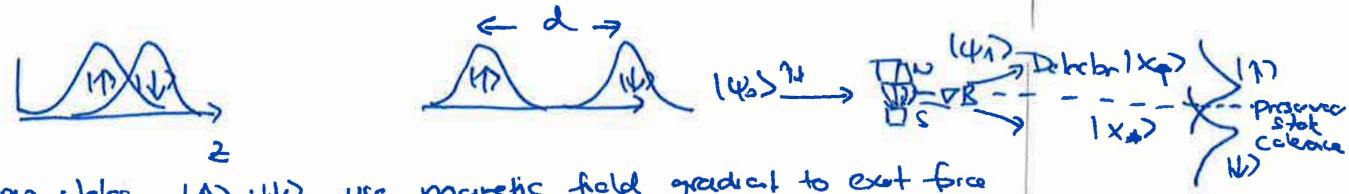
Quantum Optics Lecture 10

Quantum Measurements (basic considerations)

Consider simple example: Stern Gerlach experiment
example of a weak / strong projective measurement.



spin states $|\uparrow\rangle; |\downarrow\rangle$ use magnetic field gradient to exert force
$F \propto (\nabla \cdot B) \cdot \sigma_z$

$$|\psi_0\rangle = \underbrace{(|\uparrow\rangle + |\downarrow\rangle)\tfrac{1}{\sqrt{2}}}_{\text{spin}} \otimes |x_0\rangle \quad \text{Position eigenstate}$$

evolution by magnetic field:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|x_+\rangle + |\downarrow\rangle|x_-\rangle)$$ ie entanglement of spin and motional degree of freedom.

and $\langle r|x_+\rangle = \psi_+(x)$ is wavefunction of the atoms

if $d \gg$ spread of $|\psi_\pm(x)|^2$ we talk about <u>STRONG PROJECTIVE</u> measurement. why?

consider. $\langle\psi_0|\hat{\sigma}_x|\psi_0\rangle = 1$ for initial state $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ +1 & 0 \end{pmatrix}$

but final $\langle\psi_1|\hat{\sigma}_x|\psi_1\rangle = \frac{1}{2}(\langle x_-|x_+\rangle + \langle x_+|x_-\rangle)$

evidently $\langle x_-|x_+\rangle \to 0$ for $d \gg$ spread of $|\psi_\pm|^2$
→ PROJECTIVE

For both distributions overlapping

$$\langle\psi_1|\hat{\sigma}_x|\psi_1\rangle = 1/2 \cdot 2 = 1$$ ie spin coherence is preserved!
→ weak measurement

The destruction of the state can be viewed as a consequence of the HEISENBERG uncertainty principle.

Quantum Optics
Quantum measurements

Brief review cQED dispersive limit for $\{|e,n\rangle ; |g,n+1\rangle\}$ manifold we have $\Omega_n = \Omega_0 \sqrt{n+1}$

i.e. $\left| \begin{array}{l} (n+\frac{1}{2})\hbar\omega_c - \frac{\hbar\omega_a}{2} = (n+\frac{1}{2})\hbar\omega_c - \frac{\Delta_c \hbar}{2} \\ (n)\hbar\omega_c + \frac{\hbar\omega_a}{2} = (n+\frac{1}{2})\hbar\omega_c + \frac{\Delta_c \hbar}{2} \end{array} \right|$

$$\left| E_n^{\pm} = (n+\tfrac{1}{2})\hbar\omega_c \pm \frac{\hbar}{2}\sqrt{\Delta_c^2 + \Omega_n^2} \right|$$

or:

$$E_n^{+} \hat{=} (n)\hbar\omega_c + \hbar\omega_a/2 \left(+ \hbar/2\sqrt{\Delta_c^2+\Omega_n^2} - \frac{\hbar}{2}\Delta_c\right)$$

$$E_n^{-} \hat{=} (n+1)\hbar\omega_c - \hbar\omega_a/2 \left(+ \hbar/2\sqrt{\Delta_c^2+\Omega_n^2} + \frac{\hbar}{2}\Delta_c\right)$$

$$\hat{=} -\frac{\hbar}{2}(\omega_a + \Delta) \qquad \text{where } \Delta_c = \omega_a - \omega_c$$

Thus the levels shift according to:

Eigenstates: $E_n^{\pm} = (n+\frac{1}{2})\hbar\omega_c \pm \hbar\left(\frac{\Delta_c}{2} + \underbrace{\frac{\Omega_n^2}{4\Delta_c}}_{\frac{g^2}{\Delta_c}(n+1)}\right)$ $\quad \Omega_n^2 \hat{=} 4g^2(n+1)$

$\left| \begin{array}{l} E_{e,n} = -\hbar\omega_a/2 + n\hbar\omega - \hbar n g^2/\Delta \\ E_{g,n} = \frac{\hbar\omega_a}{2} + n\hbar\omega + \hbar(n+1)\frac{g^2}{\Delta} \end{array} \right|$

Thus: Energy shift of $|e,n\rangle$ and $|g,n\rangle$ are:

$$\Delta_{e,n} = \hbar(n+1)\frac{g^2}{\Delta_c} \qquad \Delta_{g,n} = -\hbar n \frac{g^2}{\Delta_c}$$

Note: n+1 discrepancy...

ie

Atom energy shift ω_q:

$$\left| \omega_q = \frac{1}{\hbar}(E_n^{+} - E_n^{-}) = (2n+1)\underbrace{\frac{g^2}{\Delta_c}}_{\text{Lamb shift}} \right| \quad \text{AC-stark shift}$$

where $\frac{g^2}{\Delta_c}$ is LAMB SHIFT.

Important note: how is Energy conserved?

ie energy changes → KIN ENERGY CHANGES: FORCES BY LIGHT.

photon energy shift

$$\left| \omega_{cav} = \frac{1}{\hbar}(E_n^{-} - E_{n+1}^{-}) = \hbar\left(\omega_c - \frac{g^2}{\Delta}\right) \right| \text{ for } |g\rangle$$
$$(\omega_c + g^2/\Delta) \text{ for } |e\rangle$$

→ as in a refractive index shift

→ dispersive shift.

→ DISPERSIVE LIMIT.

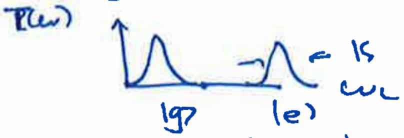
$\omega_L \rightsquigarrow \left[\!\left[\frac{|e\rangle}{|g\rangle} \right]\!\right] \rightsquigarrow T(\omega)$ (Transmission)

Consider two cases:

weak measurement

weak measurements

strong measurement



→ strong dispersive measurements

"Cavity Casimir effect"

→ VACUUM PULLS CAVITY IN $|g\rangle$ into cavity!

Next consider we make a measurement of the QUBIT via the cavity: Linear continuous dispersive quantum measurement

First define & probe cavity response:

$\hat{a}_{in}$, $\hat{a}_{out}$ $\rightleftarrows [\![\;]\!]$

$$\hat{H}_0 = \hbar\omega_c(1 + \sigma_z g^2/\Delta_c)\hat{a}^\dagger\hat{a}$$

$$\Rightarrow \left| \frac{\hat{a}_{out}}{\hat{a}_{in}} = R(\omega) = -\left(\frac{1 + 2i\, g^2/\Delta\hat{\sigma}_z}{1 - 2i\, g^2/\Delta\hat{\sigma}_z}\right) \right|$$

$$\left| \begin{array}{l} \hat{a}_{out} + \hat{a}_{in} = \sqrt{\kappa}\,\hat{a} \\ \frac{d}{dt}\hat{a} = -i\omega_c - \kappa/2\,\hat{a} + \left(\sigma_z \frac{g^2}{\Delta}\right)\hat{a} \end{array} \right|$$

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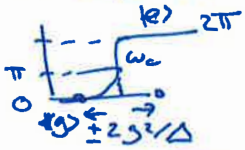
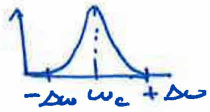
Quantum Measurements.

Thus the cavity will have a response depending on the phase shift of atom $g^2/\Delta\,\hat{\sigma}_z$ ie



phase response.



Amplitude response

$r = |r| e^{i\theta}$ reflected phase

Thus: $\boxed{\theta \approx \frac{\mathrm{re}\{r\}}{\mathrm{Im}\{r\}} = \left(\frac{g^2}{\Delta}\right)\cdot\underbrace{\left(\frac{4}{\kappa}\right)}\cdot\hat{\sigma}_z}$

$\propto$ ie slope of phase response

if $\theta \ll 1$ one cannot immediately determine qubit

NUMBER-PHASE UNCERTAINTY

If a coherent state is input $|\alpha\rangle$ on the cavity we can define its quantum noise as a classical phase fluctuation via:

$\langle\Delta\hat{X}_1^2\rangle = 1/2$

$\langle\Delta\hat{X}_1\rangle = \langle\alpha|\Delta\hat{X}_1|\alpha\rangle = 1/2$ } ie displaced VACUUM Noise.

Hence $\langle\Delta\theta^2\rangle = \frac{(1/2)\langle\Delta\hat{X}_2\rangle^2}{|\alpha|^2} = \frac{1/4}{|\alpha|^2} = \frac{1}{4\bar{N}}$

We thus obtain when considering

$\langle\Delta\hat{N}^2\rangle = \cancel{\ldots} = \langle\alpha|(\hat{a}^\dagger a) - \langle\hat{a}^\dagger a\rangle|\alpha\rangle$
$= \bar{N}$

Thus we have the phase-number uncertainty

$\underline{\langle\Delta\hat{N}^2\rangle\langle\Delta\hat{\theta}^2\rangle = 1/2}$ for a coherent state $|\alpha\rangle$

The signal to noise ratio becomes: $\theta_0 = \pm(g^2/\Delta\cdot 4/\kappa)$

$\boxed{SNR = \frac{\langle I_{signal}\rangle^2}{\langle\Delta I^2\rangle} = \frac{\theta_0^2}{(S_{\theta\theta}\,t^{-1})}}$

$S_{\theta\theta}$: phase fluctuation noise spectral density.

that is $\langle\Delta\hat{\theta}^2\rangle = \underbrace{S_{\theta\theta}}_{\text{Phase noise}}\cdot t^{-1}$ $[\mathrm{rad}^2/\mathrm{Hz}]$

t^{-1}: Bandwidth

t: measurement time.

$S_{\theta\theta} = \frac{1}{2\bar{N}}$

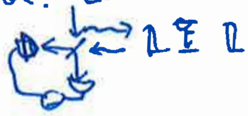
Note: $\langle\Delta\theta^2\rangle = \frac{1}{4\bar{N}}$ $\bar{N}$: number of detected photons.

relate to Flux: $\hat{a}\sqrt{\kappa} = \hat{a}_{in} + \hat{a}_{out}$, $0 = \frac{\kappa}{2}\hat{a} - a_{in}\sqrt{\kappa}$ twice
$\rightarrow \underline{|\hat{a}|^2 = \frac{\kappa^{-1}}{4}|\hat{a}_{in}|^2}$

i.e $\bar{n} = \langle|\hat{a}|\rangle = (4/\kappa)\cdot\underbrace{\dot{\bar{N}}}_{\text{Flux input.}} = \underbrace{\Phi}_{\text{Flux}}\cdot\frac{4}{\kappa}$

relation between cavity field and input

Konsider a homodyne measurement of output field:



$\langle\Delta\theta^2\rangle = \left(\frac{1}{4\,\Phi\cdot t}\right)$ t: measurement time

$= \underbrace{\left(\frac{\hbar\omega}{4P}\right)}_{S_{\theta\theta}}\cdot\left(\frac{1}{t}\right)$ $\Phi = P/\hbar\omega$

$\rightarrow$ spectral density. $[1/\mathrm{Hz}]$

Quantum Optics
Quantum Measurements

Measurement rate

$$\boxed{\Gamma_{meas} \equiv \frac{SNR}{2t} = \left(\frac{\theta_0^2}{2S_{\theta\theta}}\right)}$$ [in fact expressed in Hz]

Note: not "time" t in expression t: measurement time

Thus Γ_{meas}^{-1} corresponds to "measurement time" that makes SNR equal to unity.

We thus obtain $\Gamma_{meas} = 2\theta_0^2 \cdot S_{\dot{N}\dot{N}}$

$$\Gamma_{meas} = 2\theta_0^2 \cdot \frac{1}{4S_{\theta\theta}} = \frac{\theta_0^2}{2S_{\theta\theta}}$$ ↓ see below

Note $\boxed{S_{\theta\theta} \cdot S_{\dot{N}\dot{N}} = \frac{1}{4}}$ ie Heisenberg uncertainty for the spectral densities.

Note $\langle \Delta\theta^2 \rangle = S_{\theta\theta}(1/t)$ $\langle \Delta N^2 \rangle = S_{\dot{N}\dot{N}} \cdot (1/t)$

$S_{\theta\theta} = \frac{1}{4}\left(\frac{P}{\hbar\omega}\right)^{-1}$ $S_{\dot{N}\dot{N}} \equiv \frac{P}{\hbar\omega}$

Phase noise spectral density   Photon number fluctuation spectral density

Next consider the BACKACTION of the measurement (Despite being QND – this exists if system is not in an eigenstate of $\hat{H}_0 + \hat{H}_{int}$).

Consider $\hat{H} = \frac{\hbar}{2}(\omega_{12} + \omega_c g^2/\Delta \, \hat{a}^\dagger\hat{a})\hat{\sigma}_z + \hbar\omega_c \hat{a}^\dagger\hat{a}$

Thus photons shift levels $\Delta\omega_{12} = g^2/\Delta \cdot \hat{a}^\dagger\hat{a}\, \omega_c$

Expand field in cavity $\hat{a} = \bar{a} + \delta\hat{a}$ i.e. $\hat{u} = \bar{u} + \delta\hat{u}$

thus:

$$\Delta\omega_{12} = g^2/\Delta\, \bar{u}\,\omega_c + \underbrace{\delta\hat{u}\; g^2/\Delta \cdot \omega_c}_{\text{Fluctuations}}$$ $\hat{F}_z = \delta\hat{u} \cdot \frac{g^2}{\Delta}\omega_c$

"force" on qubit.

$[\hat{\sigma}_z, \hat{\sigma}_z] = 0$

evolution of TLS! check

$[\hat{\sigma}_+, \hat{\sigma}_z] = \hat{\sigma}_-$

consider:

$$\frac{d}{dt}\hat{\sigma}_z = -i\frac{\omega_{12}}{2}\hat{\sigma}_z + \underbrace{(\delta\hat{u} \cdot g^2/\Delta \cdot \omega_c)}_{\text{Fluctuation force}}\hat{\sigma}$$

full calculation:

Dephasing $\langle \sigma^+(t)\sigma^-(0) \rangle = \langle (e^{-i\int_0^t dt' \varphi(t')}) \rangle$

and $\varphi(t) = 2\frac{g^2}{\Delta}\int_0^t dt'\, u(t')$ phase shift between excited and ground state due to photons.

$$\simeq \exp\left(-\frac{1}{2}\left(\frac{2g^2}{\Delta}\right)^2 \iint dt_1 dt_2 \langle u(t_1)u(t_2)\rangle\right) = e^{-\Gamma_\varphi t}$$

$\hat{a}(t) = \bar{a} + \hat{d}$

and: $\langle (u(t)-\bar{u})(u(0)-\bar{u}) \rangle = \bar{u}\, \langle \hat{d}(t)d(0) \rangle = \bar{u}\, e^{-\kappa t/2}$

cavity fluctuations have memory!

$$\Rightarrow \boxed{\Gamma_\varphi = 4\theta_z \frac{\kappa}{2}\bar{u}}$$ (DEPHASING)

→ cavity decay (from QLE)

Quantum Optics
Lecture 10
- Quantum measurement -

$$\boxed{\Gamma_\varphi = 2\Theta_0^2 S_{NN} \qquad \text{and} \qquad \Gamma_{meas} = \frac{2\Theta_0^2}{4 S_{\Theta\Theta}}} \rightarrow \boxed{\Gamma_\varphi^{-1}\cdot\Gamma_{meas} = \frac{1}{2} = S_{\Theta\Theta} S_{NN}}$$

Hence both measurement rate and dephasing rate are equal. Note dephasing concerns e.g. the decay of the off diagonal COHERENCE ρ_{12} of the TLS. As in the Stern-Gerlach e.g. $|\psi\rangle = (|1\rangle+|2\rangle)\frac{1}{2}$

$\langle\psi_0|\sigma_+|\psi_0\rangle = \frac{1}{2}$

but $\langle\psi_0(t)|\sigma_+|\psi(t)\rangle = e^{-\Gamma_\varphi \cdot t}$ i.e. decays.

why? due to ENTANGLEMENT of cavity and ~~field~~ atom

Assume $|\psi_0\rangle = (|1\rangle+|2\rangle)\frac{1}{\sqrt{2}} \otimes \underset{\text{cavity}}{|\alpha\rangle}$ at $t=0$

next: $\psi_t = \frac{1}{\sqrt{2}}\left(e^{-i\omega_{12}t/2}|1\rangle \otimes |r_\uparrow\alpha\rangle + e^{+i\omega_{12}t/2}|2\rangle\otimes|r_\downarrow\alpha\rangle\right)$ i.e. entangled state.

$r_{\uparrow\downarrow} = r\left(\frac{\omega}{c} \pm i\frac{g^2}{\Delta}\hat{\sigma}_z\right)$ i.e. cavity reflection coefficient

(note: not very precise as cavity is coupled to environment... but okay)

Trace over the field:

$$\rho_{12} = \mathrm{Tr}_{field}\{\langle 1|\psi_t\rangle\langle\psi_t|2\rangle\} \qquad \left(\ldots = \mathrm{Tr}_{fl}\{\underline{|r_\downarrow\alpha\rangle e^{-i\omega_{12}t/2}}\;\underline{\langle r_\uparrow\alpha| e^{+i\omega_{12}t}}\}\right)$$

$$= \mathrm{Tr}\{\langle 1|\psi_t\rangle\langle\psi_t|2\rangle\}$$

$$= \langle r_\uparrow\alpha|r_\downarrow\alpha\rangle e^{+i\omega_{12}t/2}/2 \qquad |n\rangle\langle u|r_\downarrow\alpha\rangle \cdot \quad \langle u|\cdot\langle r_\uparrow\alpha|n\rangle$$

$$= e^{i\omega_{12}t/2}\;\underbrace{e^{-|\alpha|^2(1-r_\uparrow^* r_\downarrow)}}_{\text{overlap of two coherent states}}$$

and $\exp[-|\alpha|^2(1-r_\downarrow^* r_\uparrow)] = \exp(-2\Theta_0^2 \bar{N})$

thus again we find $\equiv \exp(-\underset{\text{dephasing}}{\Gamma_\varphi}\cdot t)$.

$S_{NN} = 2\bar{N}\left(\frac{4}{\kappa}\right)$

$S_{NN} = 2 S_{NN}\left(\frac{4}{\kappa}\right)^2$

Now there is NO DEPHASING if system starts in an EIGENSTATE. $|1\rangle$ or $|2\rangle$.

Note: if one loses information e.g. has a two sided cavity same analysis yields larger dephasing e.g. $\Gamma_\varphi \cdot t = \underset{\text{"TWO SIDED"}}{2}\cdot\Gamma_{meas} t$

e.g. $|\alpha\rangle = \underbrace{|r\alpha\rangle\otimes|t\alpha\rangle}_{\text{transmission and reflection}}$

consider state $|1\rangle$: $\sigma_z|1\rangle = -\frac{\hbar\omega_{12}}{2}|1\rangle$ equation of motion

$$d_t\sigma_z = [\hat{H}, \hat{\sigma}_z] = [\sigma_z, \sigma_z] = 0 \rightarrow \underline{\text{constant}}$$

Quantum Optics
Lecture 12

Measurement induced dephasing

$$\Gamma_\varphi = 2\Theta_0^2 S_{\dot{N}\dot{N}}$$

re-express in terms of photon number in the cavity.

Transmission on resonance: $\frac{\kappa}{2}\hat{a} = \hat{a}_{in}\sqrt{\kappa}$

$$\Rightarrow \underbrace{\langle \hat{a}^\dagger a\rangle}_{\bar{n}} = \underbrace{\langle a_{in}^\dagger a_{in}\rangle}_{\dot{\bar{N}}} \frac{4}{\kappa}$$

Thus:

$$\Gamma_\varphi = 2\Theta_0^2 \cdot \underbrace{S_{\dot{N}\dot{N}}}_{\text{Photon flux fluct.}} = 2\Theta_0^2 \left(\frac{\kappa}{4}\bar{n}\right) \qquad S_{\dot{N}\dot{N}} = \frac{\kappa}{2}4\bar{n}$$

Therefore $\boxed{\Gamma_\varphi = 2\Theta_0^2 \cdot \left(\frac{\kappa}{4}\bar{n}\right) = \left(\Theta_0^2 \kappa \bar{n}\right)} = 2\Theta_0^2 \cdot \dot{N}$ ($\dot{N} = \kappa\bar{n}$ → decay of photons ie flux)

ie this is the cavity fields that leaks from cavity

Simple intuitive result:

if $(2\Theta_0)^2$ is the phase change this requires

$\langle \Delta\Theta^2\rangle = (2\Theta_0)^2$ and $\Delta\Theta^2 = \frac{1}{4N}$ (from coherent state fluct).

The measurement time is $T_{meas} = \frac{1}{(\kappa\bar{n}\Theta_0^2)} \hat{=} \Gamma_{meas}^{-1}$

The Backaction is thus evident in the measurement.

We can also consider instead on $\hat{\sigma}_z = \hat{z}$ operator

$\hat{z}$ is the POSITION. ie mechanical position detector → LIGO.

Quantum Optics
– Lecture 8 –

$$\langle \Delta \hat{N}^2 \rangle = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2 = \bar{\alpha}^2 \langle 0 | \underbrace{d d^+}_{d^+ d + 1} | 0 \rangle = N$$

Define quadratures $\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^+)$ $\hat{x}_2 = \frac{i}{2}(\hat{a}^+ - \hat{a})$

Lets assue $\langle \hat{x}_1 \rangle = \sqrt{2}\alpha$ i.e $\langle x_1 \rangle = \sqrt{2}^{-1}(\alpha + \alpha^*) = \frac{1}{\sqrt{2}} 2\alpha = \sqrt{2}\alpha$
(α real)
$\langle \hat{x}_2 \rangle = 0$ $\left(= \frac{1}{\sqrt{2}}(\alpha - \alpha) = 0\right)$ $\wedge$ $\left(\langle \hat{x}_2^2 \rangle = \frac{1}{2} \langle 0 | d d^+ | 0 \rangle\right)$

Let us define a Phase ie $\langle \hat{\theta} \rangle = \dfrac{\langle \hat{Y} \rangle}{\langle X \rangle}$ $\theta \ll 1$

Thus $\Delta\theta^2 = \dfrac{\langle \hat{x}_1^2 \rangle}{(\langle x_2 \rangle)^2} = \dfrac{\frac{1}{2}\langle 0 | \hat{d}\hat{d}^+ | 0 \rangle}{2\bar{N}} = \dfrac{1}{4\bar{N}}$

Also note that $\langle \Delta \hat{N}^2 \rangle = \bar{N}$

Thus $\boxed{\underbrace{(\Delta\theta)^2}_{\text{classical "phase" uncertainty}} \cdot \langle \Delta \hat{N}^2 \rangle = \frac{1}{4\bar{N}} \cdot \bar{N} = \frac{1}{4}}$ i.e

Returning to the question of measurements. Assue $\langle \sigma_z \rangle = \pm 1$ state of the qubit. $\theta_\pm = A.\mathrm{arc}\left(\frac{4}{k}\right).(\pm 1)$ Phase shift of reflected input field

If $\theta \ll 1$ it will imply that many photons are necessary to determine state of the cavity [weak measurements].
*Note: photon resolving is STRONGLY measured.

The analysis can be repeated for the input fields
$$\hat{a}_{in} = \overline{a_{in}} + \underbrace{\delta\hat{a}_{in}}_{\text{vacuum noise}}$$
$\dot{\bar{N}} = \Phi = |\langle \hat{a}_{in} \rangle|^2 = (\overline{a_{in}})^2$

and $\langle \dot{\hat{N}}(t) \dot{\hat{N}}(0) \rangle - \langle \dot{\hat{N}} \rangle = \dot{\bar{N}}\,\delta(t)$ Fluctuations.

Noise spectral density $\underline{S_{\dot{N}\dot{N}} = \dot{\bar{N}}}$ (white) ie $S_{\dot{N}\dot{N}} = \int e^{i\omega t} \underbrace{C(t)}_{\text{autocorrel.}} dt$

Phase $\hat{\theta} = \dfrac{i \langle \hat{a}_{in}^+ - \hat{a}_{in} \rangle}{\langle \hat{a}_{in} + \hat{a}_{in}^+ \rangle} = \langle \theta \rangle + i \dfrac{(\delta\hat{a}_{in}^+ - \delta\hat{a}_{in})}{2\overline{a_{in}}}$

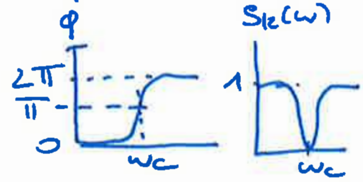
$\Rightarrow$ Noise correlator $\langle \delta\theta(t)\,\delta\theta(0) \rangle = \frac{1}{4\dot{\bar{N}}}\,\delta(t)$ $\Rightarrow S_{\theta\theta} = \frac{1}{4\dot{\bar{N}}}$ $\Rightarrow S_{\theta\theta} S_{\dot{N}\dot{N}} = \frac{1}{4}$

Quantum Optics
– Lecture 8 –

Measurements with parametrically coupled resonant cavities

→ measure phase from cavity

→ weak, continuous measurement



Two examples: • QND measurement for qubit state

• non-QND of mechanical mirror motion

First compute classical phase

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a}\,(1 + \underbrace{A\hat{\sigma}_z}_{\text{dispersive shift}}) + \underbrace{\hbar\omega_{12}\hat{\sigma}_z}_{\text{TLS}} + \hat{H}_{diss}$$

The cavities equation of motion with dissipation

(1) $\hat{a}_{out} = \hat{a}_{in} + \sqrt{\kappa}\cdot\hat{a}$; input-output relation

(2) $\partial_t \hat{a} = -i(\omega_c + A\hat{\sigma}_z)\hat{a} - \kappa/2 - \sqrt{\kappa}\,\hat{a}_{in}$

$$\Rightarrow \boxed{r\,[\omega=\omega_c] = -\left(\frac{1+2iA\hat{\sigma}_z\cdot Q_c}{1-2iA\hat{\sigma}_z Q_c}\right)}$$

(on resonance reflection)

and $Q_c = \omega/\kappa$ "Quality factor"

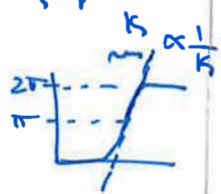
thus $|r(\omega=\omega_c)| = 1$

Phase, $r = |r|e^{i\varphi}\underbrace{e^{i\pi}}_{-1}$ φ: reflected phase.

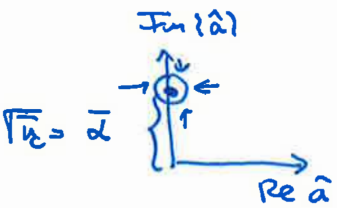
Phase: $\boxed{\varphi \approx 4Q_c A\hat{\sigma}_z}$

$$\tan\varphi = \left(\frac{\text{Re}\{r\}}{\text{Im}\{r\}}\right)^{-1} \approx \varphi$$

$$\varphi = 4\frac{\omega_c}{\kappa}A\hat{\sigma}_z = \underbrace{\left(\frac{4}{\kappa}\right)}_{\text{slope of response curve.}}\underbrace{\omega_c A\hat{\sigma}_z}_{\text{cavity shift}}$$



next: phasor diagram for the cavity amplitude:



$\hat{a} = \bar{a} + \underbrace{\hat{d}}_{\text{noise}}$ $(\delta\hat{a} \hat{=} d)$ noise operator

$[\delta\hat{a}, \delta\hat{a}^\dagger] = 1$

linearization procedure.

$$\bar{N} = \langle\hat{n}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle = \langle 0|(\bar{a} + \hat{d}^\dagger)(\bar{a} + \hat{d})|0\rangle$$
$$= \bar{a}^2 + \bar{a}\langle 0|d^\dagger|0\rangle + \bar{a}\langle 0|d|0\rangle + \langle 0|d^\dagger d|0\rangle$$
$$= \bar{a}^2$$