Quantum Electrodynamics and Quantum Optics
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.6

6.1 Quantization of coupled resonators

6.1.1

The Lagrangian of a classical LC circuit writes

$$L = \frac{1}{2}LI^2 - \frac{q^2}{2C} = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C} \tag{1}$$

6.1.2

Introduce the flux of the inductor by

$$\Phi = \frac{\partial L}{\partial \dot{q}} = LI \tag{2}$$

which is the conjugate momentum of q. Hence the Lagrangian can be rewritten in terms of Φ and q by

$$L = \frac{\Phi^2}{2L} - \frac{q^2}{2C} \tag{3}$$

The Hamiltonian of the system is given by

$$H = \Phi\dot{q} - L = \frac{\Phi^2}{2L} + \frac{q^2}{2C} \tag{4}$$

6.1.3

Due to commutation relation $[q, \Phi] = i\hbar$, introduce ladder operators by

$$\begin{aligned} a &= \frac{1}{\sqrt{2L\hbar\omega}}\Phi + i\frac{1}{\sqrt{2C\hbar\omega}}q \\ a^\dagger &= \frac{1}{\sqrt{2L\hbar\omega}}\Phi - i\frac{1}{\sqrt{2C\hbar\omega}}q \end{aligned} \tag{5}$$

where $\omega = 1/\sqrt{LC}$ is the resonance frequency of the LC circuit. The Hamiltonian is thus rewritten by

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) = \hbar\omega\left(n + \frac{1}{2}\right) \tag{6}$$

6.1.4

Let $|n\rangle$ be the energy eigenstate with eigenvalue E_n, so

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle \tag{7}$$

Therefore the energy level is given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N} \tag{8}$$

At ground state $|0\rangle$, we have

$$
\begin{aligned}
\langle 0|\Phi|0\rangle &= \sqrt{2C\hbar\omega}\left\langle 0\left|\frac{a+a^{\dagger}}{2}\right|0\right\rangle = 0\\
\langle 0|q|0\rangle &= \sqrt{2L\hbar\omega}\left\langle 0\left|\frac{a-a^{\dagger}}{2i}\right|0\right\rangle = 0\\
\left\langle 0\left|\Phi^2\right|0\right\rangle &= 2C\hbar\omega\left\langle 0\left|\frac{a^2+a^{\dagger 2}+aa^{\dagger}+a^{\dagger}a}{4}\right|0\right\rangle = \frac{1}{2}L\hbar\omega\\
\left\langle 0\left|q^2\right|0\right\rangle &= -2L\hbar\omega\left\langle 0\left|\frac{a^2+a^{\dagger 2}-aa^{\dagger}-a^{\dagger}a}{4}\right|0\right\rangle = \frac{1}{2}C\hbar\omega
\end{aligned}
\tag{9}
$$

So the zero-point fluctuations for q and Φ are

$$
\begin{aligned}
q_{\mathrm{ZPF}} &= \langle 0|\Delta q|0\rangle^{1/2} = \left(\left\langle 0\left|q^2\right|0\right\rangle - \langle 0|q|0\rangle^2\right)^{1/2} = \sqrt{\frac{1}{2}C\hbar\omega}\\
\Phi_{\mathrm{ZPF}} &= \langle 0|\Delta\Phi|0\rangle^{1/2} = \left(\left\langle 0\left|\Phi^2\right|0\right\rangle - \langle 0|\Phi|0\rangle^2\right)^{1/2} = \sqrt{\frac{1}{2}L\hbar\omega}
\end{aligned}
\tag{10}
$$

6.1.5

The Lagrangian reads

$$
L = \frac{1}{2}C_1\dot{\Phi}_1^2 + \frac{1}{2}C_2\dot{\Phi}^2 + \frac{1}{2}C_0\left(\dot{\Phi}_1-\dot{\Phi}_2\right)^2 - \frac{\Phi_1^2}{2L_1} - \frac{\Phi_2^2}{2L_2} \tag{11}
$$

or in a matrix form

$$
L = \frac{1}{2}\dot{\Phi}C\dot{\Phi} - \frac{1}{2}\Phi L^{-1}\Phi \tag{12}
$$

where

$$
C = \begin{pmatrix} C_1+C_0 & -C_0 \\ -C_0 & C_2+C_0 \end{pmatrix} \tag{13}
$$

and

$$
L^{-1} = \begin{pmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{pmatrix} \tag{14}
$$

6.1.6

The canonical conjugate of Φ is

$$
Q_i = \frac{\partial L}{\partial \dot{\Phi}_i} = C_{ij}\dot{\Phi}_j \tag{15}
$$

Therefore $\dot{\Phi} = C^{-1}Q$ and the Hamiltonian can be given in matrix form in terms of Q and Φ by

$$
H = \frac{1}{2}QC^{-1}Q - \frac{1}{2}\Phi L^{-1}\Phi \tag{16}
$$

where

$$
C^{-1} = \frac{1}{C_1C_2+C_1C_0+C_2C_0}\begin{pmatrix} C_2+C_0 & C_0 \\ C_0 & C_1+C_0 \end{pmatrix} \tag{17}
$$

and

$$
L^{-1} = \begin{pmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{pmatrix} \tag{18}
$$

6.1.7

The Hamiltonian can be written in terms of matrix components by

$$H = \frac{Q_1^2}{2}C_{11}^{-1} + \frac{\Phi_1^2}{2}L_{11}^{-1} + \frac{Q_2^2}{2}C_{22}^{-1} + \frac{\Phi_2^2}{2}L_{22}^{-1} + \frac{Q_1 Q_2}{2}\left(C_{12}^{-1} + C_{21}^{-1}\right) \tag{19}$$

where the first 4 terms represent the contribution of two independent systems and the last term represents the coupling term. Define for $i = 1, 2$

$$\begin{aligned} a_i &= \sqrt{\frac{L_{ii}^{-1}}{2\hbar\omega_i}}\Phi_i + i\sqrt{\frac{C_{ii}^{-1}}{2\hbar\omega_i}}Q_i \\ a_i^\dagger &= \sqrt{\frac{L_{ii}^{-1}}{2\hbar\omega_i}}\Phi_i - i\sqrt{\frac{C_{ii}^{-1}}{2\hbar\omega_i}}Q_i \end{aligned} \tag{20}$$

Then the Hamiltonian can be rewritten as

$$H = H_1 + H_2 + V \tag{21}$$

where

$$\begin{aligned} H_i &= \hbar\omega_i\left(a_i^\dagger a_i + \frac{1}{2}\right), \quad i = 1, 2 \\ V &= -\frac{1}{2}\beta\hbar\sqrt{\omega_1\omega_2}\left(a_1 - a_1^\dagger\right)\left(a_2 - a_2^\dagger\right) \end{aligned} \tag{22}$$

where ω_i and β are defined by

$$\begin{aligned} \omega_i &= \sqrt{C_{ii}^{-1}L_i^{-1}} \\ \beta &= \frac{C_{12}^{-1} + C_{21}^{-1}}{2\sqrt{C_{11}^{-1}C_{22}^{-1}}} = \frac{C_0}{\sqrt{(C_2 + C_0)(C_1 + C_0)}} \end{aligned} \tag{23}$$

6.1.8

As shown in (23), the system energy can be obtained by

$$E_{\text{sys}} = E_1 + E_2 = \hbar\omega_1\left(n_1 + \frac{1}{2}\right) + \hbar\omega_2\left(n_2 + \frac{1}{2}\right) \tag{24}$$

where $n_i \in \mathbb{N}$; the coefficient in V is also given by (23) and (24).

6.2 Anharmonicity of transmon qubits

(a) We simply use the Taylor expansion of the cosine to the lowest order $\cos\hat{\varphi} \approx 1 - \frac{\hat{\varphi}^2}{2}$. Plugin it in the Hamiltonian together with the definition of $\hat{n}$ and $\hat{\varphi}$ yields

$$\begin{aligned} \hat{H} &= 4E_C\hat{n}^2 - E_J\cos\hat{\varphi} \\ &\approx 4E_C\hat{n}^2 + E_J\frac{\hat{\varphi}^2}{2} - E_J \\ &= -4E_C\sqrt{\frac{E_J}{8E_C}}\frac{1}{2}\left(\hat{a} - \hat{a}^\dagger\right)^2 + E_J\sqrt{\frac{2E_C}{E_J}}\frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right)^2 - E_J \\ &= \sqrt{\frac{E_J E_C}{2}}\left(-\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - \left(\hat{a}^\dagger\right)^2 + \hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \left(\hat{a}^\dagger\right)^2\right) - E_J \\ &= \sqrt{2E_J E_C}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) - E_J = \sqrt{8E_J E_C}\left(\hat{a}\hat{a}^\dagger + \frac{1}{2}\right) - E_J. \end{aligned} \tag{25}$$

The approximation that we obtain is the Hamiltonian of a Quantum Harmonic Oscillator, and it is clear that on this level we have an equal spacing of the energy levels.

(b) The fourth order expansion leads to an additional quartic term in the cosine as $\cos\hat{\varphi} \approx 1 - \frac{\hat{\varphi}^2}{2} + \frac{\hat{\varphi}^4}{24}$, which corresponds to a Hamiltonian correction of $\hat{H}^{\text{corr}} = -E_J\frac{\hat{\varphi}^4}{24}$. Since we want to compute the energy correction of this term, only the contributions with an identical number of raising $\hat{a}^\dagger$ and lowering $\hat{a}$ operators will be non-vanishing (terms such as $\hat{a}^\dagger\hat{a}^3$ will not contribute to $\langle n|\hat{H}^{\text{corr}}|n\rangle$). Indeed, one can show that only terms in power of $\hat{n} = \hat{a}^\dagger\hat{a}$ contribute

$$\begin{aligned}
\langle n|\hat{H}^{\text{corr}}|n\rangle &= -\frac{E_J}{24}\langle n|\hat{\varphi}^4|n\rangle = -\frac{E_C}{12}\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^4|n\rangle \\
&= -\frac{E_C}{12}\langle n|\left(\hat{a}^2+\left(\hat{a}^\dagger\right)^2+2\hat{n}+1\right)^2|n\rangle \\
&= -\frac{E_C}{12}\langle n|\Big(4\hat{n}^2+4\hat{n}+1+\hat{a}^2\left(\hat{a}^\dagger\right)^2+\left(\hat{a}^\dagger\right)^2\hat{a}^2 \\
&\qquad +\hat{a}^4+\left(\hat{a}^\dagger\right)^4+2\hat{a}^2+2\left(\hat{a}^\dagger\right)^2+2\hat{a}^2\hat{n}+2\hat{n}\hat{a}^2+2\left(\hat{a}^\dagger\right)^2\hat{n}+2\hat{n}\left(\hat{a}^\dagger\right)^2\Big)|n\rangle \\
&= -\frac{E_C}{12}\langle n|\left(4\hat{n}^2+4\hat{n}+1+\hat{n}^2+3\hat{n}+2+\hat{n}^2-\hat{n}\right)|n\rangle \\
&= -\frac{E_C}{12}\langle n|\left(6\hat{n}^2+6\hat{n}+3\right)|n\rangle = -\frac{E_C}{12}\left(6n^2+6n+3\right)
\end{aligned} \tag{26}$$

Hence the n-th corrected energy level becomes

$$E_n^{\text{corr}} = \sqrt{8E_JE_C}\left(n+\frac{1}{2}\right) - E_J - \frac{E_C}{4}\left(2n^2+2n+1\right) \tag{27}$$

Additionnally, the energy difference between the n and the $n+1$ state is

$$\Delta E_n^{\text{corr}} = E_{n+1}^{\text{corr}} - E_n^{\text{corr}} = \sqrt{8E_JE_C} - E_C\left(n+1\right). \tag{28}$$

Now the anharmonicity is expressed as

$$\eta = \left(\Delta E_1^{\text{corr}} - \Delta E_0^{\text{corr}}\right)/\hbar = E_{n+1}^{\text{corr}} - E_n^{\text{corr}} = -\frac{E_C}{\hbar}. \tag{29}$$

(c) Considering the relative anharmonicity η_r, we compute that

$$\begin{aligned}
\eta_r &= \hbar\eta/E_{10} = \frac{-E_C}{\sqrt{8E_JE_C}-E_C} = \frac{1}{1-\sqrt{8E_J/E_C}} \\
&\xrightarrow[E_J/E_C\gg 1]{} -\sqrt{\frac{E_C}{8E_J}} \sim \left(E_J/E_C\right)^{-1/2}
\end{aligned} \tag{30}$$