Quantum Electrodynamics and Quantum Optics
École Polytechnique Fédérale de Lausanne (EPFL)

Exercise No.9

Solution: Quantum Langevin equation for a harmonic oscillator interacting with a heat bath

1. The full interaction Hamiltonian will contain terms like:

$$\sum_k (\hat{a}+\hat{a}^\dagger)(\hat{b}_k-\hat{b}_k^\dagger) = \sum_k \hat{a}\hat{b}_k - \hat{a}\hat{b}_k^\dagger + \hat{a}^\dagger\hat{b}_k - \hat{a}^\dagger\hat{b}_k^\dagger \,. \tag{1}$$

We can change reference frame (or, equivalently, go to the interaction picture) and do the following substitutions: $\hat{a} \to \hat{a}e^{-i\omega_s t}$ and $\hat{b} \to \hat{b}e^{-i\omega_k t}$. So that the terms above become:

$$\sum_k \hat{a}\hat{b}_k e^{-i(\omega_k+\omega_s)t} - \hat{a}\hat{b}_k^\dagger e^{i(\omega_k-\omega_s)t} + \hat{a}^\dagger\hat{b}_k e^{-i(\omega_k-\omega_s)t} - \hat{a}^\dagger\hat{b}_k^\dagger e^{i(\omega_k+\omega_s)t} \,. \tag{2}$$

Now if $|\omega_k+\omega_s| \gg |\omega_k-\omega_s|$, the terms with $e^{\pm i(\omega_k+\omega_s)t}$ will average to zero over much shorter time scales than the terms with $e^{\pm i(\omega_k-\omega_s)t}$, thus the former terms can be neglected. This is the rotating wave approximation.

2. We calculate the equations of motion using $\partial_t \hat{O} = \frac{i}{\hbar}[H,\hat{O}]$. A straightforward calculation leads to:

$$\begin{aligned} \partial_t \hat{a} &= -i\omega_s \hat{a} - i\sum_k g_k \hat{b}_k && (3)\\ \partial_t \hat{b}_k &= -i\omega_k \hat{b}_k - i g_k \hat{a} && (4) \end{aligned}$$

3. First we formally integrate the equation for $\hat{b}_k$. We obtain (can be verified by substitution or derived using variation of the constant):

$$\hat{b}_k(t) = \hat{b}_k(t)e^{-i\omega_k t} - i g_k \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{i\omega_k(t-t')} \,. \tag{5}$$

We now insert this back into the expression for $\hat{a}$:

$$\partial_t \hat{a} = -i\omega_s \hat{a}(t) \underbrace{-i\sum_k g_k \hat{b}_k(0)e^{-i\omega_k t}}_{\hat{f}_a(t)} - \sum_k g_k^2 \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{i\omega_k(t-t')} \,. \tag{6}$$

Now we transform the sum into an integral:

$$\sum_k g_k^2 \to \int_0^\infty \mathrm{d}\omega_k\, D(\omega_k)|g(\omega_k)|^2 \,; \tag{7}$$

and apply the 1st Markov approximation $g_k = g(\omega_k) = g$ and make the assumption that the density of states is slowly varying $D(\omega) = D(\omega_s)$:

$$\sum_k g_k^2 \int_0^t \mathrm{d}t'\, \hat{a}(t') e^{-i\omega_k(t-t')} = \int_0^\infty \mathrm{d}\omega_k\, D(\omega_k)|g(\omega_k)|^2 \int_0^t \mathrm{d}t'\, \hat{a}(t') e^{-i\omega_k(t-t')} \tag{8}$$

$$= D(\omega)|g(\omega)|^2 \int_0^t \mathrm{d}t'\, \hat{a}(t') \int_0^\infty \mathrm{d}\omega_k\, e^{-i\omega_k(t-t')} \tag{9}$$

$$= D(\omega)|g(\omega)|^2 \int_0^t \mathrm{d}t'\, \hat{a}(t') 2\pi\delta(t-t') \tag{10}$$

$$= 2\pi D(\omega)|g(\omega)|^2 \frac{1}{2}\hat{a}(t) \equiv \frac{\kappa}{2}\hat{a}(t)\,. \tag{11}$$

Finally we obtain the QLE for $\hat{a}$:

$$\partial_t \hat{a}(t) = -i\omega_s \hat{a}(t) - \frac{\kappa}{2}\hat{a}(t) + \hat{f}_a(t)\,. \tag{12}$$

4. Take an operator $\hat{a}$ whose equation of motion is given by $\partial_t \hat{a} = \frac{i}{\hbar}[H, \hat{a}]$. Then $\hat{\tilde{a}} = \hat{a}e^{-i\omega t}$ has the equation of motion:

$$\partial_t \hat{\tilde{a}} = i\omega\hat{\tilde{a}} + \frac{i}{\hbar}\left[H, \hat{\tilde{a}}\right]\,. \tag{13}$$

So with going to a frame rotating with ω_s ($\hat{a} = \hat{\tilde{a}}e^{i\omega_s t}$):

$$\partial_t \hat{\tilde{a}} = -\frac{\kappa}{2}\hat{\tilde{a}}(t) + \hat{f}_{\tilde{a}}(t) \tag{14}$$

5. $F(t) = \hat{f}_{\tilde{a}}(t)$

$$\left\langle F^\dagger(t) F(t') \right\rangle = \sum_k \sum_{k'} g_k g_{k'} e^{i\omega_k t - i\omega_{k'} t'} \left\langle \hat{b}_k^\dagger(0) \hat{b}_{k'}(0) \right\rangle \tag{15}$$

$$= 2\pi \sum_k g_k^2 \bar{n}_k e^{i\omega_k(t-t')} \tag{16}$$

$$= \int_0^\infty \mathrm{d}\omega\, g^2 \bar{n}(\omega) e^{i\omega_k(t-t')} D(\omega) \tag{17}$$

$$= \kappa \bar{n}_{\mathrm{th}} \delta(t-t') \tag{18}$$

6.

$$\partial_t \left[\hat{\tilde{a}}(t), \hat{\tilde{a}}^\dagger(t)\right] = -\kappa\left[\hat{\tilde{a}}(t), \hat{\tilde{a}}^\dagger(t)\right] + \left[F(t), \hat{\tilde{a}}^\dagger(t)\right] + \left[\hat{\tilde{a}}(t), F^\dagger(t)\right] \tag{19}$$

To compute $\left[\hat{\tilde{a}}(t), F^\dagger(t)\right]$ we use $\hat{\tilde{a}}(t) = \hat{\tilde{a}}(0)e^{-\frac{\kappa}{2}t} + \int_0^t \mathrm{d}t'\, e^{-\frac{\kappa}{2}(t-t')} F(t')$ and obtain:

$$\left[\hat{\tilde{a}}(t), F^\dagger(t)\right] = \int_0^t \mathrm{d}t'\, e^{-\frac{\kappa}{2}(t-t')} \left[F(t'), F^\dagger(t)\right] \tag{20}$$

$$= \frac{\kappa}{2} = \left[F(t), \hat{\tilde{a}}^\dagger(t)\right]\,. \tag{21}$$

So

$$\partial_t \left[\hat{\tilde{a}}(t), \hat{\tilde{a}}^\dagger(t)\right] = -\kappa\left[\hat{\tilde{a}}(t), \hat{\tilde{a}}^\dagger(t)\right] + \kappa = 0\ \forall t \tag{22}$$

knowing that $\left[\hat{\tilde{a}}(0), \hat{\tilde{a}}^\dagger(0)\right]$.

7. $$\langle \partial_t \hat{\tilde{a}} \rangle = \partial_t \langle \hat{\tilde{a}} \rangle = -\frac{\kappa}{2} \langle \hat{\tilde{a}}(t) \rangle + \langle F(t) \rangle = -\frac{\kappa}{2} \langle \hat{\tilde{a}}(t) \rangle \Rightarrow \langle \hat{\tilde{a}}(t) \rangle = \langle \hat{\tilde{a}}(0) \rangle e^{-\frac{\kappa}{2}t} \tag{23}$$

8. Using all the information up until now it is a straightforward calculation to find the equation of motion

$$\langle t|N|t \rangle = -\kappa \langle N(t) \rangle + \left\langle F^\dagger(t)\hat{\tilde{a}}(t) \right\rangle + \left\langle \hat{\tilde{a}}^\dagger(t)F(t) \right\rangle = -\kappa \langle N(t) \rangle + \kappa \bar{n}_{\mathrm{th}} \tag{24}$$

which solves to:

$$\langle N(t) \rangle = (\langle N(0) \rangle - \bar{n}_{\mathrm{th}})\, e^{-\kappa t} + \bar{n}_{\mathrm{th}} \tag{25}$$

9.1 Purcell enhancement

9.1.1 $D(\omega)$ of vacuum

c.f. Solution for Homework 1. The answer is:

$$D(\omega) = \frac{\omega^2}{\pi^2 c^3} \tag{26}$$

9.1.2 $D_c(\omega)$ in cavity

The Langevin equation for cavity mode a reads:

$$\partial_t a = -i\omega_c a - \frac{\omega_c}{2Q} a + F \tag{27}$$

From the above Langevin equation, we can solve out the cavity mode:

$$a(t) = a(0)e^{-(i\omega_c + \frac{\omega_c}{2Q})t} + \int_0^t d\tau\, a(0)e^{-(i\omega_c + \frac{\omega_c}{2Q})(t-\tau)} F(\tau) \tag{28}$$

Thus:

$$\langle a^\dagger(t)a(0) \rangle = \langle a^\dagger(0)a(0) \rangle e^{-(i\omega_c + \frac{\omega_c}{2Q})t} + \langle n \rangle e^{-(i\omega_c + \frac{\omega_c}{2Q})t} \tag{29}$$

The power spectral density reads:

$$S(\omega) = \frac{1}{\pi} \frac{\langle n \rangle \frac{\omega_c}{2Q}}{(\omega - \omega_c)^2 + (\frac{\omega_c}{2Q})^2} \tag{30}$$

Thus:

$$D_c(\omega) = \frac{S(\omega)}{\langle n \rangle} = \frac{1}{\pi} \frac{\frac{\omega_c}{2Q}}{(\omega - \omega_c)^2 + (\frac{\omega_c}{2Q})^2} \tag{31}$$

9.1.3 High-Q limit and two-level system decay rate

In the high-Q limit,

$$D_c(\omega) \approx \frac{2}{\pi} \frac{Q}{\omega_c} \tag{32}$$

Thus the decay rate enhancement:

$$\frac{\Gamma_{cav}}{\Gamma_{vac}} = \frac{2}{\pi} \frac{Q}{\omega_c} \frac{\pi^2 c^3}{\omega^2} 1/V = 2\pi (\frac{c}{\omega})^3 \frac{Q}{V} \tag{33}$$