For example, if we consider the electronic cousin to (1.5.28) in the nonrelativistic limit we have, for example, a plane wave relation of the form

$$\varphi_{\text{electron}}(\mathbf{r}, t) = \frac{1}{\sqrt{V}}e^{i(k_z z - \omega_k t)}, \tag{1.5.34}$$

where $k_z = p_z/\hbar$ and $\omega_k = p_z^2/2m\hbar$. Now to give the electron a momentum kick in the $\hat{x}$-direction we need only apply the boost operation $\exp(ik_x x)$ where now $x = \partial/\partial k_x$. Thus the new momentum is given by

$$\mathbf{k} = \hat{\mathbf{e}}_x k_x + \hat{\mathbf{e}}_z k_z. \tag{1.5.35}$$

But now consider the same sort of operation applied to (1.5.20). That is if we initially write $\varphi(\mathbf{r}, t)$ for a plane wave propagating in the $\hat{z}$-direction with polarization in the $\hat{x}$-direction, that is

$$\varphi(\mathbf{r}, t) = \hat{\mathbf{e}}_x \frac{e^{i(k_z z - \omega_k t)}}{\sqrt{V}}, \tag{1.5.36}$$

and we then apply a boost operation as before, we might think we could write the new function

$$\tilde{\varphi}(\mathbf{r},t) = \hat{\mathbf{e}}_x \frac{e^{i(k_z z + k_x x - \omega t)}}{\sqrt{V}}. \tag{1.5.37}$$

But now the Maxwell equation (1.5.27b) is no longer satisfied

$$\nabla \cdot \tilde{\varphi} = \frac{\partial}{\partial x}\left[\frac{e^{i(k_z z + k_x x - \omega t)}}{\sqrt{V}}\right] \neq 0. \tag{1.5.38}$$

This is just one example of how the photon 'wave function' is different from that of a nonrelativistic massive particle, and a 'photon-as-a-particle' picture can be misleading.

As another, even more dramatic example, we turn to the question of two-photon events. Specifically we have in mind two photon emission and detection as in Fig. 1.7. As discussed in detail in Chapter 4, the probability of two photoelectrons being counted at detectors D_1 and D_2 is governed by the two-photon correlation function. To that end, we calculate the two-photon correlation function

$$\begin{aligned} &G^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \\ &\quad = \langle\psi|E^{(-)}(\mathbf{r}_1,t_1)E^{(-)}(\mathbf{r}_2,t_2)E^{(+)}(\mathbf{r}_2,t_2)E^{(+)}(\mathbf{r}_1,t_1)|\psi\rangle, \end{aligned} \tag{1.5.39}$$

corresponding to two detectors at points $\mathbf{r}_1$ and $\mathbf{r}_2$ and where the interaction with the photon field, described by $|\psi\rangle$, is switched on at times t_1 and t_2, respectively as in Fig. 1.7.

We note that for the radiation from a single atom only two photons are involved so that

$$\begin{aligned} \langle\psi|E_1^{(-)}E_2^{(-)}E_2^{(+)}E_1^{(+)}|\psi\rangle &= \sum_{\{n\}}\langle\psi|E_1^{(-)}E_2^{(-)}|\{n\}\rangle\langle\{n\}|E_2^{(+)}E_1^{(+)}|\psi\rangle \\ &= \langle\psi|E_1^{(-)}E_2^{(-)}|0\rangle\langle 0|E_2^{(+)}E_1^{(+)}|\psi\rangle, \end{aligned} \tag{1.5.40}$$

and therefore it is the two-photon detection amplitude

$$\Psi^{(2)}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) \equiv \langle 0|E^{(+)}(\mathbf{r}_2,t_2)E^{(+)}(\mathbf{r}_1,t_1)|\psi\rangle \tag{1.5.41}$$

which we must now consider.

First consider the case in which the atomic decay rates from level $|a\rangle$ to level $|b\rangle$, γ_a, and from level $|b\rangle$ to level $|c\rangle$, γ_b, are such that $\gamma_a \gg \gamma_b$, that is the atom decays very quickly to level $|b\rangle$ and then after some time decays to level $|c\rangle$. In such a case we find, see Section 21.4.1, that

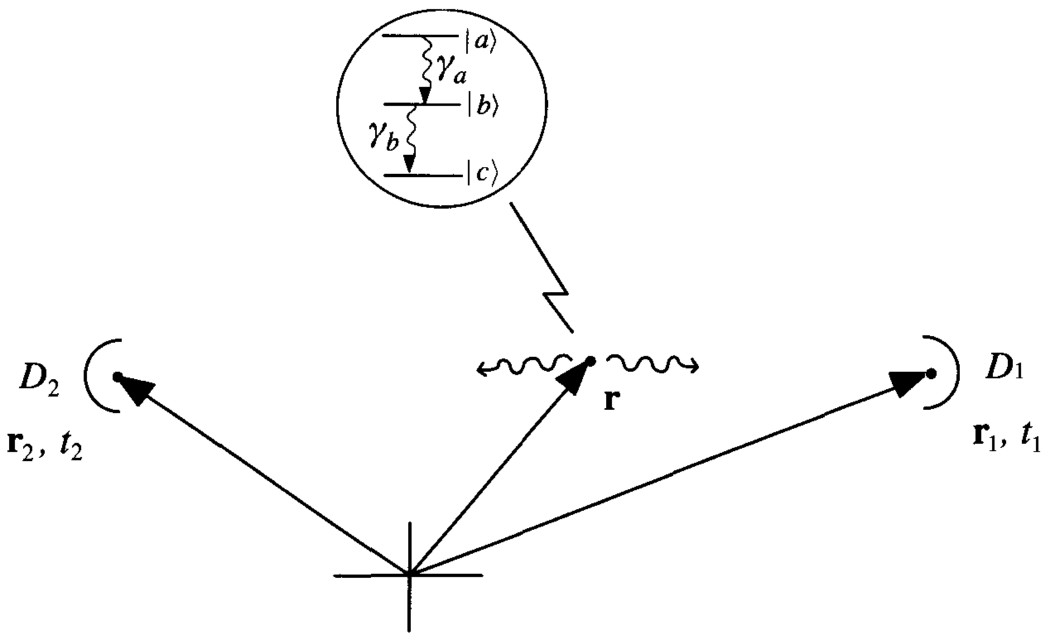


Fig. 1.7
Three-level atom located at $\mathbf{r}$ decays from $|a\rangle \to |b\rangle$ with rate γ_a and $|b\rangle \to |c\rangle$ with rate γ_b. Detectors D_1 and D_2 at $\mathbf{r}_1$ and $\mathbf{r}_2$ are switched on at times t_1 and t_2.

$$\Psi^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$$
$$= \Psi_\alpha(\mathbf{r}_1, t_1)\Psi_\beta(\mathbf{r}_2, t_2) + \Psi_\beta(\mathbf{r}_1, t_1)\Psi_\alpha(\mathbf{r}_2, t_2), \tag{1.5.42}$$

where

$$\Psi_\alpha(\mathbf{r}_i, t_i) = \frac{\mathscr{E}_a}{\Delta r_i}\Theta\left(t_i - \frac{\Delta r_i}{c}\right) e^{-\gamma_a\left(t_i - \frac{\Delta r_i}{c}\right)} e^{-i\omega_{ab}\left(t_i - \frac{\Delta r_i}{c}\right)}, \tag{1.5.43a}$$

and

$$\Psi_\beta(\mathbf{r}_i, t_i) = \frac{\mathscr{E}_b}{\Delta r_i}\Theta\left(t_i - \frac{\Delta r_i}{c}\right) e^{-\gamma_b\left(t_i - \frac{\Delta r_i}{c}\right)} e^{-i\omega_{bc}\left(t_i - \frac{\Delta r_i}{c}\right)}, \tag{1.5.43b}$$

in which $i = 1, 2$, ω_{ab} and ω_{bc} are the atomic frequencies for the $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$ transitions, Δr_i is the distance from the atom to the ith detector and $\mathscr{E}_a$ and $\mathscr{E}_b$ are uninteresting constants. The immediate comparison between Eqs. (1.5.43a) and (1.5.43b) and the single photoelectron detection amplitude (1.5.17) is apparent. We clearly have here a Bose–Einstein expression of the type we might write for two helium atoms.

But things are very different when we make the simple change to the case $\gamma_b \gg \gamma_a$. That is when the atoms which decay at some time to level $|b\rangle$ rapidly decay to level $|c\rangle$. Then Section 21.4.1, we find a two-photon detection amplitude of the form

$$\Psi^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2; t_2)$$
$$= \frac{-\kappa}{\Delta r_1 \Delta r_2} \exp\left[-(i\omega_{ac} + \gamma_a)\left(t_1 - \frac{\Delta r_1}{c}\right)\right] \Theta\left(t_1 - \frac{\Delta r_1}{c}\right)$$
$$\times \exp\left\{-(i\omega_{bc} + \gamma_b)\left[\left(t_2 - \frac{\Delta r_2}{c}\right) - \left(t_1 - \frac{\Delta r_1}{c}\right)\right]\right\}$$
$$\times \Theta\left[\left(t_2 - \frac{\Delta r_2}{c}\right) - \left(t_1 - \frac{\Delta r_1)}{c}\right)\right] + (1 \leftrightarrow 2). \tag{1.5.44}$$

The message is clear. When $\gamma_a \gg \gamma_b$ we have essentially independent photons emitted. But when $\gamma_a \ll \gamma_b$ the two events are strongly correlated and the 'photon-as-a-particle' picture is very misleading.

In conclusion, we can say that while we have perhaps overcome the main objection of Kramers (the probability amplitude $\langle 0|E|\psi\rangle \sim e^{-i\omega t}$ only) and partially overcome that of Bohm (photodetection events are indeed localized* to distances smaller than the wavelength), naively visualizing $\varphi(\mathbf{r}, t)$ as a literal particle-like wave function can be misleading. "Photon" physics is very different from that of Schrödinger particles.

The proper operational "photon" philosophy is well summarized by Willis Lamb who says:

> What do we do next? We can, and should, use the Quantum Theory of Radiation. Fermi showed how to do this for the case of Lippmann fringes. The idea is simple, but the details are somewhat messy. A good notation and lots of practice makes it easier. Begin by deciding how much of the universe needs to be brought into the discussion. Decide what normal modes are needed for an adequate treatment. Decide how to model the light sources and work out how they drive the system.

This is what we will be doing in the next 20 chapters.

1.A Equivalence between a many-particle Bose gas and a set of quantized harmonic oscillators

In Section 1.1 we quantized the radiation field by associating each mode of the field with a quantized simple harmonic oscillator. This procedure led to the introduction of the Fock or number state of the field containing, for each oscillator, n photons and the associated operators a and $a^\dagger$ which annihilate and create photons, respectively. In this section, we argue that a set of harmonic oscillators is dynamically equivalent to a many-particle Bose gas.

Consider a Bose gas of N particles inside a volume V. The N-particle wave function can be written by symmetrizing the product of

* We note however that the localization of the photon (as opposed to the localizability of the photon-detection probability amplitude) is qualitatively different from the localization of a massive particle e.g., an electron. For an electron it is possible to 'fit' the electron into a small box greater than or equal to the Compton wavelength. For the photon, however, it is not possible to 'fit' or 'force' the photon into a box smaller than its wavelength. See Deutsch and Garrison [1991].

the single-particle wave functions $\psi_{\mathbf{s}}(\mathbf{r})$:

$$\Psi^N_{n_{\mathbf{p}},n_{\mathbf{q}},\ldots,n_{\mathbf{k}},\ldots}(\mathbf{r}_1,\mathbf{r}_2,\ldots\mathbf{r}_N) = \left[\frac{n_{\mathbf{p}}!n_{\mathbf{q}}!\ldots n_{\mathbf{k}}!\ldots}{N!}\right]^{1/2}$$

$$\times\sum_{\mathbf{P}}\begin{bmatrix} \psi_{\mathbf{p}}(\mathbf{r}_1)\psi_{\mathbf{p}}(\mathbf{r}_2)\ldots\psi_{\mathbf{p}}(\mathbf{r}_{n_{\mathbf{p}}}) \\ \times\psi_{\mathbf{q}}(\mathbf{r}_{n_{\mathbf{p}}+1})\psi_{\mathbf{q}}(\mathbf{r}_{n_{\mathbf{p}}+2})\ldots\psi_{\mathbf{q}}(\mathbf{r}_{n_{\mathbf{p}}+n_{\mathbf{q}}}) \\ \vdots \\ \times\psi_{\mathbf{k}}(\mathbf{r}_{\sigma+1})\psi_{\mathbf{k}}(\mathbf{r}_{\sigma+2})\ldots\psi_{\mathbf{k}}(\mathbf{r}_{\sigma+n_{\mathbf{k}}}) \\ \vdots \end{bmatrix} \tag{1.A.1}$$

where $\mathbf{P}$ denotes the permutation on N objects. The integers $n_{\mathbf{s}}$ ($\mathbf{s} = \mathbf{p},\mathbf{q},\ldots,\mathbf{k},\ldots$) are the occupation numbers of the single-particle wave functions $\psi_{\mathbf{s}}(\mathbf{r}_j)$ such that

$$\sum_{\mathbf{s}} n_{\mathbf{s}} = N, \tag{1.A.2}$$

and $n_{\mathbf{s}}$ can take the values $0, 1, 2, \ldots, N$. The single-particle wave function for a free particle is given by

$$\psi_{\mathbf{s}}(\mathbf{r}) = \frac{1}{\sqrt{V}}e^{i\mathbf{s}\cdot\mathbf{r}}. \tag{1.A.3}$$

Here $\hbar\mathbf{s}$ is the momentum of the particle.

Let the N particles interact with each other via a potential

$$\mathscr{V} = \sum_{j=1}^{N} v(\mathbf{r}_j). \tag{1.A.4}$$

A particle in the state $\psi_{\mathbf{p}}(\mathbf{r}_j)$ can go to the state $\psi_{\mathbf{k}}(\mathbf{r}_j)$ by interacting with the potential. The transition amplitude for this process is proportional to

$$v_{\mathbf{kp}} = \int d\mathbf{r}_j\psi^*_{\mathbf{k}}(\mathbf{r}_j)v(\mathbf{r}_j)\psi_{\mathbf{p}}(\mathbf{r}_j). \tag{1.A.5}$$

As an example, if a free particle with momentum $\hbar\mathbf{k}$ scatters with a phonon wave with wave vector $\tilde{\mathbf{k}}$, i.e.,

$$v(\mathbf{r}_j) = v_0 e^{i\tilde{\mathbf{k}}\cdot\mathbf{r}_j}, \tag{1.A.6}$$

to a state with momentum $\hbar\mathbf{p}$, we have

$$v_{\mathbf{kp}} = v_0\delta(\mathbf{p} + \tilde{\mathbf{k}} - \mathbf{k}). \tag{1.A.7}$$

We now consider the many-particle analysis of the problem.

Before considering the general case of a Bose gas of N particles inside a volume V, we will consider the simple case of a three-boson system. The wave function for a three-particle system initially having $n_{\mathbf{p}} = 2, n_{\mathbf{k}} = 1$ is given by

$$\psi^3_{n_{\mathbf{p}}=2,n_{\mathbf{k}}=1}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \frac{1}{\sqrt{3}}[\psi_{\mathbf{p}}(\mathbf{r}_1)\psi_{\mathbf{p}}(\mathbf{r}_2)\psi_{\mathbf{k}}(\mathbf{r}_3) + \psi_{\mathbf{p}}(\mathbf{r}_3)\psi_{\mathbf{p}}(\mathbf{r}_1)\psi_{\mathbf{k}}(\mathbf{r}_2) + \psi_{\mathbf{p}}(\mathbf{r}_2)\psi_{\mathbf{p}}(\mathbf{r}_3)\psi_{\mathbf{k}}(\mathbf{r}_1)]. \tag{1.A.8}$$

An interaction between the particles via a potential (Eq. (1.A.4)) with $N = 3$ transforms one particle in state $\mathbf{p}$ to state $\mathbf{k}$, i.e.,

$$\psi^3_{n_{\mathbf{p}}=1,n_{\mathbf{k}}=2}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \frac{1}{\sqrt{3}}[\psi_{\mathbf{p}}(\mathbf{r}_1)\psi_{\mathbf{k}}(\mathbf{r}_2)\psi_{\mathbf{k}}(\mathbf{r}_3) + \psi_{\mathbf{p}}(\mathbf{r}_3)\psi_{\mathbf{k}}(\mathbf{r}_1)\psi_{\mathbf{k}}(\mathbf{r}_2) + \psi_{\mathbf{p}}(\mathbf{r}_2)\psi_{\mathbf{k}}(\mathbf{r}_3)\psi_{\mathbf{k}}(\mathbf{r}_1)]. \tag{1.A.9}$$

The three-particle matrix element for the process is then

$$\mathcal{M}_3 = \int\int\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \psi^{3*}_{n_{\mathbf{p}}=1,n_{\mathbf{k}}=2}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \times \sum_{i=1}^{3} v(\mathbf{r}_i)\psi^3_{n_{\mathbf{p}}=2,n_{\mathbf{k}}=1}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3). \tag{1.A.10}$$

Now each particle in the sum $\sum_{i=1}^{3} v(\mathbf{r}_i)$ contributes equally so that we may simply choose a particle, say particle 1, and replace $\sum_{i=1}^{3}$ by the factor 3. Then we have

$$\mathcal{M}_3 = 3\int\int\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \frac{1}{\sqrt{3}}[\psi^*_{\mathbf{k}}(\mathbf{r}_2)\psi^*_{\mathbf{p}}(\mathbf{r}_3) + \psi^*_{\mathbf{k}}(\mathbf{r}_3)\psi^*_{\mathbf{p}}(\mathbf{r}_2)] \times \psi^*_{\mathbf{k}}(\mathbf{r}_1)v(\mathbf{r}_1)\psi_{\mathbf{p}}(\mathbf{r}_1)\frac{1}{\sqrt{3}}[\psi_{\mathbf{k}}(\mathbf{r}_2)\psi_{\mathbf{p}}(\mathbf{r}_3) + \psi_{\mathbf{k}}(\mathbf{r}_3)\psi_{\mathbf{p}}(\mathbf{r}_2)]. \tag{1.A.11}$$

If we multiply and divide by $\sqrt{2}$ each of the expressions in square brackets and use the definition of $\Psi^2_{n_{\mathbf{p}}=1,n_{\mathbf{k}}=1}(\mathbf{r}_2,\mathbf{r}_3)$ from Eq. (1.A.1), we obtain

$$\mathcal{M}_3 = \sqrt{2}\sqrt{2}\int d\mathbf{r}_1 \psi^*_{\mathbf{k}}(\mathbf{r}_1)v(\mathbf{r}_1)\psi_{\mathbf{p}}(\mathbf{r}_1) \times \int\int d\mathbf{r}_2 d\mathbf{r}_3 |\Psi^2_{n_{\mathbf{p}}=1,n_{\mathbf{k}}=1}(\mathbf{r}_2,\mathbf{r}_3)|^2. \tag{1.A.12}$$

Since the two-particle wave function is normalized we have

$$\mathcal{M}_3 = \sqrt{2}\sqrt{2}\, v_{\mathbf{kp}}, \tag{1.A.13}$$

where $v_{\mathbf{kp}}$ is defined by Eq. (1.A.5).

Consider next the same process for scattering a single particle from the initial state (Eq. (1.A.1)) via the interaction (Eq. (1.A.4)) to the final state

$$\Psi^N_{n_\mathbf{p}-1,n_\mathbf{q},\ldots,n_\mathbf{k}+1,\ldots}(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=\left[\frac{(n_\mathbf{p}-1)!n_\mathbf{q}!\ldots(n_{\mathbf{k}_1}+1)!\ldots}{N!}\right]^{1/2}$$
$$\times\sum_\mathbf{P}\begin{bmatrix}\psi_\mathbf{p}(\mathbf{r}_1)\psi_\mathbf{p}(\mathbf{r}_2)\ldots\psi_\mathbf{p}(\mathbf{r}_{n_\mathbf{p}-1})\\ \times\psi_\mathbf{q}(\mathbf{r}_{n_\mathbf{p}})\psi_\mathbf{q}(\mathbf{r}_{n_\mathbf{p}+1})\ldots\psi_\mathbf{q}(\mathbf{r}_{n_\mathbf{p}+n_\mathbf{q}-1})\\ \vdots\\ \times\psi_\mathbf{k}(\mathbf{r}_\sigma)\psi_\mathbf{k}(\mathbf{r}_{\sigma+1})\ldots\psi_\mathbf{k}(\mathbf{r}_{\sigma+n_\mathbf{k}})\\ \vdots\end{bmatrix}. \tag{1.A.14}$$

Now, as in the three-particle case, we want to evaluate the matrix element

$$\mathcal{M}_N=\int\ldots\int d\mathbf{r}_1\ldots d\mathbf{r}_N\Psi^{N*}_{n_\mathbf{p}-1,n_\mathbf{q},\ldots,n_\mathbf{k}+1,\ldots}(\mathbf{r}_1,\ldots,\mathbf{r}_N)$$
$$\times\sum_j v(\mathbf{r}_j)\Psi^N_{n_\mathbf{p},n_\mathbf{q},\ldots,n_\mathbf{k},\ldots}(\mathbf{r}_1,\ldots,\mathbf{r}_N). \tag{1.A.15}$$

Again, as in the three-particle case, we recognize that all permutations are identical and replace $\sum_{i=1}^N v(\mathbf{r}_i)$ by $Nv(\mathbf{r}_1)$. Equation (1.A.15) can then be rewritten in terms of $(N-1)$-particle wave functions as

$$\mathcal{M}_N=\int\ldots\int d\mathbf{r}_1\ldots d\mathbf{r}_N\sqrt{\frac{n_\mathbf{k}+1}{N}}\psi^*_\mathbf{k}(\mathbf{r}_1)\Psi^{N-1\,*}_{n_\mathbf{p}-1,n_\mathbf{q},\ldots,n_\mathbf{k},\ldots}(\mathbf{r}_2,\ldots,\mathbf{r}_N)$$
$$\times Nv(\mathbf{r}_1)\sqrt{\frac{n_\mathbf{p}}{N}}\psi_\mathbf{p}(\mathbf{r}_1)\Psi^{N-1}_{n_\mathbf{p}-1,n_\mathbf{q},\ldots,n_\mathbf{k},\ldots}(\mathbf{r}_2,\ldots,\mathbf{r}_N)$$
$$=\int d\mathbf{r}_1\sqrt{\frac{n_\mathbf{k}+1}{N}}\psi^*_\mathbf{k}(\mathbf{r}_1)Nv(\mathbf{r}_1)\psi_\mathbf{p}(\mathbf{r}_1)\sqrt{\frac{n_\mathbf{p}}{N}}. \tag{1.A.16}$$

Thus we see that the multi-particle character of the problem is contained in the $\sqrt{n_\mathbf{p}}$ and $\sqrt{n_\mathbf{k}+1}$ factors, associated with the removal (annihilation) of a particle in state $\psi_\mathbf{p}$ and addition (creation) of a particle in state $\psi_\mathbf{k}$.

It is natural (and much easier!) to introduce a multi-particle state vector

$$|n_\mathbf{p},n_\mathbf{q},\ldots,n_\mathbf{k},\ldots\rangle, \tag{1.A.17}$$

and operators which transform state vectors into one another by changing the numbers of particles in the various states. To this end, we introduce annihilation (or destruction or absorption) operators for our boson system as

$$a_\mathbf{p}|n_\mathbf{p},n_\mathbf{q},\ldots,n_\mathbf{k},\ldots\rangle=\sqrt{n_\mathbf{p}}|n_\mathbf{p}-1,n_\mathbf{q},\ldots,n_\mathbf{k},\ldots\rangle, \tag{1.A.18}$$

and the corresponding creation operators

$$a_{\mathbf{p}}^{\dagger}|n_{\mathbf{p}}, n_{\mathbf{q}}, \ldots, n_{\mathbf{k}}, \ldots\rangle = \sqrt{n_{\mathbf{p}} + 1}\,|n_{\mathbf{p}} + 1, n_{\mathbf{q}}, \ldots, n_{\mathbf{k}}, \ldots\rangle. \tag{1.A.19}$$

From the definitions it is clear that we have the commutation relations

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}] = \delta_{\mathbf{p},\mathbf{p}'}, \tag{1.A.20}$$

and

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}'}^{\dagger}] = 0, \tag{1.A.21}$$

as is apparent from the action of such ordered operations on the state vectors. In order to regain the results of our matrix element calculation we are thus led to introduce the interaction Hamiltonian

$$\mathscr{V} = \sum_{\mathbf{k},\mathbf{p}} a_{\mathbf{p}+\mathbf{k}}^{\dagger} a_{\mathbf{p}} v_{\mathbf{k}} \tag{1.A.22}$$

and the free particle Hamiltonian

$$\mathscr{H} = \sum_{p} \frac{p^2}{2m} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}. \tag{1.A.23}$$

To summarize: the physics is in the occupancy of the number states where information is contained in the states $|n_{\mathbf{p}}\rangle$ and the matrix elements $v_{\mathbf{kp}} = \int d\mathbf{r}\psi_{\mathbf{k}}^{*}(\mathbf{r})v(\mathbf{r})\psi_{\mathbf{p}}(\mathbf{r})$. That is, we never have to worry about complicated combinations, the operator formalism takes care of all that in a very neat way.

The main point of this section, however, is not the convenience of the operator approach but rather the deep connection between many-boson quantum mechanics and that of quantized harmonic oscillators. In the words of Dirac

> The dynamical system consisting of an ensemble of similar bosons is equivalent to the dynamical system consisting of a set of oscillators – the two systems are just the same system looked at from two different points of view. There is one oscillator associated with each independent boson state. We have here one of the most fundamental results of quantum mechanics, which enables a unification of the wave and corpuscular theories of light.

However, as compelling as the ‘boson’ ↔ ‘oscillator set’ comparison is, there are fundamental differences. For example, in the oscillator problem we end up with a vacuum fluctuation contribution that does not appear in the boson collection argument. In Section 1.3 we used this vacuum state of the electromagnetic field to obtain the Lamb shift.

Problems

1.1 The radiation field in an empty cubic cavity of side L satisfies the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0,$$

together with the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. Show that the solution that satisfies the boundary conditions has components

$$A_x(\mathbf{r}, t) = A_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z),$$
$$A_y(\mathbf{r}, t) = A_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z),$$
$$A_z(\mathbf{r}, t) = A_z(t)\sin(k_x x)\sin(k_y y)\cos(k_z z),$$

where $\mathbf{A}(t)$ is independent of position and the wave vector $\mathbf{k}$ has components given by Eq. (1.1.21). Hence show that the integers n_x, n_y, n_z in Eq. (1.1.21) are restricted in that only one of them can be zero at a time.

1.2 If A and B are two noncommuting operators that satisfy the conditions

$$[[A, B], A] = [[A, B], B] = 0,$$

then show that

$$\begin{aligned} e^{A+B} &= e^{-\frac{1}{2}[A,B]}e^A e^B, \\ &= e^{+\frac{1}{2}[A,B]}e^B e^A. \end{aligned}$$

This is a special case of the so-called Baker–Hausdorff theorem of group theory.

1.3 If A and B are two noncommuting operators and α is a parameter, then show that

$$e^{-\alpha A} B e^{\alpha A} = B - \alpha[A, B] + \frac{\alpha^2}{2!}[A, [A, B]] + \ldots$$

1.4 If $f(a, a^\dagger)$ is a function which can be expanded in a power series of a and $a^\dagger$, then show that

(a) $[a, f(a, a^\dagger)] = \frac{\partial f}{\partial a^\dagger}$,

(b) $[a^\dagger, f(a, a^\dagger)] = -\frac{\partial f}{\partial a}$,

(c) $e^{-\alpha a^\dagger a} f(a, a^\dagger) e^{\alpha a^\dagger a} = f(a e^\alpha, a^\dagger e^{-\alpha})$,

where α is a parameter.

1.5 Show that

$$[a, e^{-\alpha a^\dagger a}] = (e^{-\alpha} - 1)e^{-\alpha a^\dagger a}a,$$
$$[a^\dagger, e^{-\alpha a^\dagger a}] = (e^{\alpha} - 1)e^{-\alpha a^\dagger a}a^\dagger,$$

where α is a parameter.

1.6 Show that the free-field Hamiltonian

$$\mathscr{H} = \hbar\nu\left(a^\dagger a + \frac{1}{2}\right)$$

can be written in terms of the number states as

$$\mathscr{H} = \sum_n E_n|n\rangle\langle n|,$$

and hence

$$e^{i\mathscr{H}t/\hbar} = \sum_n e^{iE_nt/\hbar}|n\rangle\langle n|.$$

1.7 Show that Maxwell's equations in free space may be written in the form of Eqs. (1.5.27a) and (1.5.27b) by first showing that

$$\frac{1}{c}\frac{\partial\tilde{\mathbf{E}}}{\partial t} = \nabla \times \tilde{\mathbf{H}}, \qquad \nabla \cdot \tilde{\mathbf{E}} = 0,$$

$$-\frac{1}{c}\frac{\partial\tilde{\mathbf{H}}}{\partial t} = \nabla \times \tilde{\mathbf{E}}, \qquad \nabla \cdot \tilde{\mathbf{H}} = 0,$$

where $\tilde{\mathbf{E}} = \sqrt{\epsilon_0}\,\mathbf{E}$ and $\tilde{\mathbf{H}} = \sqrt{\mu_0}\,\mathbf{H}$. Then prove that

$$\mathbf{s} \cdot \nabla\mathbf{V} = \nabla \times \mathbf{V},$$

$$s_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad s_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$s_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\mathbf{s}$ and $\mathbf{V}$ on the left-hand side are regarded as 1×3 column vectors. Use this identity to obtain Eqs. (1.5.27a) and (1.5.27b).

1.8 Derive the current density (1.5.32) by writing the equations of motion for φ_γ and χ_γ in the form

$$\begin{aligned}
\dot{\varphi}_\gamma &= c\mathbf{s}\cdot\nabla\chi_\gamma, \\
\dot{\chi}_\gamma &= -c\mathbf{s}\cdot\nabla\varphi_\gamma, \qquad \dot{\varphi}_\gamma^\dagger = c\nabla\chi_\gamma^\dagger\cdot\mathbf{s}^\dagger, \\
\dot{\chi}_\gamma^\dagger &= -c\nabla\varphi_\gamma^\dagger\cdot\mathbf{s}^\dagger,
\end{aligned}$$

and noting that $\mathbf{s}^\dagger = -\mathbf{s}$.

1.9 Verify that $\sum_i \hat{\mathbf{e}}_i\hat{\mathbf{e}}_i = \mathbf{1}$ by taking the dot product with any vector $\mathbf{v}$. Thus if $\hat{\mathbf{e}}_1 = \hat{\epsilon}_{\mathbf{k}}^{(1)}$, $\hat{\mathbf{e}}_2 = \hat{\epsilon}_{\mathbf{k}}^{(2)}$, and $\hat{\mathbf{e}}_3 = \mathbf{k}/k$ we have equation (1.1.36). It is also possible to prove (1.1.37) by letting k, θ, ϕ be the polar coordinates of the wave vector $\mathbf{k}$, so that

$$\mathbf{k} \equiv k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

The two transverse polarization vectors can then be represented by

$$\begin{aligned}
\hat{\epsilon}_{\mathbf{k}}^{(1)} &\equiv (\sin\phi, -\cos\phi, 0), \\
\hat{\epsilon}_{\mathbf{k}}^{(2)} &\equiv (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta),
\end{aligned}$$

and it can be verified that

$$\epsilon_{\mathbf{k}i}^{(1)}\epsilon_{\mathbf{k}j}^{(1)} + \epsilon_{\mathbf{k}i}^{(2)}\epsilon_{\mathbf{k}j}^{(2)} = \delta_{ij} - \frac{k_i k_j}{k^2},$$

where i, j represent the Cartesian components. Demonstrate this by direct substitution.

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