Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.9

9.1 Quantum Langevin equation for an optical cavity interacting with a heat bath [1]

Consider a cavity coupled to a heat bath, consisting of a large ensemble of harmonic oscillators. The Hamiltonian of the system and the bath can be expressed as (for simplicity we consider only one cavity mode)

$$\hat{H}_{\text{sys}} = \hbar\omega_s \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \tag{1}$$

$$\hat{H}_{\text{bath}} = \sum_k \hbar\omega_k \left(\hat{b}_k^\dagger \hat{b}_k + \frac{1}{2}\right) \tag{2}$$

Here, $\hat{a}$ and $(\hat{a}^\dagger)$ are the annihilation and creation operators which satisfy the commutator relation $[\hat{a}, \hat{a}^\dagger] = 1$. Assume that the bath and system are interacting in a bilinear way, i.e. that the interaction Hamiltonian takes the form:

$$\hat{H}_{int} = \hbar \sum_k g_k (\hat{a}\hat{b}_k^\dagger + \hat{b}_k \hat{a}^\dagger) \tag{3}$$

We also assume that the heat bath is in thermal equilibrium and has a finite temperature. This implies that $\langle \hat{b}_k^\dagger(0)\rangle = \langle \hat{b}_k(0)\rangle = 0$, as well as $\langle \hat{b}_k^\dagger(0)\hat{b}_l(0)\rangle = \delta_{k,l}\bar{n}_k$ and $\langle \hat{b}_k(0)\hat{b}_l^\dagger(0)\rangle = \delta_{k,l}(\bar{n}_k + 1)$ where $\bar{n}_k$ is the effective occupation number of the k^{th} mode. Finally, we assume that the bath modes are initially uncorrelated with each other, i.e. $\langle \hat{b}_k(0)\hat{b}_l^\dagger(0)\rangle = 0$ for $k \neq l$.

1. Derive the Heisenberg equations of motion for the bath and system operators separately.
2. Next, eliminate the bath operators from the equation of motion for $\hat{a}(t)$ by inserting the equation of the bath. Moreover, introduce the *density of states* $D(\omega)$ (that is, the number of modes in a given volume between ω and $\omega + \mathrm{d}\omega$) of the bath modes and convert the summation over k to an integral over ω, assuming that the coupling is frequency independent, i.e. $g_k = g(\omega_k) = g$ (this is called the 1st Markov approximation). Carry out the integration over ω using $\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t')$.[2]
3. Transform the equations of motion to a frame rotating with ω_s with respect to the original Hamiltonian $H_0 = H_{\text{sys}} + H_{\text{bath}}$. Show that the equation of motion for the system operator obeys the **Quantum Langevin Equation**:

$$\boxed{\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + \sqrt{\kappa}\hat{a}_{\text{in}}(t)}$$

   Give an expression for the input noise term $\hat{a}_{\text{in}}(t)$ in terms of the bath operators. What is the physical intuition behind the "input noise" term?
4. Show that the input noise operator $\hat{a}_{\text{in}}(t)$ satisfies

$$\langle \hat{a}_{\text{in}}^\dagger(t)\hat{a}_{\text{in}}(t')\rangle = \bar{n}_{th}\delta(t-t'),$$

   $\langle \hat{a}_{\text{in}}(t)\rangle = 0$ and show that the system's decay rate (κ) is given by: $\kappa = 2\pi D(\omega_s)|g|^2$.

[1] See e.g. Scully, *Quantum Optics*, Chapter 9 or Gardiner and Collet, *"Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation"*, Physical Review A (1984)

[2] Note that this property also gives rise to the equality $\int_{t_0}^{t} c(t')\delta(t-t')dt' = \frac{1}{2}c(t)$, which is needed to derive the exact form of the quantum Langevin equation (see next point).

5. Show that the quantum Langevin equation - despite containing now dissipation and damping of the system operator - preserves the commutator, $[\hat{a}(t), \hat{a}^{\dagger}(t)] = 1$. We have achieved a quantum mechanically consistent description of damping.

6. Derive and solve the equations of motion for the mean field $\langle \hat{a}(t) \rangle$.

7. How does the (average) number of photons evolve in time? Calculate $\frac{d}{dt}\hat{N} = \frac{d}{dt}\left(\hat{a}^{\dagger}\hat{a}\right)$ and $\frac{d}{dt}\langle \hat{N} \rangle = \frac{d}{dt}\langle \hat{a}^{\dagger}\hat{a} \rangle$.

9.2 Purcell enhancement

In this exercise we study the Purcell effect [3] [4].
Consider a two-level system in vacuum. The spontaneous emission rate is given by:

$$\Gamma = 2\pi \langle |g(\omega)|^2 \rangle D(\omega)$$

Where $g(\omega)$ is atom-cavity coupling and $D(\omega)$ is the density of states at the atomic transition frequency ω.

(a) Find $D(\omega)$ for the vacuum in terms of mode volume V, and express vacuum spontaneous emission rate (You may check the exercise 1 in HW 1).

(b) Find density of states $D_c(\omega)$ for a cavity with the central frequency ω_c and quality factor $Q = \omega_c/\kappa$ (cavity frequency / line-width). Express the spontaneous emission rate Γ, for a two-level system in a cavity. In order to derive density of states of a cavity you may follow these steps [5] :

   i. Write down the Langevin equation for the field in the cavity ($a_{(t)}$). Consider the input field as $F_{(t)}$ and show:
   $$\dot{a}_{(t)} = -i\omega_c a_{(t)} - \frac{\omega_c}{2Q} a_{(t)} + F_{(t)}$$

   ii. Calculate field's auto-correlation $\langle a^{\dagger}_{(t)} a_{(t+\tau)} \rangle$ in terms of τ and mean number of photons in the cavity $\langle n \rangle$.

   iii. Calculate power spectral density of the field in terms of $\langle n \rangle$ based on the definition:
   $$S(\omega) = \frac{1}{\pi}\mathrm{Re}[\int_0^{\infty} \langle a^{\dagger}_{(t)} a_{(t+\tau)} \rangle e^{i\omega\tau}]d\tau$$

   iv. We can express energy of the system using power spectral density:
   $$E = \frac{1}{\pi}\int_0^{\infty} S(\omega) d\omega$$
   Use this property and definition of density of states to calculate $D_c(\omega)$.

(c) Calculate the previous result in the limit of the high Q cavity. Compare the spontaneous emission rate for a two-level system in two cases and explain the difference.

9.3 Quantization of a Cooper pair box coupled to an LC resonator(*) [6]

Let's consider the circuit showed in Fig.1, which shows a Cooper pair box coupled to an LC resonator. In this exercise we will quantize this system and recover the Jaynes-Cummings Hamiltonian.

[3] E. M. Purcell, Phys. Rev. 69, 681 (1946)
[4] S. Haroche; D. Kleppner (1989). "Cavity Quantum Dynamics". Physics Today. 42 (1): 24–30.
[5] Scull y, chapter 9.3
[6] Circuit QED Girvin Les Houches - Appendix A, Oxford University Press

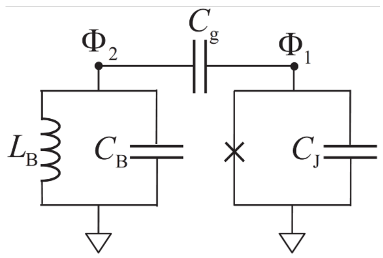

Figure 1: Circuit Diagram of a Cooper pair box capacitively coupled to a parallel LC resonator

1. Starting from the circuit Lagrangian, derive the Hamiltonian in term of the magnetic fluxes $\hat{\Phi}_1, \hat{\Phi}_2$ and charges $\hat{Q}_1, \hat{Q}_2$ as $\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$, where

$$\begin{aligned} H_1 &= \frac{\hat{Q}_1^2}{2C_{1\Sigma}} - E_J \cos \frac{2e}{\hbar}\hat{\Phi}_1, \\ H_2 &= \frac{\hat{Q}_2^2}{2C_{2\Sigma}} + \frac{1}{2L_B}\hat{\Phi}_2^2, \\ H_{12} &= \frac{\beta}{C_{2\Sigma}}\hat{Q}_1\hat{Q}_2. \end{aligned}$$

   Here $C_{1\Sigma} = C_J + \left(\frac{1}{C_g} + \frac{1}{C_B}\right)^{-1}$ and $C_{2\Sigma} = C_B + \left(\frac{1}{C_g} + \frac{1}{C_J}\right)^{-1}$ are the capacitances seen by the flux $\hat{\Phi}_1$ and $\hat{\Phi}_2$ respectively.

2. We treat the two branches differently. For the buffer branch (LC resonator), we express the Hamiltonian in ladder operators, i.e. $\hat{Q}_2 = -iQ_{2ZPF}(\hat{a} - \hat{a}^\dagger)$. For the Josephson junction branch, we take advantage of the sprctral theorem and expand $\hat{Q}_1, \hat{\Phi}_1$ with the eigenstates $|i\rangle$ of $\hat{Q}_1$. Show that the full Hamiltonian reads:

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \sum_{k=0}^{\infty} \epsilon_k |k\rangle\langle k| - \sum_{j,k=0}^{\infty} i\frac{\beta Q_{2\mathrm{ZPF}} Q_{jk}}{C_{2\Sigma}} |j\rangle \left(\hat{a} - \hat{a}^\dagger\right) \langle k|,$$

   where $\omega_c = \frac{1}{\sqrt{L_B C_{2\Sigma}}}$, $Q_{2ZPF} = \sqrt{C_{2\Sigma}\hbar\omega_c/2}$ and $Q_{jk} = \langle j|\hat{Q}|k\rangle$.

3. Now suppose the spectrum of the qubit is sufficiently anharmonic. We restrict our attention to its lowest two state $|0\rangle$ and $|1\rangle$. Represent the Hamiltonian with Pauli matrix:

$$\begin{aligned} |0\rangle\langle 0| &= \frac{1-\sigma^z}{2}, \\ |1\rangle\langle 1| &= \frac{1+\sigma^z}{2}, \\ |1\rangle\langle 0| &= \sigma^+, \\ |0\rangle\langle 1| &= \sigma^-, \end{aligned}$$

   and recover the celebrated Jaynes-Cummings Hamiltonian:

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_{01}}{2}\sigma^z - i\hbar g \left(\hat{a} - \hat{a}^\dagger\right) \left(\sigma^+ + \sigma^-\right),$$

   where $\hbar\omega_{01} \equiv \epsilon_1 - \epsilon_0$. *(Hint: You can choose the arbitrary phases of the eigenstates $|i\rangle$, and set $Q_{01} = Q_{10}$ to be real.)* Finally, give the expression of the coupling strength g.