Quantum Electrodynamics and Quantum Optics
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.6

6.1 Quantization of coupled resonators[1]

In this exercise we aim to quantize a coupled pair of electrical oscillators using the second quantization principle.

1. Consider a single LC circuit. Write down the Lagrangian of the system in terms of q (the capacitor's charge), and I (the inductor's current). (Hint: you may use the circuit-mechanics analogy of $L \sim$ "mass" and $\frac{1}{C} \sim$ "spring constant")

2. Rewrite the Lagrangian in terms of inductor's flux Φ and q. Show that Φ and q are canonically conjugate variables, by checking the derivatives of the Lagrangian. Then, express the Hamiltonian of the system.

3. Following the second quantization principle, consider canonical conjugate variables as operators with the standard canonical commutation relation ($[\hat{q}, \hat{\Phi}] = -i\hbar$). Derive the quantized Hamiltonian and express it in terms of unit-less creation and annihilation operators ($\hat{a}^\dagger$ and $\hat{a}$). Write down the definition of $\hat{a}^\dagger$ and $\hat{a}$ in terms of $\hat{q}$ and $\hat{\Phi}$.

4. Calculate energy levels of the system and zero-point-fluctuation of charge q_{ZPF} and flux Φ_{ZPF}, i.e. $\langle 0|\Delta\hat{O}^2|0\rangle^{1/2}$.

5. Now consider a pair of capacitively coupled resonators (Fig. 1). Write down the Lagrangian of the system in terms of Φ_1, Φ_2, $\dot{\Phi}_1$, and $\dot{\Phi}_2$. Express it in the matrix form ($\mathcal{L} = \frac{1}{2}\dot{\Phi}C\dot{\Phi} - \frac{1}{2}\Phi L^{-1}\Phi$), and find C and L^{-1} matrices. (Hint: $\dot{\Phi}_i$ is the voltage difference of L_i)

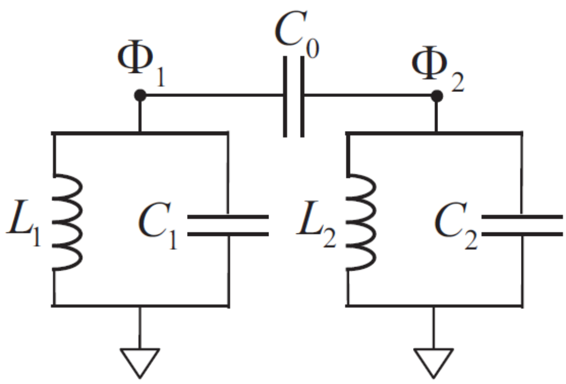

Figure 1: Capacitively coupled electricalresonators.

6. Show that the canonical conjugate is $Q = C\dot{\Phi}$, and derive the Hamiltonian in the matrix form in terms of canonical conjugate variables.

7. Given the definitions below, expand the Hamiltonian in terms of Q_1, Q_2, Φ_1, and Φ_2. Show that the Hamiltonian can be expressed as system and coupling terms ($H = H_1 + H_2 + V$).

$$\omega_i^2 \equiv \frac{(C^{-1})_{ii}}{L_{ii}}, \quad \beta \equiv \frac{C_0}{\sqrt{(C_1 + C_0)(C_2 + C_0)}}$$

[1] Circuit QED Girvin Les Houches - chap 2-4

8. Using the second quantization principle, express the quantized Hamiltonian in terms of annihilation an creation operators. Find energy levels of the system Hamiltonians ($\hat{H}_1$ and $\hat{H}_2$) and the coupling coefficient in $\hat{V}$.

6.2 Anharmonicity of transmon qubits[2]

The name 'Transmon qubit' is an abbreviation of the term transmission line shunted plasma oscillation qubit. It is closely related to a Cooper-pair box, while operating in a regime where $E_J/E_C \gg 1$. The Hamiltonian of a transmon qubit is[3]

$$H = 4E_C\hat{n}^2 - E_J \cos\hat{\varphi},$$

where $\hat{n} = -i\left(\frac{E_J}{8E_C}\right)^{1/4}\frac{1}{\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right)$ and $\hat{\varphi} = \left(\frac{2E_C}{E_J}\right)^{1/4}\left(\hat{a} + \hat{a}^\dagger\right)$ is the conjugate pair of position and momentum, $E_C = \frac{e^2}{2C_\Sigma}$ is the Coulomb charging energy corresponding to one electron on the total junction capacitance C_Σ[4], and E_J is the Josephson energy. [5]

(a) Expand for small $\hat{\varphi}$ and show the Hamiltonian to the lowest order is $\hat{H}_0 \approx \sqrt{8E_JE_C}(\hat{a}^\dagger\hat{a} + 1/2)$. Therefore, under the lowest level of approximation, energy levels are equally spaced and anharmonicity is absent.

(b) Now we apply perturbation theory to calculate the anharmonicity defined as $\eta \equiv (E_{21} - E_{10})/\hbar$. Expand $\cos\hat{\varphi}$ up to the fourth order of $\hat{\varphi}$ in the Hamiltonian, derive the corrected energy level for transmon qubit. (in terms of E_C, E_J, and state index m)

(c) Define relative anharmonicity η_r as $\eta_r \equiv \hbar\eta/E_{10}$. Show how η_r scales with E_J/E_C as $E_J/E_C \gg 1$.

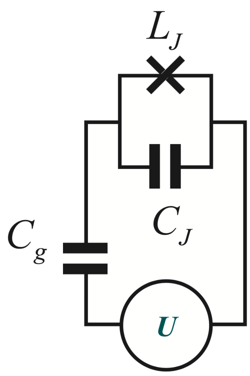

Figure 2: Circuit diagram of the transmon qubit.[2]

6.3 Numerical solution of the Cooper-pair box Hamiltonian(*)[6]

The Hamiltonian of a Cooper-pair box with bias voltage V_g is

$$H = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi},$$

[2] Koch, Jens, et al. "Charge-insensitive qubit design derived from the Cooper pair box." Physical Review A 76.4 (2007): 042319.

[3] Note that in some literature $\hat{\varphi}$ is refered to as $\hat{\delta}$. Also $\hat{\varphi}$ is not to be confused with the magnetic flux Φ.

[4] The definition of E_C may vary in different literatures. If E_C is defined as the charging energy of one cooper-pair, i.e. $E_C = \frac{(2e)^2}{2C_\Sigma}$, the factor 4 will be omitted.

[5] refer to M. H. Devoret et al., Superconducting Qubits: A Short Review, for more information

[6] Graded exercise

where $n_g = \frac{C_g V_g}{2e}$ is the reduced gate charge.

In this exercise, we will numerically diagnolise the CPB Hamiltonian in charge basis and plot the energy levels. To diagnolise the CPB Hamiltonian, we need to obtain the representation of $\cos\hat{\varphi}$ in charge basis.

(a) Show that $U_\pm \hat{n} U_\pm^\dagger = \hat{n} \mp 1$, where $U_\pm = e^{\pm i\hat{\varphi}} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\varphi e^{i\varphi}|\varphi\rangle\langle\varphi|$. Further show that the Hamiltonian in charge basis is

$$H = \sum_n [4E_C(n-n_g)^2|n\rangle\langle n| - \frac{E_J}{2}(|n\rangle\langle n+1| + |n+1\rangle\langle n|)]$$

(*Hint: use conjugation relationships* $|\varphi\rangle = \sum_{n=-\infty}^{+\infty} e^{in\varphi}|n\rangle, |n\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi e^{-in\varphi}|\varphi\rangle$ to show that $e^{i\hat{\varphi}}|n\rangle = |n-1\rangle$)

(b) Numerically diagnolise the Hamiltonian obtained in (a), plot the first 3 energy levels in terms of n_g for $E_J/E_C = \{1, 5, 10, 50\}$.

(c) Finally compare the numerical result to the asymptotic expressions of the charge dispersion and anharmonicity given in the Koch 2007 paper[7]. Reproduce fig. 4(a) and fig. 5(a) shown in the paper.

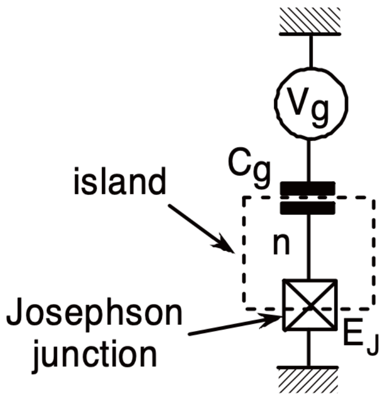

Figure 3: Electrical scheme of a Cooper-pair box.[8]

[7] Koch, Jens, et al. "Charge-insensitive qubit design derived from the Cooper pair box." Physical Review A 76.4 (2007): 042319.