Quantum Electrodynamics and Quantum Optics

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Exercise No.8

8.1 Equivalence of rE- and pA-Hamiltonians

We consider a single mode electric field

$$\mathbf{E}(t) = \mathbf{E}_0 \cos(\omega t) \tag{1}$$

The vector potential is given by

$$\mathbf{A}(t) = -\frac{1}{\omega}\mathbf{E}_0 \sin(\omega t) \tag{2}$$

The rE and pA Hamiltonian are then replaced by

$$\hat{H}_{\mathbf{r}\cdot\mathbf{E}} = -\, q\mathbf{r}\cdot\mathbf{E} = -q\mathbf{r}\cdot\mathbf{E}_0\cos(\omega t) \tag{3}$$

$$\hat{H}_{\mathbf{p}\cdot\mathbf{A}} = -\,\frac{q}{m}\mathbf{p}\cdot\mathbf{A} = \frac{q}{m\omega}\mathbf{p}\cdot\mathbf{E}_0\sin(\omega t) \tag{4}$$

Now we calculate the off-diagonal elements of the transition matrix of the two Hamiltonian:

$$\langle f|\hat{H}_{\mathbf{r}\cdot\mathbf{E}}|i\rangle = -q\,\langle f|\mathbf{r}\cdot\mathbf{E}_0|i\rangle\cos(\omega t) \tag{5}$$

$$\begin{aligned}\langle f|\hat{H}_{\mathbf{p}\cdot\mathbf{A}}|i\rangle &= \langle f|\frac{q}{m\omega}\mathbf{p}\cdot\mathbf{E}_0|i\rangle\sin(\omega t)\\ &= \langle f|\frac{q}{m\omega}\left(\frac{im}{\hbar}\right)[\hat{H},\mathbf{r}]\cdot\mathbf{E}_0|i\rangle\sin(\omega t)\\ &= \frac{iq}{\omega\hbar}[\langle f|\hat{H}\mathbf{r}\cdot\mathbf{E}_0|i\rangle - \langle f|\mathbf{r}\cdot\mathbf{E}_0\hat{H}|i\rangle]\sin(\omega t)\\ &= \frac{iq}{\omega}(\omega_f-\omega_i)\,\langle f|\mathbf{r}\cdot\mathbf{E}_0|i\rangle\sin(\omega t)\end{aligned} \tag{6}$$

Therefore, the off-diagonal elements are different by only a factor related to the frequency.

Solution: Wigner-Weisskopf theory

The solution can be find in Scully, M.O. and Zubairy, M.S., 1999. Quantum optics, Chapter 6.3.

8.2 Synthesizing arbitrary quantum states

8.2.1 Describe the Hamiltonian

The Ω_{JC} term decribes the interaction between atom('resonator mode') and photons. The Ω_{Rabi} term and the Ω_D terms describes the external drive applied on the resonator and the waveguide, which can be understood per writing down the evolution equation.

8.2.2 State Evolution

Only Rabi Ω_{Rabi}:

$$|\psi(t)\rangle = \cos(1/2\Omega_{Rabi}t)|1,0\rangle + \sin(1/2\Omega_{Rabi}t)|0,0\rangle \tag{7}$$

Only Drive Ω_D:

$$|\psi(t)\rangle = e^{iHt/\hbar}|1,0\rangle = |1,\Omega_D t/2\rangle \;\; \text{(Coherent state)} \tag{8}$$

Only JC-interaction Ω_{JC}:
The equation reads:

$$\begin{aligned}\partial_t a &= -i\Omega_{JC}/2\, b\\ \partial_t b &= -i\Omega_{JC}/2\, a\end{aligned} \tag{9}$$

Thus:

$$|\psi(t)\rangle = \cos(1/2\,|\Omega_{JC}|t)|1,0\rangle + \sin(1/2\,|\Omega_{JC}|t)|0,1\rangle \tag{10}$$

8.2.3 Arbitrary state generation

Please refer to *Synthesizing arbitrary quantum states in a superconducting resonator.*

Table 1 | Sequence to generate the resonator state $|\psi\rangle = |1\rangle + i|3\rangle$

Sequence of states, operations	Operational parameter	System state, parameter value
$\|\psi\rangle$		$\|g\rangle(0.707\|1\rangle+0.707i\|3\rangle)$
S_3	$\tau_3\Omega$	1.81
Q_3	q_3	3.14
$\|\psi_2\rangle$		$\|g\rangle(-0.557i\|0\rangle+0.707\|2\rangle)+0.436\|e\rangle\|1\rangle$
Z_2	$t_2\Delta$	4.71
S_2	$\tau_2\Omega$	1.44
Q_2	q_2	$-2.09 - 2.34i$
$\|\psi_1\rangle$		$(0.553-0.62i)\|g\rangle\|1\rangle-(0.371+0.416i)\|e\rangle\|0\rangle$
Z_1	$t_1\Delta$	3.26
S_1	$\tau_1\Omega$	1.96
Q_1	q_1	$-2.71-1.59i$
$\|\psi_0\rangle$		$(0.197-0.98i)\|g\rangle\|0\rangle$

8.3 Solution : Bloch-Siegert shift: An example of non-RWA effect

Here we attempt to give the solution using the Laplace Transform method. Considering the equations for the coherences given by the coupled system

$$\begin{cases} \dot\rho_{12} = \left(i\omega_0 - \frac{\Gamma}{2}\right)\rho_{12} + \Gamma\rho_{21}, \\ \dot\rho_{21} = \left(-i\omega_0 - \frac{\Gamma}{2}\right)\rho_{21} + \Gamma\rho_{12}, \end{cases} \tag{11}$$

and the intitial conditions $\rho_{12}(0), \rho_{21}(0)$, we perform a Laplace transformation of the system. It is given by

$$\mathcal{L}[f(t)](s) = \int_0^\infty \mathrm{d}t\, f(t)e^{-st}, \tag{12}$$

where the parameter $s = \sigma + i\omega \in \mathbb{C}$. We will use the following properties for $\mathcal{L}$:

$$\begin{cases} \mathcal{L}[cf](s) = c\mathcal{L}[f](s), \text{ with } c \in \mathbb{C} \\ \mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0). \end{cases} \tag{13}$$

The tranformed system is given by

$$\begin{cases} s\mathcal{L}[\rho_{12}](s) - \rho_{12}(0) = \left(i\omega_0 - \frac{\Gamma}{2}\right)\mathcal{L}[\rho_{12}](s) + \Gamma\mathcal{L}[\rho_{21}](s), \\ s\mathcal{L}[\rho_{21}](s) - \rho_{21}(0) = \left(-i\omega_0 - \frac{\Gamma}{2}\right)\mathcal{L}[\rho_{21}](s) + \Gamma\mathcal{L}[\rho_{12}](s), \end{cases} \tag{14}$$

then

$$\begin{cases} \mathcal{L}[\rho_{12}](s) = \frac{1}{s-i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\mathcal{L}[\rho_{21}](s) + \rho_{12}(0)\right) = \frac{1}{s-i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\frac{\Gamma\mathcal{L}[\rho_{12}](s)+\rho_{21}(0)}{s+i\omega_0+\frac{\Gamma}{2}} + \rho_{12}(0)\right), \\ \mathcal{L}[\rho_{21}](s) = \frac{1}{s+i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\mathcal{L}[\rho_{12}](s) + \rho_{21}(0)\right) = \frac{1}{s+i\omega_0+\frac{\Gamma}{2}}\left(\Gamma\frac{\Gamma\mathcal{L}[\rho_{21}](s)+\rho_{12}(0)}{s-i\omega_0+\frac{\Gamma}{2}} + \rho_{21}(0)\right). \end{cases} \tag{15}$$

We can now isolate the Laplace transform of the coherences to obtain

$$\begin{cases} \mathcal{L}[\rho_{12}](s) = \frac{\left(s+\frac{\Gamma}{2}+i\omega_0\right)\rho_{12}(0)+\Gamma\rho_{21}(0)}{\left|s+\frac{\Gamma}{2}+i\omega_0\right|^2-\Gamma^2}, \\ \mathcal{L}[\rho_{21}](s) = \frac{\left(s+\frac{\Gamma}{2}-i\omega_0\right)\rho_{21}(0)+\Gamma\rho_{12}(0)}{\left|s+\frac{\Gamma}{2}-i\omega_0\right|^2-\Gamma^2}. \end{cases} \tag{16}$$

Assuming that $\Gamma \ll \omega_0$ (we will add a term $\propto (\Gamma/\omega_0)^4$), we write the denominator as

$$\begin{aligned} D(s) &= \left|s + \frac{\Gamma}{2} - i\omega_0\right|^2 - \Gamma^2 = \left(s + \frac{\Gamma}{2}\right)^2 + \omega_0^2\left(1 - \frac{\Gamma^2}{\omega_0^2}\right) \\ &= \left(s + \frac{\Gamma}{2}\right)^2 + \omega_0^2\left(1 - \frac{\Gamma^2}{\omega_0^2} + \left(\frac{\Gamma^2}{2\omega_0^2}\right)^2\right) = \left(s + \frac{\Gamma}{2}\right)^2 + \left(\omega_0 - \frac{\Gamma^2}{2\omega_0}\right)^2 \\ &= (s - z_+)(s - z_-), \end{aligned} \tag{17}$$

where $z_\pm = -\frac{\Gamma}{2} \pm i\left(\omega_0 - \frac{\Gamma^2}{2\omega_0}\right)$ are the poles of both Laplace transforms. The real part of $z_\pm$ is linked to the damping of the solution, while the imaginary part corresponds to the oscillating part. We see that the frequency changes from $\omega_0 \to \omega_0 - \frac{\Gamma^2}{2\omega_0}$, which is the so-called Bloch-Siegert shift when $\Gamma \ll \omega_0$. To gain a better understanding, we consider the following two Laplace transforms :

$$\begin{cases} \mathcal{L}[e^{-at}\cos(bt)](s) = \frac{s+a}{(s+a)^2+b^2}, \\ \mathcal{L}[e^{-at}\sin(bt)](s) = \frac{b}{(s+a)^2+b^2}, \end{cases} \tag{18}$$

Recall that the Laplace transform of ρ_{12} (a similar expression was found for ρ_{21}) was found to be

$$\mathcal{L}[\rho_{12}](s) = \frac{\left(s + \frac{\Gamma}{2}\right)\rho_{12}(0) + i\omega_0\rho_{12}(0) + \Gamma\rho_{21}(0)}{\left(s + \frac{\Gamma}{2}\right)^2 + \left(\omega_0 - \frac{\Gamma^2}{2\omega_0}\right)^2}, \tag{19}$$

from the linearity of the Laplace transform, we deduce that in the time-domain, the expression of the coherence is

$$\rho_{12}(t) = e^{-\Gamma t/2}\left\{\rho_{12}(0)\cos\left[\left(\omega_0 - \frac{\Gamma^2}{2\omega_0}\right)t\right] + \frac{i\omega_0\rho_{12}(0) + \Gamma\rho_{21}(0)}{\omega_0 - \frac{\Gamma^2}{2\omega_0}}\sin\left[\left(\omega_0 - \frac{\Gamma^2}{2\omega_0}\right)t\right]\right\}. \tag{20}$$

With this final expression, we can clearly observe the decaying part with a rate $\Gamma/2$ and the oscillating part with the Bloch-Siegert shift with corrected frequency $\omega_0 - \frac{\Gamma^2}{2\omega_0}$.