

Quantum Electrodynamics and Quantum Optics
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.7

Solution: Semiclassical Atom-Field interaction

1. The dynamic equation of a charged particle in an electromagnetic field is given by

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (1)$$

Introducing the electromagnetic potentials U and $\mathbf{A}$, then

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = -\nabla U - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \quad (2)$$

Use the relation that

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - \mathbf{v} \cdot \nabla \mathbf{A} \quad (3)$$

we can rewrite the dynamic equation as

$$\frac{d\mathbf{p}}{dt} = e(-\nabla(U - \mathbf{v} \cdot \mathbf{A}) - \frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \cdot \nabla \mathbf{A}) \quad (4)$$

Since the particle is moving,

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{A} \quad (5)$$

Then the dynamic equation can be written as

$$\frac{d}{dt}(\mathbf{p} + e\mathbf{A}) = -e\nabla(U - \mathbf{v} \cdot \mathbf{A}) \quad (6)$$

Comparing with the Lagrangian equation

$$\frac{d}{dt} \frac{\partial \mathbf{L}}{\partial \mathbf{v}_i} - \frac{\partial \mathbf{L}}{\partial \mathbf{x}_i} = 0 \quad (7)$$

We can easily identify the Lagrangian

$$L = \frac{1}{2}m\mathbf{v}^2 - eU + e\mathbf{A} \cdot \mathbf{v} \quad (8)$$

describes the dynamics of this particle.

- 2.

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + e\mathbf{A} \quad (9)$$

Then the Hamiltonian is

$$H = \mathbf{p} \cdot \dot{\mathbf{r}} - L = \frac{1}{2}m\mathbf{v}^2 + eU = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + eU \quad (10)$$

3. The Schrodinger equation for the new wave function ϕ is given by

$$i\hbar \left[i \frac{e}{\hbar} \dot{\mathbf{A}} \cdot \mathbf{r} \phi(r, t) + \dot{\phi}(r, t) \right] \exp\left(\frac{ie}{\hbar} \mathbf{A} \cdot \mathbf{r}\right) = \exp\left(\frac{ie}{\hbar} \mathbf{A} \cdot \mathbf{r}\right) \left[\frac{p^2}{2m} + V(r) \right] \phi(r, t) \quad (11)$$

This equation, after the cancellation of the exponential factor and some rearrangement, takes the simple form

$$i\hbar \dot{\phi}(r, t) = H\phi(r, t) \quad (12)$$

where $H = H_0 + H_I$, $H_0 = \frac{(-i\hbar\nabla)^2}{2m} + V(r)$, $H_I = -e\mathbf{r} \cdot \mathbf{E}(\mathbf{r}_0, t)$

4. (a) Using the completeness relation, $|1\rangle\langle 1| + |2\rangle\langle 2| = 1$, we can write H_0 as

$$H_0 = \hbar\omega_1|1\rangle\langle 1| + \hbar\omega_2|2\rangle\langle 2| \quad (13)$$

The interaction Hamiltonian can be written as

$$H_I = -e\mathbf{r} \cdot \mathbf{E}(t) \quad (14)$$

$$= -e(|1\rangle\langle 1| + |2\rangle\langle 2|)\mathbf{r}(|1\rangle\langle 1| + |2\rangle\langle 2|)\mathbf{E}(t) \quad (15)$$

Note only 2 terms in the right hand side are nonzero because of the parity, i.e.,

$$\langle 1|\hat{\mathbf{r}}|1\rangle = \iint d^3r d^3r' \langle 1|r\rangle \langle r|\hat{\mathbf{r}}|r'\rangle \langle r'|1\rangle \quad (16)$$

$$= \int d^3r \psi(r)^* \psi(r') \delta(r - r') r \quad (17)$$

$$= \int d^3r |\psi(r)|^2 r \quad (18)$$

$$= 0 \quad (19)$$

Therefore,

$$H_I = -(\mathbf{d}_{12}|1\rangle\langle 2| + \mathbf{d}_{21}|2\rangle\langle 1|) \cdot \mathbf{E}(t) \quad (20)$$

where $\mathbf{d}_{12} = \mathbf{d}_{21} = \langle 1|e\mathbf{r}|2\rangle = \langle 2|e\mathbf{r}|1\rangle$.

Using the Schrodinger equation, the evolution of state $|\Psi(t)\rangle = a(t)|1\rangle + b(t)|2\rangle$ is given by

$$i\hbar(\dot{a}|1\rangle + \dot{b}|2\rangle) = \hbar\omega_1 a|1\rangle + \hbar\omega_2 b|2\rangle - \mathbf{d}_{12} \cdot \mathbf{E}_0 \cos(\omega t)b|1\rangle - \mathbf{d}_{21} \cdot \mathbf{E}_0 \cos(\omega t)a|2\rangle \quad (21)$$

Thus,

$$\dot{a} = -i\omega_1 a + i\frac{\Omega_R}{2}(e^{i\omega t} + e^{-i\omega t})\beta \quad (22)$$

$$\dot{b} = -i\omega_2 b + i\frac{\Omega_R}{2}(e^{i\omega t} + e^{-i\omega t})a \quad (23)$$

Making the substitutions $a = \alpha e^{-i\omega_1 t}$, and $b = \beta e^{-i\omega_2 t}$

$$\dot{\alpha} = i\frac{\Omega_R}{2}(e^{i\omega t} + e^{-i\omega t})\beta e^{-i\omega_{21}t} \quad (24)$$

$$\dot{\beta} = i\frac{\Omega_R}{2}(e^{i\omega t} + e^{-i\omega t})\alpha e^{i\omega_{21}t} \quad (25)$$

Then making the rotating wave approximation(ignore terms proportional to $e^{\pm i(\omega + \omega_{21})t}$), we have

$$\dot{\alpha} = i\frac{\Omega_R}{2}e^{i(\omega - \omega_{21})t}\beta \quad (26)$$

$$\dot{\beta} = i\frac{\Omega_R}{2}e^{-i(\omega - \omega_{21})t}\alpha \quad (27)$$

where $\Omega_R = \frac{\mathbf{d}_{21} \cdot \mathbf{E}_0}{\hbar}$ is the Rabi frequency and $\omega_{21} = \omega_2 - \omega_1$ is the transition frequency. From the above two equations, we have

$$\ddot{\alpha} - i(\omega - \omega_{21})\dot{\alpha} + \frac{\Omega_R^2}{4}\alpha = 0 \quad (28)$$

The solution has the form of $C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, where the λ_1 and λ_2 are the solutions to the equation

$$\lambda^2 - i(\omega - \omega_{21})\lambda + \frac{\Omega_R^2}{4} = 0 \quad (29)$$

$$\lambda_{1,2} = \frac{i(\omega - \omega_{21}) \pm \sqrt{-(\omega - \omega_{21})^2 - \Omega_R^2}}{2} = i \frac{(\omega - \omega_{21}) \pm \sqrt{\Omega_R^2 + (\omega - \omega_{21})^2}}{2} \quad (30)$$

Define $\Delta = \omega - \omega_{21}$ and $\Omega = \sqrt{\Omega_R^2 + \Delta^2}$, the solution of α and β (similar to α but with a sign different) can then be written as

$$\alpha = (C_1 e^{i\Omega t/2} + C_2 e^{-i\Omega t/2}) e^{i\Delta t/2} \quad (31)$$

$$\beta = (C_3 e^{i\Omega t/2} + C_4 e^{-i\Omega t/2}) e^{-i\Delta t/2} \quad (32)$$

C_1, C_2, C_3, C_4 is determined by the initial condition and . If $|\Psi(t=0)\rangle = |1\rangle$, we have

$$C_1 + C_2 = 1 \quad (33)$$

$$C_3 + C_4 = 0 \quad (34)$$

$$C_1(\Omega + \Delta) + C_2(-\Omega + \Delta) = 0 \quad (35)$$

$$C_3(\Omega - \Delta) + C_4(-\Omega - \Delta) = \Omega_R \quad (36)$$

which gives

$$C_1 = \frac{\Omega - \Delta}{2\Omega} \quad (37)$$

$$C_2 = \frac{\Omega + \Delta}{2\Omega} \quad (38)$$

$$C_3 = \frac{\Omega_R}{2\Omega} \quad (39)$$

$$C_4 = \frac{-\Omega_R}{2\Omega} \quad (40)$$

(b)

$$\beta(t) = \frac{\Omega_R}{2\Omega} (e^{i\Omega t/2} - e^{-i\Omega t/2}) e^{-i\Delta t} = i \frac{\Omega_R}{\Omega} \sin\left(\frac{\Omega t}{2}\right) e^{-i\Delta t} \quad (41)$$

The probability of being in the excited states is

$$P_2(t) = |\beta(t)|^2 = \frac{\Omega_R^2}{\Omega^2} \sin^2\left(\frac{\Omega t}{2}\right) = \frac{1}{1 + (\Delta/\Omega_R)^2} \sin^2\left(\frac{\Omega t}{2}\right) \quad (42)$$

5. For the resonant case, $\Omega = \Omega_R, \Delta = \omega - \omega_{21} = 0$, the expectation value of the dipole moment is given by

$$e\langle\psi(t)|\mathbf{r}|\psi(t)\rangle = \alpha^* \beta \mathbf{d}_{1,2} e^{-i\omega_{21}t} + \alpha \beta^* \mathbf{d}_{2,1} e^{i\omega_{21}t} = 2\text{Re}\{\alpha^* \beta \mathbf{d}_{1,2} e^{-i\omega_{21}t}\} \quad (43)$$

$$= 2\text{Re}\{(C_1^* e^{-i\Omega t/2} + C_2^* e^{i\Omega t/2})(C_3 e^{i\Omega t/2} + C_4 e^{-i\Omega t/2}) \mathbf{d}_{1,2} e^{-i\omega t}\} \quad (44)$$

$$= 2\text{Re}\{(C_1^* C_3 + C_2^* C_4 + C_1^* C_4 e^{-i\Omega t} + C_2^* C_3 e^{i\Omega t}) \mathbf{d}_{1,2} e^{-i\omega t}\} \quad (45)$$

Obvisouly, the dipole moment exhibits a fast modulation at the oscillation frequency(ω) of the field as well as a slow amplitude modulation at Rabi frequency.

6. When is on resonace and if the initial condition is $|\Psi(t=0)\rangle = |1\rangle$, $C_1 = C_2 = C_3 = -C_4 = 1/2$.

$$e\langle\psi(t)|\mathbf{r}|\psi(t)\rangle = 2 \sin(\Omega t/2) \cos(\Omega t/2) \sin \omega t \mathbf{d}_{1,2} = \sin(\Omega t) \sin \omega t \mathbf{d}_{1,2} \quad (46)$$

$$U = \int_0^{\pi/\Omega} \int_V \mathbf{E}(\mathbf{r}_0, t) \frac{d}{dt} \langle \mathbf{P}(\mathbf{r}_0, t) \rangle dV dt \quad (47)$$

Using the definition of Rabi frequency and integration over the space, U can be written as

$$U = \int_0^{\pi/\Omega} \hbar\Omega \cos(\omega t) \{ \Omega \cos(\Omega t) \sin(\omega t) + \omega \sin(\Omega t) \cos(\omega t) \} dt \quad (48)$$

Typically $\omega \gg \Omega$, thus we could use $\overline{\sin(\omega t) \cos(\omega t)} = 0$ and $\overline{\cos(\omega t) \cos(\omega t)} = \frac{1}{2}$. Therefore, from 0 to π/Ω ,

$$U = \int_0^{\pi/\Omega} \hbar\Omega\omega \sin(\Omega t) dt = \hbar\omega \quad (49)$$

atom absorb energy from the field. Similarly, from π/Ω to $2\pi/\Omega$

$$U = \int_{\pi/\Omega}^{2\pi/\Omega} \hbar\Omega\omega \sin(\Omega t) dt = -\hbar\omega \quad (50)$$

same amount of energy is given back to the field.

7.1 Solution : Coherent population trapping

Beforehand, we start by reminding what is meant by Rotating Wave Approximation. Under a unitary transformation of the quantum states $|\psi(t)\rangle_{\text{new}} = \hat{U}(t) |\psi(t)\rangle$, the unitary part of the dynamics of the system described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$ is modified. For a general change of frame $\hat{U}(t)$, in the Schrödinger picture we have the usual equation $i\hbar\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$, where the partial derivative with respect to time is noted as ∂_t . We can derive the modified Schrödinger equation in the new frame as (we omit the explicit time dependency)

$$\begin{aligned} i\hbar\partial_t |\psi(t)\rangle_{\text{new}} &= i\hbar\partial_t (\hat{U} |\psi\rangle) \\ &= i\hbar(\partial_t \hat{U}) |\psi\rangle + i\hbar\hat{U}\partial_t |\psi\rangle \\ &= (i\hbar\partial_t \hat{U} + \hat{U}\hat{H}) |\psi\rangle \\ &= (i\hbar(\partial_t \hat{U})\hat{U}^\dagger + \hat{U}\hat{H}\hat{U}^\dagger) |\psi\rangle_{\text{new}} \\ &= \hat{H}_{\text{new}} |\psi\rangle_{\text{new}}. \end{aligned} \quad (51)$$

This equation with $\hat{H}_{\text{new}} = i\hbar(\partial_t \hat{U})\hat{U}^\dagger + \hat{U}\hat{H}\hat{U}^\dagger$ represents the same system but in a different frame. In the case of the RWA, this change of frame is performed by the operator $\hat{U}(t) = \exp(i\hat{H}_0 t/\hbar)$ (keep only interaction terms in the Schrödinger equation) and furthermore, we neglect all the counter rotative terms (not energy conserving). These two operations lead in our case to the Hamiltonian $\hat{H}_{\text{new}}^{\text{RWA}} = \hat{H}_D$. Note that we did not go back to the original frame, we shall keep working in this so-called new frame for this exercise and we will drop the "new" for convenience.

1. Now that the setup is clear, we want to use the Schrödinger equation for the evolution of $|\Psi\rangle = a(t)|a\rangle + b(t)|b\rangle + c(t)|c\rangle$ in this new frame (under the RWA). When both fields are resonant $\omega_1 = \omega_{ab}$ and $\omega_2 = \omega_{ac}$, the interaction Hamiltonian simplifies to

$$\hat{H}_D = -\frac{\hbar}{2} (\Omega_{R1} |a\rangle \langle b| + \Omega_{R2} |a\rangle \langle c| + \text{h.c.}) \quad (52)$$

Upon projection of the Schrödinger equation for $|\Psi\rangle$ onto the three states $|a\rangle$, $|b\rangle$ and $|c\rangle$, we obtain a set of coupled differential equations for the amplitudes.

$$\begin{cases} i\hbar\partial_t a(t) = \langle a|\hat{H}_D|\Psi(t)\rangle = -\frac{\hbar\Omega_{R1}}{2} b(t) - \frac{\hbar\Omega_{R2}}{2} c(t) \\ i\hbar\partial_t b(t) = \langle b|\hat{H}_D|\Psi(t)\rangle = -\frac{\hbar\Omega_{R1}}{2} a(t) \\ i\hbar\partial_t c(t) = \langle c|\hat{H}_D|\Psi(t)\rangle = -\frac{\hbar\Omega_{R2}}{2} a(t) \end{cases} \quad (53)$$

This system can also be written in its matrix form as

$$i\hbar\partial_t \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{R1} & \Omega_{R2} \\ \Omega_{R1} & 0 & 0 \\ \Omega_{R2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} \quad (54)$$

Now we verify that the three states given are indeed eigenstates (otherwise just diagonalise the matrix Hamiltonian).

$$\begin{cases} |\Psi_+\rangle = \frac{1}{\sqrt{2}} \left(|a\rangle + \frac{\Omega_{R1}^*}{\Omega} |b\rangle + \frac{\Omega_{R2}^*}{\Omega} |c\rangle \right) \\ |\Psi_0\rangle = \frac{\Omega_{R2}}{\Omega} |b\rangle - \frac{\Omega_{R1}}{\Omega} |c\rangle \\ |\Psi_-\rangle = \frac{1}{\sqrt{2}} \left(|a\rangle - \frac{\Omega_{R1}^*}{\Omega} |b\rangle - \frac{\Omega_{R2}^*}{\Omega} |c\rangle \right) \end{cases} \quad (55)$$

We find that they are indeed normalised eigenstates, with respective eigenvalues of $+\frac{\hbar\Omega}{2}, 0, -\frac{\hbar\Omega}{2}$, where $\Omega = \sqrt{|\Omega_{R1}|^2 + |\Omega_{R2}|^2}$.

2. Computing the transition amplitudes on resonance from $|NC\rangle$ and $|C\rangle$ to $|a\rangle$ yields

$$\begin{aligned} \langle a|\hat{H}_D|NC\rangle &= -\frac{\hbar}{2} (\Omega_{R1} \langle b| + \Omega_{R2} \langle c|) \left(\frac{\Omega_{R2}}{\Omega} |b\rangle - \frac{\Omega_{R1}}{\Omega} |c\rangle \right) \\ &= -\frac{\hbar}{2} \left(\frac{\Omega_{R1}\Omega_{R2}}{\Omega} - \frac{\Omega_{R2}\Omega_{R1}}{\Omega} \right) = 0 \\ \langle a|\hat{H}_D|C\rangle &= -\frac{\hbar}{2} (\Omega_{R1} \langle b| + \Omega_{R2} \langle c|) \left(\frac{\Omega_{R1}}{\Omega} |b\rangle + \frac{\Omega_{R2}}{\Omega} |c\rangle \right) \\ &= -\frac{\hbar}{2} \left(\frac{\Omega_{R1}^2}{\Omega} + \frac{\Omega_{R2}^2}{\Omega} \right) = -\frac{\hbar\Omega}{2} \neq 0 \end{aligned} \quad (56)$$

This shows exactly what we expect, namely that the state $|NC\rangle$ is not coupled to $|a\rangle$, whereas $|C\rangle$ is. It is the superposition of the two ground states that leads to a destructive interference of population in the excited state.