Quantum Electrodynamics and Quantum Optics
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.8

8.1 Equivalence of rE- and pA-Hamiltonians

Show that the two forms of interaction Hamiltonians, i.e. rE-Hamiltonian:

$$\hat{H}^{\mathrm{rE}}_{\mathrm{int}} = -q\hat{\mathbf{r}}\hat{\mathbf{E}}(a_0, t), \tag{1}$$

and pA-Hamiltonian:

$$\hat{H}^{\mathrm{pA}}_{\mathrm{int}} = -\frac{q}{m}\hat{\mathbf{p}}\hat{\mathbf{A}}(a_0, t), \tag{2}$$

have similar transition matrix element, by calculating their off-diagonal elements $\langle f| \cdots |i\rangle$.

We assume $|i\rangle$ and $|f\rangle$ are two eigenstates of the particle free-Hamiltonian $H_0|i\rangle = \hbar\omega_i|i\rangle$ and $H_0|i\rangle = \hbar\omega_f|f\rangle$, and we assume

$$\hat{\mathbf{E}}(a_0, t) = \hat{E}\cos\omega t \qquad \hat{\mathbf{A}}(a_0, t) = -\frac{1}{\omega}\hat{E}\sin\omega t \tag{3}$$

The following relation also needs to be used:

$$\hat{\mathbf{p}} = m\frac{i}{\hbar}\left[\hat{H}_0, \hat{r}\right], \tag{4}$$

8.2 Wigner-Weisskopf theory

The interaction picture Hamiltonian of a two-level excited system in multimode (vacuum) field has the following form:

$$\hat{H}^{\mathrm{int}} = \hbar\sum_{\mathbf{k}}\left[g^*_{\mathbf{k}}(r_0)\hat{\sigma}_+ a_{\mathbf{k}} e^{i(\omega_0-\omega_{\mathbf{k}})t} + \mathrm{H.c.}\right], \tag{5}$$

where $g_{\mathbf{k}}(r_0)$ is the coupling between levels, and r_0 and ω_0 are the location and resonance frequency of an atom. The state vector of such a system has the form:

$$|\psi(t)\rangle = c_a(t)\,|a, 0\rangle + \sum_{\mathbf{k}} c_{b,\mathbf{k}}(t)\,|b, 1_{\mathbf{k}}\rangle, \tag{6}$$

with initial conditions $c_a(0) = 1$ and $c_{b,\mathbf{k}}(0) = 0$.

1. From the Schrödinger equation in interaction picture, derive an differential-integral equation for the population of excited state $c_a(t)$.

2. Assuming that the modes of the field are close to each other in frequency, replace the summation over wavevector by integration:

$$\sum_{\mathbf{k}} \to 2\frac{V}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk k^2, \tag{7}$$

where V is a quantization volume. Use the following relation:

$$|g_{\mathbf{k}}(r_0)|^2 = \frac{\omega_{\mathbf{k}}}{2\hbar\epsilon_0 V} d^2_{ab}\cos^2\theta, \tag{8}$$

where θ is the angle between the atomic dipole moment d_{ab} and field polarization.

3. Obtain the dynamics of excited state population in the form:

$$\dot{c}_a(t) = -\frac{\Gamma}{2} c_a(t), \tag{9}$$

and show the explicit expression for Γ.[1]

4. What is the physical meaning of Γ?

8.3 Synthesizing arbitrary quantum states

So far we have studied the basic interaction between a two level system (atom) with a bosonic field (photons). Even though these interactions are very basic, we can already use these interaction to create complex quantum states[2] by realizing these interactions in real world experiment (e.g. superconducting qubits coupled to microwave cavities). Imagine that we realized the following system Hamiltonian consisting of an atom (σ) and a photon field (a) in our thought experiment:

$$\frac{\hat{H}}{\hbar} = \frac{\Delta(t)}{2}\sigma_z + \left(\frac{\Omega_{\mathrm{JC}}(t)}{2}\sigma_+ a + \frac{\Omega_{\mathrm{Rabi}}(t)}{2}\sigma_+ + \frac{\Omega_{\mathrm{D}}(t)}{2}a^\dagger\right) + \mathrm{h.c.},$$

where $\Delta(t)$ is the frequency difference between the atom and the photon field, $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$, $\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$ are the operators that act on the two level system, and a is the annihilation operator acting on the photon field. Worth noticing that the coupling coefficients Ω are complex valued.

1. Briefly describe what the effect of the Hamiltonian interaction terms $\frac{\Omega_{\mathrm{JC}}(t)}{2}\sigma_+ a + \mathrm{h.c.}$, $\frac{\Omega_{\mathrm{Rabi}}(t)}{2}\sigma_+ + \mathrm{h.c.}$, $\frac{\Omega_{\mathrm{D}}(t)}{2}a^\dagger + \mathrm{h.c.}$ is when they are applied to our atom-photon system.

2. Assuming that initially the state of the system is prepared in the ground state $|1_{\mathrm{atom}}, 0_{\mathrm{photon}}\rangle$ and $\Delta(t) = 0$, derive the state evolution when only the following interactions are applied seperately (assuming constant interaction strength and keep in mind the Fock state dependent Rabi frequency):

    - $\frac{\Omega_{\mathrm{Rabi}}}{2}\sigma_+ + \mathrm{h.c.}$
    - $\frac{\Omega_{\mathrm{JC}}}{2}\sigma_+ a + \mathrm{h.c.}$
    - $\frac{\Omega_{\mathrm{D}}}{2}a^\dagger + \mathrm{h.c.}$

3. Now that we understand how different interaction terms evolve the quantum state of the system, try to design an experimental interaction sequence (e.g. draw a sequence diagram with graphic illustration of how the state of the system evolves through your designed interaction sequence) that could generate the following states of the photon field (keep in mind that if your photon is entangled with the atom, the photon state will actually be a mixed state):

    - $|3\rangle$
    - $\frac{|1\rangle + i|2\rangle}{\sqrt{2}}$
    - $\frac{|1\rangle - i|3\rangle}{\sqrt{2}}$

[1] for help see Scully, Marlan O., and M. Suhail Zubairy *Quantum optics* (1999), Chapter 6.

[2] Hofheinz, Max, et al."Synthesizing arbitrary quantum states in a superconducting resonator." Nature 459.7246 (2009): 546.

8.4 Bloch-Siegert shift: An example of non-RWA effect

The rotating wave approximation (RWA) is not always correct, since counter-rotating terms can contribute to the evolution of atomic states. For instance, they produce a small shift in atomic levels, known in literature as the Bloch-Siegert shift.[3] The exact version of non-RWA Hamiltonian has the following form:

$$\hat{H}^{\text{int}} = -\frac{1}{2} i\hbar \sum_{\mathbf{k}} g_{\mathbf{k}} \left[\hat{\sigma}^{+} \hat{a}_{\mathbf{k}}(t) - \hat{\sigma}^{-} \hat{a}^{\dagger}_{\mathbf{k}}(t) + \hat{\sigma}^{-} \hat{a}_{\mathbf{k}}(t) - \hat{\sigma}^{+} \hat{a}^{\dagger}_{\mathbf{k}}(t) \right]. \tag{10}$$

Using this Hamiltonian, one can obtain the equations for coherences:

$$\begin{cases} \dot{\rho}_{12} &= i\omega_0 \rho_{12} - \frac{1}{2}\Gamma \rho_{12} + \Gamma \rho_{21}, \\ \dot{\rho}_{21} &= -i\omega_0 \rho_{21} - \frac{1}{2}\Gamma \rho_{21} + \Gamma \rho_{12}, \end{cases} \tag{11}$$

where ω_0 is a resonance frequency and Γ is a spontaneous emission rate. Due to the fact that counter-propogating terms are now included into consideration, these equations imply mutual transfer between coherences ρ_{12} and ρ_{21}.

The task can be done in two ways. Either **solve equations** using Laplace transform method and find for the imaginary roots, which contribute to population oscillations, **the expression for additional shift to the atomic frequency ω_0, known as Bloch-Siegert shift**. Another way is to directly treat the non-RWA Hamiltonian using time-dependent perturbation theory.

8.5 A semi-classical treatment of Electromagnetically Induced Transparency (EIT) (*)[4]

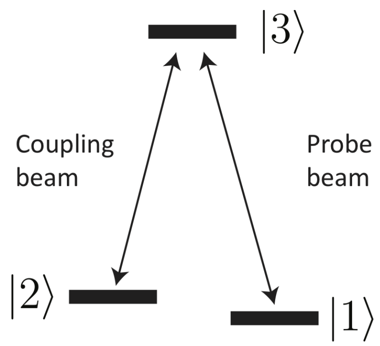

Figure 1: Level diagram of a 3-level lambda atom

Electromagnetically induced transparency [5] (EIT) is an atomic phenomenon where an atomic medium is rendered transparent for an electromagnetic field with a certain frequency ('probe field') in the presence of another field ('coupling field'). Consider an atom with 3 levels (Fig1) with energies ω_1, ω_2 and ω_3, interacting with an external field, $\vec{E}(t)$. In a rotating frame rotating with the frequency ω_1, the Hamiltonian of the system is:

$$\hat{H} = \hbar\omega_{21} \left|2\right\rangle \left\langle 2\right| + \hbar\omega_{31} \left|3\right\rangle \left\langle 3\right| + e\hat{\vec{r}} \cdot \vec{E} \tag{12}$$

where $\omega_{21} = \omega_2 - \omega_1$ and $\omega_{31} = \omega_3 - \omega_1$. Also consider that the states $|2\rangle$ and $|3\rangle$ have phenomenological decay rates of Γ_2 and Γ_3. There are two external fields: the coupling field with amplitude E_c and frequency ω_c and the probe field with amplitude E_p and frequency ω_p. The total field then can be written as:

$$\vec{E}(t) = \frac{1}{2}\vec{E}_c e^{-i\omega_c t} + \frac{1}{2}\vec{E}_p e^{-i\omega_p t} + \text{C.C.} \tag{13}$$

[3] Ficek, Zbigniew, and Mohamed Ridza Wahiddin. *Quantum optics for beginners*. Jenny Stanford Publishing, 2016.

[4] Graded exercise

[5] c.f S. E. Harris, J. E. Field, and A. Imamoglu "Nonlinear optical processes using electromagnetically induced transparency" Phys. Rev. Lett. 64, 1107

1. Write the state of the atom in the general form of $|\psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle + a_3(t)|3\rangle$ and use Schrödinger equation to obtain the equations of motion for the amplitudes $a_1(t)$, $a_2(t)$ and $a_3(t)$. Use the definitions of the dipole moments: $\vec{\mu}_{ij} = \langle i| e\hat{\vec{r}} |j\rangle$ for $i, j = 1, 2, 3$ and $i \neq j$. Introduce ad hoc the terms that describe decay of levels $|2\rangle$ and $|3\rangle$.

2. Move to an interaction picture described by the transformations below:

$$a_2(t) = \tilde{a}_2(t)e^{-i(\omega_p-\omega_c)t} \tag{14}$$

$$a_3(t) = \tilde{a}_3(t)e^{-i\omega_p t} \tag{15}$$

Consider that ω_c is tuned close to $\omega_3 - \omega_2$ and that ω_p is tuned close to ω_{31}. Make proper rotating wave approximations to get *time-independent* equations of motion for the amplitudes in the following form:

$$\dot{\tilde{a}}_1 = -i\frac{\Omega_{13}^*}{2}\tilde{a}_3 \tag{16}$$

$$\dot{\tilde{a}}_2 = -i(\Delta_2 - i\frac{\Gamma_2}{2})\tilde{a}_2 - i\frac{\Omega_{23}^*}{2}\tilde{a}_3 \tag{17}$$

$$\dot{\tilde{a}}_3 = -i(\Delta_3 - i\frac{\Gamma_3}{2})\tilde{a}_3 - i\frac{\Omega_{13}}{2}\tilde{a}_1 - i\frac{\Omega_{23}}{2}\tilde{a}_2 \tag{18}$$

3. Find the steady-state solution of the equations of motion Eq. 17,Eq. 18. Assume that probe power is weak enough $|\vec{E}_p| \ll |\vec{E}_c|$ so that Eq. 16 is satisfied automatically, and $\tilde{a}_1(t) = a_0$ is constant.

4. The polarization vector of an atomic medium with number density of N is given by $\vec{P}(t) = N\langle\psi(t)| e\hat{\vec{r}} |\psi(t)\rangle$. Express $\vec{P}(t)$ as a function of $\tilde{a}_1(t)$, $\tilde{a}_2(t)$ and $\tilde{a}_3(t)$ and recast it in the following form: $\sum_i \vec{P}(\omega_i)e^{i\omega_i t} + C.C.$, where the sum is over different involved frequency components.

5. The susceptibility for a frequency component ω_p is defined as $\vec{P}(\omega_p) = \epsilon_0\chi(\omega_p)\vec{E}_p$. Use the solution of part 3 to derive an expression for $\chi(\omega_p)$. Separate the real and imaginary part of $\chi(\omega_p)$and sketch them as a function of ω_p.

6. For the case of a resonant pump ($\omega_c = \omega_3 - \omega_2$), compute the group velocity for the probe field at $\omega_p = \omega_{31}$. Consider the case of a strong coupling field; i.e. where the corresponding Rabi frequency ($\Omega_{23} = \frac{|\mu_{23}||E_c|}{\hbar}$) is much greater than other frequency scales in the problem ($\Omega_{23} \gg \Gamma_2,\ \Gamma_3,\ etc.$). Simplify the expression for the group velocity and show it is equal to[6]:

$$v_g = \frac{\hbar c\epsilon_0}{2\omega_p}\frac{\Omega_{23}^2}{|\mu_{13}|^2 N} \tag{19}$$

7. Can the group velocity exceed the speed of light? Under what conditions? What does it mean for the transfer of information? Refer to the following references for more details:
   - Wang, L., Kuzmich, A. & Dogariu, A. Gain-assisted superluminal light propagation. Nature 406, 277–279 (2000)
   - Stenner, M., Gauthier, D. & Neifeld, M. The speed of information in a 'fast-light' optical medium. Nature 425, 695–698 (2003)

[6] c.f Hau, L., Harris, S., Dutton, Z. et al. Light speed reduction to 17 metres per second in an ultracold atomic gas. Nature 397, 594-598 (1999)