Lecture 11

Quantum Optics

Quantum Measurements and Measurement Quantum Backaction (QND-Measurement)

Consider simple example of measurement BACKACTION / STANDARD QUANTUM LIMIT.

(I) consider a "free mass" $\hat{H}_0 = \frac{\hat{p}^2}{2m}$

Heisenberg uncertainty states $\langle\Delta\hat{p}^2\rangle\langle\Delta\hat{x}^2\rangle \geq \hbar^2/4$

~~definition~~ $[\hat{x},\hat{p}] = i\hbar$

NOTE: definition of $x_{0}$ for free mass? $\sqrt{\frac{\hbar}{2m\omega}}(b+b^\dagger)$? → not a H.O. ($\hat{x}$) basis (plane waves!)

Free evolution: $\frac{d}{dt}\hat{x} = \frac{1}{i\hbar}[\hat{x},\hat{H}] = \hat{p}/m$

$$\Rightarrow \hat{x}(T) = \hat{x}(0) + \frac{\hat{p}(0)T}{m}$$

Thus: $$\langle\Delta\hat{x}(T)^2\rangle = \langle\Delta\hat{x}(0)^2\rangle + \frac{\langle\hat{p}(0)^2\rangle T^2}{m^2} + \underbrace{\langle\Delta\hat{x}(0)\Delta\hat{p}(0) + \Delta p(0)\Delta x(0)\rangle\frac{T}{m}}_{\text{correlations in } \hat{x},\hat{p}}$$

→ neglected here.

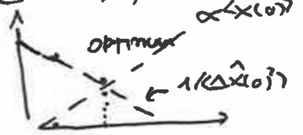
utilize the uncertain principle principle:

$$\Delta\hat{x}(T)^2\rangle \geq \langle\Delta\hat{x}(0)^2\rangle + \left[\langle\Delta x(0)^2\rangle/(2\hbar)^2\right]^{-1}\frac{T^2}{m^2}$$

$$\to \boxed{\langle\Delta\hat{x}(T)^2\rangle \geq \langle\Delta\hat{x}(0)^2\rangle + \frac{\hbar^2}{4m^2\langle\Delta x(0)^2\rangle}\cdot T^2}$$

Key point: measurement at a later time (ie 2nd measurement) yields a result that is different, ie increases with $\propto \frac{1}{\langle\Delta x(0)^2\rangle}$

One can minimize the expression:




$$\boxed{\Delta x^2(T) \geq \frac{\hbar T}{m}}$$

FREE MASS STANDARD QUANTUM LIMIT

Concrete exp. setting: gravitational wave detection

event durations ~ 1 ms  mass ~ 4 kg → yield $\langle\Delta x^2(T)\rangle \sim 10^{-34}\cdot\frac{10^{-3}}{O(1)} \sim 10^{-37}$

→ requirement on the detector  $\langle\Delta x^2\rangle^{1/2} \sim 10^{-18}\,m$

ultimate sensitivity for linear measurement.

NOTE: Energy (ie $|n\rangle$ Fock states) and momentum $|p\rangle$ ie $\psi(r) \propto e^{ip\cdot r/\hbar}$ are eigenstates of $\hat{H}_0$ and repeated measurement of an eigenstate give same results. (But: not superpositions! $(|0\rangle + |3\rangle) \neq |4\rangle$) $\hat{H}_0|4\rangle \neq |4\rangle$ )

Quantum Optics
Lecture 11

## HARMONIC OSCILLATOR QUANTUM LIMIT

(II) consider $\hat{H}_0 = \hat{p}^2/2m + m\Omega_m^2 \hat{x}^2/2$ $\Omega_m$: eigenfrequency.

Solution:

$$[\hat{x}(0), \hat{x}(T)] = \frac{i\hbar}{m\omega} \cdot \sin(\omega T) \qquad \langle\hat{x}(0)^2\rangle\langle\hat{p}(0)^2\rangle \geq \frac{\hbar^2}{4}$$

ie: $\hat{x}(T) = \hat{x}(0)\cos(\omega T) + \hat{p}(0)\frac{1}{m\omega}\sin(\omega T)$

ie $\dot{\hat{x}} = \frac{1}{i\hbar}[\hat{x}, \hat{H}_0]$ $\hat{x}(T) = \int_0^T \dot{\hat{x}}\,dt$

Hence $[\hat{x}(0), \hat{x}(0)\cos(\omega T) + \hat{p}(0)\frac{1}{m\omega}\sin(\omega T)] =$

$$[\hat{x}(0), \hat{p}(0)\frac{1}{m\omega}\sin(\omega T)] = \frac{i\hbar}{m\omega}\sin(\omega T)$$

Therefore when measuring at a later time, at 1/2 period $(\omega T = \pi/2)$

one obtains: $T = \frac{\pi}{2\omega}$

$\langle\hat{x}^2(T)^2\rangle \geq \left(\frac{\hbar}{2m\omega}\right)$ ie <u>zero point motion</u> $(\omega T = \pi/2)$

$$\boxed{\text{ie } \langle x(T)^2\rangle^{1/2} \geq \sqrt{\frac{\hbar}{2m\omega}}}$$

Limit for measurements of a H.O. separated by $\omega T = \pi/2$ (periodic measurement)

H.O. SQL.

(note: velocity and position also cannot be measured simultaneously), since: uncertainty principle holds.

SQL: refers to:

$$\langle\Delta\hat{X}_1^2\rangle = \langle\Delta\hat{X}_2^2\rangle = \frac{\hbar}{2m\omega}$$

→ EQUAL UNCERTAINTY.

$\hat{X}_1 = \hat{x}$
$\hat{X}_2 = \hat{p}/m\omega$
$\hat{a} = \hat{X}_1 + i\hat{X}_2 = \hat{x} + i\hat{p}$

$$\langle\Delta\hat{x}^2\rangle\langle\Delta\hat{p}^2\rangle = \hbar^2/4$$

(precision on $\langle\Delta\hat{x}^2\rangle$ causes uncertainty in $\langle\Delta\hat{p}^2\rangle$ → backaction.

How can this be avoided? → Topic of QND measurements.

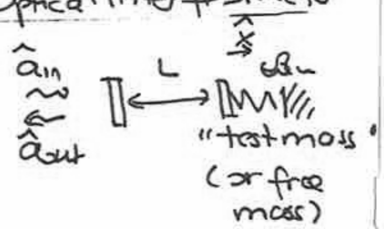
Note

$\Delta x^2(T) =$

Quantum Optics

– Lecture 11 –

Concrete historical example: (Quantum) Optical interferometer for grav. wave detection.




Coupling of mass to cavity is described by

$$\omega_c = m \cdot \frac{c}{2L} \qquad m \in \mathbb{N} \qquad L = L + \hat{x}$$

$$= m \cdot \frac{c}{2(L+\hat{x})} \simeq m \cdot \frac{c}{2L(1-\hat{x}/L)} \cong \frac{mc}{2L}\left(1-\frac{\hat{x}}{L}\right) = \omega_c - \frac{\hat{x}}{L}\omega_c$$

Thus the Hamiltonian is:

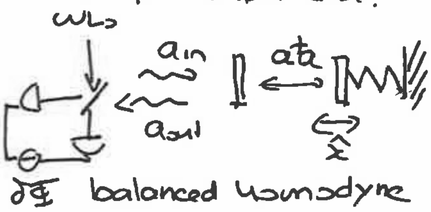
$$\hat{H} = \hbar\omega_c \hat{a}^\dagger\hat{a} + \hbar\Omega_m \hat{b}^\dagger\hat{b} = \hbar\left(\omega_c - \frac{\hat{x}}{L}\omega_c\right)\hat{a}^\dagger\hat{a} + \hbar\Omega_m \hat{b}^\dagger\hat{b}$$

$$\boxed{\hat{H}_{int} = \frac{\hbar\omega_c}{L}\hat{x}\,(\hat{a}^\dagger\hat{a}) = \frac{\hbar\omega_c}{L}(\hat{b}+\hat{b}^\dagger)\,x_{ZPF}\cdot\hat{a}^\dagger\hat{a}}$$

PONDEROMOTIVE COUPLING

Note: a classical force can be added $\hat{H}_{classical} = \underbrace{F(t)}_{\text{force on mass / oscillator.}}\cdot\hat{x}$

Measurement scheme: similar to the qubit/atom case, ie measure reflected phase




$$r(\omega) = \frac{\cancel{\Delta}\ \omega-\omega_c-i\kappa/2}{\omega-\omega_c+i\kappa/2} = \frac{+\hat{x}/L\,\omega_c - i\kappa/2}{+\hat{x}/L\,\omega_c + i\kappa/2} = \frac{\cancel{\hat{x}\ldots/2} - i\frac{\hat{x}}{L}\frac{\omega}{\kappa} - 1}{\cancel{\hat{x}\ldots/2} - i\frac{\hat{x}}{L}\frac{\omega}{\kappa} + 1}$$

Phase shift $\boxed{\theta \simeq 2\cdot\frac{\hat{x}}{L}\frac{\omega}{\kappa} \quad \text{Phase shift of reflected field } \theta \propto \hat{x}.}$

Quantum Langevin Equations of Harmonic Osc. coupled to cavity.

$$\boxed{\hat{H} = \hbar\omega_c \hat{a}^\dagger\hat{a} + \hat{p}^2/2m + \hat{q}^2 m\Omega^2/2 + \hbar\omega_c/L\cdot\hat{q}\cdot\hat{a}^\dagger\hat{a}} \quad \underbrace{\ }_{\text{Drive}}$$

$$\frac{d}{dt}\hat{a} = -\frac{\kappa}{2}\hat{a} + i\omega_c\hat{a} + \underbrace{\left(i\omega_c\frac{\hat{x}}{L}\right)}_{\text{mech. resonance shift}}\hat{a} + \underbrace{\sqrt{\kappa}\cdot\hat{a}_{in}}_{\text{input drive and noise}} + \hat{a}_{in}e^{\ldots} \qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b}+\hat{b}^\dagger)$$

input noise

$$\frac{d}{dt}\hat{q} = \frac{1}{i\hbar}[\hat{H},\hat{q}] = -i\Omega\hat{p} - \Gamma_m/2\,\hat{q} + \underbrace{\sqrt{\Gamma_m}\,\hat{q}_{in}}_{\text{input noise (thermal noise)}}$$

$$\frac{d}{dt}\hat{p} = \frac{i}{i\hbar}[\hat{H},\hat{p}] = -im\Omega\hat{q} + \underbrace{i\frac{\omega_c}{L}}_{\text{rad pressure force}}\hat{a}^\dagger\hat{a} + \underbrace{\sqrt{\Gamma_m}\,\hat{p}_{in}}_{\ } - \underbrace{\Gamma_m/2\,p}_{\text{decay}}$$

$G = \frac{\omega_c}{L}$ : force per photon

Quantum Optics_Lecture 11 –

Equivalently:

$$\left| \begin{aligned} \partial_t \hat{a} &= -i\omega_c \hat{a} - \kappa/2\, \hat{a} + \sqrt{\kappa}\, \underbrace{a_{in}}_{\text{drive } s e^{-i\omega_L t}} + i\frac{\omega}{L}\underbrace{(\hat{b}+\hat{b}^+)}_{\text{position } q}\hat{a}\, x_{ZPF} \\ \partial_t \hat{b} &= -i\Omega_m \hat{b} - \frac{\Gamma_m}{2}\hat{b} + \sqrt{\Gamma_m}\, \hat{b}_{in} + \underbrace{i\frac{\omega}{L}\hat{a}^+ a}_{\text{Radiation pressure force}} \end{aligned} \right|$$

First move to rotating frame!
$\hat{a} \to \hat{a} e^{-i\omega_L t}$ slowly varying.

Linearization of equations:

$\hat{a} \to \bar{\alpha} + \delta\hat{a} \qquad (\bar{\alpha} + \hat{a}) \qquad a_{in} = \underbrace{\bar{a}_{in}}_{\text{drive}} + \delta a_{in}$

$\hat{b} \to \bar{b} + \delta\hat{b} \qquad (\bar{b} + \hat{b})$

This leads to: $\hat{a}^+ a = (\bar{\alpha}^* + \delta\hat{a}^+)(\bar{\alpha} + \delta a) \qquad \alpha \in \mathbb{R}$

$= \alpha(\delta a^+ + \delta a)$

$\hat{=}\ \alpha\underbrace{(a + a^+)}_{\text{amplitude quadrature.}}$

choose input phase such that $\alpha \in \mathbb{R}$.
$a_{in} = \bar{a}_{in} e^{i\varphi} e^{i\omega_L t}$ Phase.

LINEARIZATION

we obtain these equations

$$\left| \begin{aligned} \partial_t \hat{a} &= -\kappa/2\, \hat{a} + \sqrt{\kappa}\, \hat{a}_{in} + i\frac{\omega}{L} x_{ZPF} (\hat{b}+\hat{b}^+)\bar{\alpha} \\ \partial_t \hat{b} &= -\Gamma_m/2\, \hat{b} + \underbrace{\sqrt{\Gamma_m}\, \hat{b}_{in}}_{\text{input noises}} + i\frac{\omega}{L}\underbrace{(\delta a + \delta a^+)}_{\text{only the amplitude quadrature drives now}}\bar{\alpha} \end{aligned} \right| \quad (*_1)$$

This is the slowly varying operator!

Note and recognize

$X = (\hat{a} e^{+i\omega t} + \hat{a}^+ e^{-i\omega t})$ : AM quadrature ($\hat{=}\ a + \hat{a}^+$) here, as $a$ is slowly varying. $\hat{a}$: original operator.

$Y = (\hat{a} e^{+i\omega t} - \hat{a}^+ e^{-i\omega t})$ : Phase quadrature. ($\hat{=}\ a - \hat{a}^+$) here as a rotation.

Therefore it is evident that $(*_1)$ only causes the ~~note~~: PHASE quadrature to be coupled: $\underbrace{}_{\hat{=}\ g_0 \bar{\alpha}}$

$$\left| \partial_t (\hat{a}^+ \hat{a}) = \partial_t \cancel{\hat{Y}} = -\kappa/2\, \cancel{Y} + \sqrt{\kappa}\, Y_{in} + \frac{i\omega_c}{L x_{ZPF}}\bar{\alpha} \cdot \underbrace{(\hat{b}+\hat{b}^+)}_{q\text{ : mech position}} \right| \quad \text{PHASE}$$

→ mech. position is encoded into the Phase of the light → allows measurement.

$$\text{But} \left| \partial_t (a + a^+) = \partial_t \hat{X} = -\kappa/2\, \hat{X} + \sqrt{\kappa}\, \hat{X}_{in} \right|$$

AMPLITUDE decoupled from mech oscillator

inspecting the mechanical oscillator position $\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b} + \hat{b}^+)$:

$$\left| \partial_t \hat{q} = -\Gamma_m/2\, \hat{q} + \underbrace{\sqrt{\Gamma_m}\, q_{in}}_{\text{input thermal Noise}} + i\frac{\omega}{L} 2\sqrt{\frac{\hbar}{2m\omega}} \cdot \bar{\alpha}\underbrace{(\delta\hat{a} + \delta\hat{a}^+)}_{\text{amplitude quadrature}} \right|$$

Lecture 11

– Quantum Optics –

First let's consider the output field:

Note that $\boxed{\hat{a}_{out} = \hat{a}_{in} + \sqrt{\kappa}\,\hat{a}}$

Therefore we have $\delta\hat{X}_{out} = \delta\hat{X}_{in} + \sqrt{\kappa}\,\hat{X}_1$ for the quadratures too!

Consequentially; first move to the Fourier domain:

$$\boxed{\hat{a}_{in}(\omega) = \int_{-\infty}^{\infty} \hat{a}_{in}(t)\, e^{-i\omega t}}$$ Fourier operator.

$\hat{X} = (a + a^{+})/\sqrt{2}$ Phase

$\hat{Y} = (a - a^{+})/\sqrt{2}$ Ampl

Thus: $\partial_t \hat{X} = -\kappa/2\,\hat{x} + \sqrt{\kappa}\,\hat{X}_{in} + \frac{i\omega_c}{L}\bar{\alpha}\, x_{ZPF}\,(\hat{b} + \hat{b}^{+})$ (should be $\hat{Y}$)

$$-i\omega\hat{X} = -\kappa/2\,\hat{X} + \sqrt{\kappa}\,\hat{X}_{in} + \frac{i\omega_c}{L} x_{ZPF}\,\bar{\alpha}\,\underbrace{(b + b^{+})}_{b(\omega) + b^{+}(\omega)}$$

$$\hat{X}_{out}(\omega) = \hat{X}_{in} + \sqrt{\kappa}\,\hat{X}(\omega)$$

$b(\omega) + b^{+}(\omega)$ Fourier operators.

Fourier operators

thus the output quadrature is (phase):

$$\hat{X}_{out}(\omega) = \hat{X}_{in}(\omega) + \sqrt{\kappa}\cdot\frac{(\hat{X}_{in}\sqrt{\kappa} + i\omega_c/L\, x_{ZPF}\,\bar{\alpha}\,(\hat{b} + \hat{b}^{+}))}{(-i\omega + \kappa/2)}$$ Fourier operators.

ASSUME bad cavity limit $\kappa \gg \omega$

$$\Rightarrow \hat{X}_{out} \cong -\hat{X}_{in} + \underbrace{\frac{i\omega_c}{L} x_{ZPF}\,\bar{\alpha}}_{x_{ZPF}\, G\,\bar{\alpha} = g_0\sqrt{n_c}} \cdot \frac{2}{\sqrt{\kappa}}\,(b + b^{+})$$

we define the measurement rate

$$\boxed{\Gamma_{meas} \equiv \frac{4\,(g_0^2\,\bar{n}_c)}{\kappa}}$$ Measurement rate.

$$\boxed{\hat{X}_{out} = -\hat{X}_{in} - \sqrt{\Gamma_{meas}}\cdot(b(\omega) + b^{+}(\omega))}$$ output phase quadrature of the light.

PHASE QUADRATURE (should be $\hat{Y}_{out}$!)

From this we can compute the output spectrum via the: Wiener Khinchin theorem:

$$S_{out} \equiv \int \langle \hat{X}_{out}^{+}(0)\,\hat{X}_{out}(t)\rangle\, e^{-i\omega t}\, dt$$ Spectral density of photocurrent

Input noise correlators. (time domain)

$$\langle \delta a_{in}(t)\,\delta a_{in}^{+}(t')\rangle = \delta(t - t')$$

$$\langle \delta a_{in}^{+}(t)\,\delta a_{in}(t')\rangle = 0$$

$$\boxed{\hat{X}_{out}(\omega) = -\hat{X}_{in}(\omega)\,\frac{(4 g_0^2\,\bar{\alpha}^2)^{1/2}}{\sqrt{\kappa}\, x_{ZPF}}\cdot\hat{q}(\omega)}$$

if $\langle \hat{X}_{in}(t)\,\hat{q}(t)\rangle = 0$ no correlations

$\langle \hat{X}_{in}(\omega)\,\hat{q}(\omega)\rangle = 0$

$g_0 \hat{=} G\cdot x_{ZPF}$

$$\rightarrow \boxed{S_{X_{out}X_{out}}(\omega) = \underbrace{1}_{\substack{\text{shot noise / quantum phase}\\ S_{\hat{X}_{in}\hat{X}_{in}}}} + \underbrace{\frac{(4 g_0^2\,|\bar{\alpha}|^2)}{\kappa^2\, x_{ZPF}^2}\cdot S_{\hat{q}\hat{q}}(\omega)}_{\text{signal}}}$$

$$\Rightarrow S_{qq}^{imp}(\omega) \equiv \left(\frac{\kappa}{8}\left(\frac{x_{ZPF}}{g_0\bar{\alpha}}\right)\right)^2 S_{\hat{X}_{in}\hat{X}_{in}}$$

defined as SNR equal to one.

Quantum Optics

— Lecture 11 —

Mechanical oscillator: (1st order eq.)

$$\boxed{\partial_t \hat{q} = -\Gamma_m/2\, \hat{q} + \sqrt{\Gamma_m}\, q_{in} + 2i\frac{\omega_c}{L} x_{ZPF}\, \underbrace{\bar{\alpha}(\delta\hat{a}+\delta\hat{a}^+)}_{\text{quantum noise driving the mech oscillator}} + i\Omega_m \hat{q}}$$

Force driving oscillator

Solve in Fourier domain

$$\hat{q}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\hat{q}(t)\,dt$$

intracavity fluctuations

$$\Rightarrow \boxed{\hat{q}(\omega) = \Big(\underbrace{\frac{1}{-i\underbrace{\omega}_{+\Omega_m} + \Gamma/2}}_{\chi_m(\omega)}\Big)\Big(\sqrt{\Gamma_m}\,\hat{q}_{in} - i\frac{\omega_c}{L}2x_{ZPF}\,\overset{\sim}{\alpha}\,\underbrace{(\delta a + \delta\hat{a}^+)}_{\text{AMPLITUDE}}\Big)}$$

~~susceptibility~~! $\quad \chi_m(\omega) \hat{=} \frac{1}{-i(\omega-\Omega_m)+\Gamma/2}$ SUSCEPTIBILITY: $\quad \hat{X}_A \quad \hat{q}(\omega) = \chi_m(\omega)\cdot \underbrace{F(\omega)}_{\text{force.}}$

Thus: when computing the SPECTRAL DENSITY: (Power spectral density)

$$\boxed{\begin{aligned} S_{\hat{q}\hat{q}}(\omega) &\equiv \lim_{T\to\infty} \frac{1}{T}\langle O_T^+(\omega)\, O_T(\omega)\rangle \\ S_{\hat{q}\hat{q}}(\omega) &= \int d\tau\, e^{+i\omega\tau}\langle \hat{O}^+(\tau)\hat{O}(0)\rangle \hat{=} \int_{-\infty}^{\infty} d\omega \langle \hat{O}^+(-\omega)\hat{O}(\omega)\rangle \end{aligned}}$$

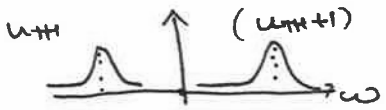
This yields:

$$\boxed{S_{\hat{q}\hat{q}}(\omega) = |\chi_m(\omega)|^2\, 2\Gamma_m\Big(\underbrace{S_{\hat{q}_{in}\hat{q}_{in}}}_{\text{input thermal noise} \to \text{thermal occupancy.}} + \overbrace{\Big|\frac{\omega_c}{L}\Big|^2 4\, x_{ZPF}^2\, \bar{\alpha}^2}^{\equiv C_{eff}} \cdot \underbrace{S_{\hat{X}_A\hat{X}_A}}_{\text{fluctuations of the intracavity field! Amplitude quadrature}}\Big)}$$

$$S_{\hat{q}\hat{q}}(\omega) = 2\Gamma_m |\chi(\omega)|^2 (\bar{n}_{TH} + C_{eff} + 1)$$
$$S_{\hat{q}\hat{q}}(-\omega) = 2\Gamma_m |\chi(\omega)|^2 (\bar{n}_{TH} + \underbrace{C_{eff}}_{\text{quantum backaction}})$$

} assymmetric Noise spectral density.

→ driving the resonator.




→ Measurement backaction $n_{Added} = C_{eff}$.

$\hat{F}_L$ = force on the oscillator:

$$S_{\hat{F}\hat{F}} = \frac{8}{\kappa}\left(\frac{\hbar\,\omega/L\cdot x_{ZPF}\bar{\alpha}}{x_{ZPF}}\right)^2 \cdot \underbrace{S_{\hat{X}_{in}\hat{X}_{in}}}$$

| expressed as input noise! not $S_{\hat{X}\hat{X}}$ intracavity!

$$S_{\hat{X}\hat{X}} = \left(\frac{2}{\sqrt{\kappa}}\right)^2 S_{\hat{X}_{in}\hat{X}_{in}}$$

Key result:

$$\boxed{S_{\hat{F}\hat{F}} \cdot S^{imp}_{\hat{X}\hat{X}} = \frac{\hbar^2}{4} \quad \text{Heisenberg uncertainty product.}}$$

i.e.

$$(\sqrt{\kappa})^2 S_{\hat{X}\hat{X}_{in}} \hat{=} S_{XX}$$

$$\hat{X} = \frac{\hat{X}_{in}\sqrt{\kappa}}{i(\omega-\omega_c)-\kappa/2}$$

$$\Rightarrow \hat{X} = 2\hat{X}_{in}/\sqrt{\kappa}$$

Quantum Optics - Lecture 11 -

**Equivalently:** (express in operator $\hat{b}$)

$$\boxed{d_t \hat{b} = -i\Omega_m \hat{b} - \underbrace{\frac{\Gamma_m}{2}\hat{b}}_{\text{decay}} - \underbrace{\sqrt{\Gamma_m}\hat{b}_{in}}_{\text{input noise}} - i\, \underbrace{g}_{\frac{\omega_c}{L}} x_{ZPF} \cdot \hat{a}^+ \hat{a}}$$ Mech Mode eq.

(and eq. for $\hat{b}^+$)

The equations of motion can be solved by linearization.

Note: the equation for $\hat{p}$ ∧ $\hat{q}$ can be written as second order equation in $\hat{q}$:

$$\ddot{\hat{Q}} + \Gamma \dot{\hat{Q}} + m\Omega^2 \hat{Q} = \sqrt{2\Gamma}\Omega_m \hat{P}_{in} - 2\left(\frac{\omega_c}{L}\right)\cdot q$$

( $Q$ : normalized units $\quad Q = \sqrt{\frac{\hbar}{2m\Omega_m}}^{-1} \hat{q}$ dim.less )

LINEARIZATION $\quad \hat{a} \to \alpha + d\hat{a} \triangleq \alpha + \underbrace{\hat{a}}_{\text{linear part}}$

Define first the quadratures:

$$\boxed{\begin{aligned} \hat{x} &\equiv (\hat{a}e^{i\omega t} + \hat{a}^+)\tfrac{1}{\sqrt{2}} \\ \hat{y} &\equiv (\hat{a}e^{i\omega t} - \hat{a}^+)\tfrac{1}{\sqrt{2}i} \end{aligned}}$$

Amplitude

Phase-Quadrature.

time dependence!

$\hat{x} = \hat{a}e^{+i\omega t} + \hat{a}^+ e^{-i\omega t}$

Thus: $\quad$ ~~$d_t\hat{x} = -\kappa/2$~~ $\quad \partial_t \hat{a} = -\frac{\kappa}{2}\hat{a} - i\omega_c\hat{a} + i\frac{\omega_c}{L}\hat{x}\cdot\hat{a} + \sqrt{\kappa}(\hat{a}_{in} + \bar{a})e^{-i\omega_L t}$

~~$d_t\hat{y} = -\kappa/2$~~

assume $\omega_L = \omega_c$ $\quad = (\partial_t \hat{a} + i\omega_c \hat{a})e^{-i\omega_c t} = -\frac{\kappa}{2}\hat{a}e^{-i\omega_c t} + \sqrt{\kappa}\,\bar{a}e^{-i\omega t} + i\omega_c \hat{a}e^{-i\omega_c t} + i\frac{\omega_c}{L}\hat{x}\hat{a}e^{-i\omega_c t}$

$\hat{a}' \to \hat{a}e^{+i\omega_c t}$ $\quad$ ~~$\partial_t\hat{a}$~~ $\quad i\omega_c/L\,\hat{x}\hat{a} = i\omega_c/L\,\hat{x}\bar{\alpha}$

substitution $\quad \omega = \omega_c$

$$\boxed{\partial_t \hat{a} = -\kappa/2\,\hat{a} + \sqrt{\kappa}(\hat{a}_{in} + \bar{a}) + i\frac{\omega_c}{L}\hat{x}\,\hat{a}}$$ in Rotating frame with $\bar{a} \to \bar{a}e^{-i\omega_c t}$

Rotating frame has been chosen by: $\quad \hat{a}(t) \equiv \underbrace{\hat{a}(t)}_{\text{slowly varying}} e^{-i\omega_c t}$

Thus

$$\hat{x} = (\hat{a} + \hat{a}^+)\frac{1}{\sqrt{2}}$$
$$\hat{y} = (\underbrace{a}_{\text{slowly varying part}} - \hat{a}^+)\frac{1}{\sqrt{2}i}$$

or

$$\hat{x} = \hat{a}e^{+i\omega_c t} + \hat{a}^+ e^{-i\omega_c t}$$
$$\hat{y} = a e^{+i\omega_c t} + \hat{a}^+ e^{+i\omega_c t}$$

This leads to the equation:

optical field quadratures

$$\frac{d}{dt}\hat{x} = -\kappa/2\,\hat{x} + \sqrt{\kappa}\,\hat{x}_{in} \quad \wedge \quad \hat{x}_{in} = (\hat{a}_{in} + \hat{a}^+_{in})\frac{1}{\sqrt{2}}$$

no coupling to $\hat{x}$ → only to phase quadrature

$$\frac{d}{dt}\hat{y} = -\kappa/2\,\hat{y} + \sqrt{\kappa}\,\hat{y}_{in} - 2\left(\frac{\omega_c}{L}\right)\hat{Q}\cdot\bar{\alpha}$$

$$\frac{d}{dt^2}\hat{Q}^2 + \Gamma_m\dot{\hat{q}} + \Omega_m^2\hat{q} = \underbrace{\sqrt{2\Gamma}\,\Omega\,\hat{P}_{in}}_{\text{input noise}} - \underbrace{2g\Omega\,\hat{x}}_{\text{OPT. coupl.}}$$

amplitude noise drives mech. oscillator.