Quantum Electrodynamics and Quantum Optics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to Exercise No.4

4.1 Quasi-probability distributions: Wigner-Weyl distribution

First focus on the definition of the characteristic function:

$$C^{(s)}\left(\beta,\beta^*\right) = \operatorname{Tr}\left(e^{i\beta\hat{a}^\dagger + i\beta^*\hat{a}}\rho\right) \tag{1}$$

Expand the exponential term inside:

$$\begin{aligned}
e^{i\beta a^\dagger + i\beta^* a} &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(i\beta a^\dagger + i\beta^* a\right)^m \\
&= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}\left(i\beta\right)^n\left(i\beta^*\right)^{m-n}\left(a^{\dagger n}a^{m-n}\right)_S \\
&= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{\left(i\beta\right)^n\left(i\beta^*\right)^{m-n}}{n!(m-n)!}\left(a^{\dagger n}a^{m-n}\right)_S \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(i\beta\right)^n\left(i\beta^*\right)^{m}}{n!m!}\left(a^{\dagger n}a^{m}\right)_S
\end{aligned} \tag{2}$$

Where the $()_S$ means symmetry ordered, for example:

$$\begin{aligned}
\left(a^\dagger a\right)_S &\equiv \frac{1}{2}\left(a^\dagger a + aa^\dagger\right) \\
\left(a^{\dagger 2} a\right)_S &\equiv \frac{1}{3}\left(a^{\dagger 2} a + a^\dagger a a^\dagger + aa^{\dagger 2}\right) \\
\left(a^\dagger a^2\right)_S &\equiv \frac{1}{3}\left(a^\dagger a^2 + aa^\dagger a + a^2 a^\dagger\right)
\end{aligned} \tag{3}$$

Then we consider the expectation values of those operator averages, from the above expansion of the characteristic function, we know that:

$$\begin{aligned}
\left\langle\left(a^{\dagger p}a^q\right)_S\right\rangle &\equiv \operatorname{tr}\left[\rho\left(a^{\dagger p}a^q\right)_S\right] \\
&= \frac{\partial^{p+q}}{\partial\left(i\beta\right)^p\partial(i\beta^*)^q}C^{(s)}\left(\beta,\beta^*\right)\Big|_{\beta^*=\beta=0}
\end{aligned} \tag{4}$$

Now we go back to the definition of the Wigner function, which is a Fourier transform of the characteristic function:

$$W\left(\alpha,\alpha^*\right) = \frac{1}{\pi^2}\int d^2\beta e^{-i\beta\alpha^* - i\beta^*\alpha}C^{(s)}\left(\beta,\beta^*\right) \tag{5}$$

Let's do an inverse Fourier transform:

$$C^{(s)}\left(\beta^*,\beta\right) = \int d^2\alpha W\left(\alpha,\alpha^*\right)e^{i\beta\alpha^*}e^{i\beta^*\alpha} \tag{6}$$

Now it becomes obvious that:

$$\begin{aligned}
\left\langle\left(a^{\dagger p}a^q\right)_S\right\rangle &= \frac{\partial^{p+q}}{\partial\left(i\beta\right)^p\partial(i\beta^*)^q}\int d^2\alpha W\left(\alpha,\alpha^*\right)e^{i\beta\alpha^*}e^{i\beta^*\alpha}\Big|_{\beta^*=\beta=0} \\
&= \left(\overline{\alpha^{*p}\alpha^q}\right)_W,
\end{aligned} \tag{7}$$

with:

$$\left(\overline{\alpha^{*p}\alpha^q}\right)_W \equiv \int d^2\alpha W\left(\alpha,\alpha^*\right)\alpha^{*p}\alpha^q \tag{8}$$

For a special case in the problem we thus have:

$$\frac{1}{2}\left\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right\rangle = \int W\left(\alpha,\alpha^*\right)|\alpha|^2 d^2\alpha \tag{9}$$

4.2 Solution : Hanbury Brown–Twiss effect for thermal and laser light

Considering the light emitted by to independent sources S and S' into the modes with wave vectors $\mathbf{k}$ and $\mathbf{k}' \neq \mathbf{k}$ respectively (otherwise there is no interference pattern). Assuming that $\omega = c|\mathbf{k}| = c|\mathbf{k}'|$ and that the polarization vectors (ϵ_k) are parallel. The electromagnetic fields can be expressed simply by using

$$\begin{aligned} E^{(+)}(\mathbf{r}_i,t) &= E_k^{\mathrm{vac}} e^{-i\omega t}\left(\hat{a}_k e^{i\mathbf{k}\cdot\mathbf{r}_i} + \hat{a}_{k'} e^{i\mathbf{k}'\cdot\mathbf{r}_i}\right) \\ E^{(-)}(\mathbf{r}_j,t) &= E_k^{\mathrm{vac}} e^{i\omega t}\left(\hat{a}_k^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}_j} + \hat{a}_{k'}^\dagger e^{-i\mathbf{k}'\cdot\mathbf{r}_j}\right). \end{aligned} \tag{10}$$

We can now compute the second order correlation function with:

$$G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t,t) = \langle E^{(-)}(\mathbf{r}_1,t)E^{(-)}(\mathbf{r}_2,t)E^{(+)}(\mathbf{r}_2,t)E^{(+)}(\mathbf{r}_1,t)\rangle \tag{11}$$

Beforehand, we notice that because this is the identical time correlation function and similar frequencies, all time phases will vanish / be compensated so we will not write them. Additionally, let us adopt the shorthand notation $\hat{a}_k(j) = \hat{a}_k e^{i\mathbf{k}\cdot\mathbf{r}_j}$, then we compute

$$\begin{aligned} G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t,t) &= \langle E^{(-)}(\mathbf{r}_1,t)E^{(-)}(\mathbf{r}_2,t)E^{(+)}(\mathbf{r}_2,t)E^{(+)}(\mathbf{r}_1,t)\rangle \\ &= (E_k^{\mathrm{vac}})^4\left\langle \left(\hat{a}_k^\dagger(1)+\hat{a}_{k'}^\dagger(1)\right)\left(\hat{a}_k^\dagger(2)+\hat{a}_{k'}^\dagger(2)\right)\left(\hat{a}_k(2)+\hat{a}_{k'}(2)\right)\left(\hat{a}_k(1)+\hat{a}_{k'}(1)\right)\right\rangle \\ &= (E_k^{\mathrm{vac}})^4\left\langle \left(\hat{a}_k^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_k(2)\hat{a}_k(1) + \hat{a}_k^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_k(2)\hat{a}_{k'}(1) + \hat{a}_{k'}^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_{k'}(2)\hat{a}_k(1)\right.\right. \\ &\qquad \left. +\hat{a}_k^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_{k'}(2)\hat{a}_k(1) + \hat{a}_{k'}^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_k(2)\hat{a}_{k'}(1) + \hat{a}_{k'}^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_{k'}(2)\hat{a}_{k'}(1)\right) \\ &\quad + \left(\hat{a}_k^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_k(2)\hat{a}_{k'}(1) + \hat{a}_{k'}^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_{k'}(2)\hat{a}_k(1) + \hat{a}_k^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_{k'}(2)\hat{a}_k(1)\right. \\ &\qquad + \hat{a}_{k'}^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_k(2)\hat{a}_{k'}(1) + \hat{a}_k^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_{k'}(2)\hat{a}_{k'}(1) + \hat{a}_{k'}^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_k(2)\hat{a}_k(1) \\ &\qquad + \hat{a}_{k'}^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_{k'}(2)\hat{a}_{k'}(1) + \hat{a}_k^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_k(2)\hat{a}_k(1) + \hat{a}_k^\dagger(1)\hat{a}_k^\dagger(2)\hat{a}_{k'}(2)\hat{a}_{k'}(1) \\ &\qquad \left.\left. + \hat{a}_{k'}^\dagger(1)\hat{a}_{k'}^\dagger(2)\hat{a}_k(2)\hat{a}_k(1)\right)\right\rangle \end{aligned}$$

Using the commutation relations, $[\hat{a}_k,\hat{a}_{k'}^\dagger] = \delta(\mathbf{k}-\mathbf{k}')$ and $[\hat{a}_k,\hat{a}_{k'}] = [\hat{a}_k^\dagger,\hat{a}_{k'}^\dagger] = 0$, we notice that the second parentheses contains only terms such as $\left(\hat{a}_k^\dagger\right)^2\hat{a}_k\hat{a}_{k'}$ or $\hat{a}_{k'}^\dagger\hat{a}_k^\dagger\hat{a}_k^2$ or $\left(\hat{a}_k^\dagger\right)^2\hat{a}_{k'}^2$. Since we will use a diagonal density matrix of the form $\hat{\rho} = \sum_n P(n)|n\rangle\langle n|$, all these non-diagonal terms will vanish when $\mathbf{k} \neq \mathbf{k}'$, leaving only the diagonal elements of the first parenthesis, which we can rewrite as

$$\begin{aligned} G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t,t) = (E_k^{\mathrm{vac}})^4 \Big\langle & \hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_k\hat{a}_k + \hat{a}_k^\dagger\hat{a}_{k'}^\dagger\hat{a}_k\hat{a}_{k'}\left(1+e^{-i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)}\right) \\ & +\hat{a}_{k'}^\dagger\hat{a}_{k'}^\dagger\hat{a}_{k'}\hat{a}_{k'} + \hat{a}_{k'}^\dagger\hat{a}_k^\dagger\hat{a}_{k'}\hat{a}_k\left(1+e^{i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)}\right)\Big\rangle \end{aligned} \tag{12}$$

Now if the average photon number is the same for both modes $\langle n\rangle = \langle n_{\mathbf{k}}\rangle = \langle n_{\mathbf{k}'}\rangle$ and $\langle n^2\rangle = \langle n_{\mathbf{k}}^2\rangle = \langle n_{\mathbf{k}'}^2\rangle$, using the commutation relations we get

$$\begin{aligned} G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t,t) &= (E_k^{\mathrm{vac}})^4\left\langle \hat{n}_k^2 - \hat{n}_k + \hat{n}_{k'}^2 - \hat{n}_{k'} + 2\hat{n}_k\hat{n}_{k'}\left(1+\cos\{(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)\}\right)\right\rangle \\ &= 2\left(E_k^{\mathrm{vac}}\right)^4\left(\langle\hat{n}^2\rangle - \langle\hat{n}\rangle + \langle\hat{n}\rangle^2\left\{1+\cos[(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)]\right\}\right) \end{aligned} \tag{13}$$

1. For a thermal light, which follows the Bose-Einstein statistics given by the density matrix $\rho = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} |n\rangle\langle n| = \sum_n (1-e^{-\beta})e^{-n\beta}|n\rangle\langle n|$, where $\langle n\rangle = 1/(e^{\hbar\omega/k_BT}-1) = 1/(e^{\beta}-1)$, we compute the average $\langle n^2\rangle$ as

$$\begin{aligned}\langle n^2\rangle &= \mathrm{Tr}\left\{\hat{\rho}\hat{n}^2\right\} = \sum_n \langle n|\rho\hat{n}^2|n\rangle = \sum_n n^2 \langle n|\rho|n\rangle = \sum_n n^2 \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} = \sum_n n^2(1-e^{-\beta})e^{-n\beta} \\ &= (1-e^{-\beta})\frac{\partial^2}{\partial\beta^2}\sum_n e^{-n\beta} = (1-e^{-\beta})\frac{\partial^2}{\partial\beta^2}(1-e^{-\beta})^{-1} = (1-e^{-\beta})^{-2}\left[(1-e^{-\beta})e^{-\beta} + 2e^{-2\beta}\right] \\ &= \frac{1}{e^{\beta}-1} + 2\left(\frac{1}{e^{\beta}-1}\right)^2 = \langle n\rangle + 2\langle n\rangle^2\end{aligned} \tag{14}$$

It follows that the correlation function can now be expressed as

$$G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t,t) = 2\,(E_k^{\mathrm{vac}})^4\left(2\langle\hat{n}\rangle^2 + \langle\hat{n}\rangle^2\left\{1+\cos[(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)]\right\}\right) \tag{15}$$

2. Finally, we use an identical procedure for a laser source high above threshold (i.e. a phase-diffused coherent state) given by the density matrix $\rho = e^{-\langle n\rangle}\sum_n \frac{\langle n\rangle^n}{n!}|n\rangle\langle n|$ (notice that it is not possible to express the density matrix as $\hat{\rho} = |\alpha\rangle\langle\alpha|$ due to the phase diffusion). Since the probabilities now follow the Poisson distribution, we use the property $\langle\Delta n^2\rangle = \langle\Delta n^2\rangle - \langle n\rangle^2 = \langle n\rangle$ to infer the value of $\langle n^2\rangle = \langle n\rangle + \langle n\rangle^2$. Therefore, the correlation function is

$$G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t,t) = 2\,(E_k^{\mathrm{vac}})^4\left(\langle\hat{n}\rangle^2 + \langle\hat{n}\rangle^2\left\{1+\cos[(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{r}_1-\mathbf{r}_2)]\right\}\right) \tag{16}$$

Solution: $g^{(2)}(\tau)$, second order intensity autocorrelation: Measuring quantum statistics and effects of light

One of the applications of the second order correlation function is to distinguish between different light sources, particularly classical from non-classical light. The normalized second order correlation function is defined as:

$$g^{(2)}(\tau) = \frac{\langle\hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t)\rangle}{\langle\hat{E}^{(-)}(t)\hat{E}^{(+)}(t)\rangle^2} = \frac{\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle^2} \tag{17}$$

1. *Show that $g^{(2)}(\tau\to 0)$ has the following form:*

$$g^{(2)}(0) = 1 + \frac{\Delta\hat{n}^2 - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^2} \tag{18}$$

where $\hat{n}$ is the number operator with $\Delta\hat{n}^2$, being the variance in photon number.

According to the definition of second order autocorrelation function Eq. 17, we have

$$g^{(2)}(0) = \frac{\langle\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2} = \frac{\langle(\hat{a}^\dagger\hat{a})^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2} = \frac{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 + \langle\hat{n}\rangle^2 - \langle n\rangle}{\langle\hat{n}\rangle^2} = 1 + \frac{\Delta\hat{n}^2 - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^2} \tag{19}$$

2. *Re-calculate Eq. (18) for:*

   (a) a coherent state $|\alpha\rangle$

   (b) a Fock state $|n\rangle$

   (c) a state $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, where $\alpha_i (i=0,1)$ are complex coefficients of vacuum and single photon states.

(d) a squeezed vacuum state $|\epsilon\rangle$

Using your results, determine if these states are classical or non-classical, knowing the fact that for a non-classical state $g^{(2)}(0) < 1$.

(a) For a coherent state $|\alpha\rangle$, $\langle\hat{n}\rangle = \Delta\hat{n}^2 = |\alpha|^2$, therefore we have

$$g^{(2)}(0) = 1 + \frac{|\alpha|^2 - |\alpha^2|}{|\alpha|^4} = 1 \tag{20}$$

It is a **classical state**.

(b) For a Fock state $|n\rangle$, $\langle\hat{n}\rangle = n,\ \langle\hat{n}^2\rangle = n^2, \Delta\hat{n}^2 = 0$, therefore we have:

$$g^{(2)}(0) = 1 + \frac{0-n}{n^2} = \begin{cases} 1 - \dfrac{1}{n} < 1 & n > 0 \\ 1 & n = 0 \end{cases} \tag{21}$$

It is a **non-classical state** if $n > 0$ and a **classical state** if $n = 0$, i.e. a vacuum state.

(c) For a superposition state $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, $\langle\hat{n}\rangle = |\alpha_1|^2,\ \langle\hat{n}^2\rangle = |\alpha_1|^2, \Delta\hat{n}^2 = |\alpha_1|^2 - |\alpha_1|^4$, therefore we have

$$g^{(2)}(0) = 1 + \frac{|\alpha_1|^2 - |\alpha_1|^4 - |\alpha_1|^2}{|\alpha_1|^4} = 0 \tag{22}$$

It is a **non-classical state**.

(d) For a squeezed state $|\epsilon\rangle = \hat{S}(\epsilon)|0\rangle$, where $\epsilon = re^{i\phi}$, we have $\langle\hat{n}\rangle = \sinh^2(r)$, $\Delta\hat{n}^2 = \sinh^2(2r)/2$, therefore we have

$$\begin{aligned} g^{(2)}(0) =& 1 + \frac{\sinh^2(2r)/2 - \sinh^2(r)}{\sinh^4(r)} \\ =& 1 + \frac{2\sinh^2(r)\cosh^2(r) - \sinh^2(r)}{\sinh^4(r)} \\ =& 1 + \frac{2\cosh^2(r) - 1}{\sinh^2(r)} \\ =& 2 + \coth^2(r) > 1 \end{aligned} \tag{23}$$

Even though $g^{(2)}(0) > 1$, we cannot say that the squeezed vacuum is a classical state since squeezing is indeed a non-classical quantum signature. Here we cite a more general criterion of non-classical state: a state represented with negative or singlar value in Glauber-Sudarshan quasidistribution can be referred as a non-classical state [1].

Solution: Hong-Ou-Mandel effect

1. (a) From the homework 3 we know the relation between annihilation/creation operators of the modes

$$\hat{a}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_3^\dagger + i\hat{a}_4^\dagger) \tag{24}$$

$$\hat{a}_2^\dagger = \frac{1}{\sqrt{2}}(i\hat{a}_3^\dagger + \hat{a}_4^\dagger) \tag{25}$$

[1] D.N. Klyshko. Observable signs of nonclassical light. Physics Letters A, 213(1):7–15, 1996.

Therefore,

$$
\begin{aligned}
|\psi\rangle_{out} &= \hat{a}^\dagger_{1,H}\hat{a}^\dagger_{2,V}|0\rangle_1|0\rangle_2 && (26)\\
&= \frac{1}{2}(\hat{a}^\dagger_{3,H} + i\hat{a}^\dagger_{4,H})(i\hat{a}^\dagger_{3,V} + \hat{a}^\dagger_{4,V})|0\rangle_1|0\rangle_2 && (27)\\
&= \frac{1}{2}(i\hat{a}^\dagger_{3,H}\hat{a}^\dagger_{3,V} + \hat{a}^\dagger_{3,H}\hat{a}^\dagger_{4,V} - \hat{a}^\dagger_{4,H}\hat{a}^\dagger_{3,V} + i\hat{a}^\dagger_{4,H}\hat{a}^\dagger_{4,V})|0\rangle_1|0\rangle_2 && (28)\\
&= \frac{1}{2}(i|1,H\rangle_3|1,V\rangle_3 + |1,H\rangle_3|1,V\rangle_4 - |1,V\rangle_3|1,H\rangle_4 + i|1,H\rangle_4|1,V\rangle_4) && (29)
\end{aligned}
$$

(b) The probability of detecting one photon in each output port is

$$|(\langle 1,H|_3\langle 1,V|_4)|\psi_{out}\rangle|^2 + |(\langle 1,V|_3\langle 1,H|_4)|\psi_{out}\rangle|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{30}$$

(c) The probability of detecting photon pairs in each output port is

$$|(\langle 1,H|_3\langle 1,V|_3)|\psi_{out}\rangle|^2 = |(\langle 1,V|_4\langle 1,H|_4)|\psi_{out}\rangle|^2 = \frac{1}{4} \tag{31}$$

2. When the two single photons are indistinguishable,the output state

$$
\begin{aligned}
|\psi\rangle_{out} &= \hat{a}^\dagger_{1,H}\hat{a}^\dagger_{2,H}|0\rangle_1|0\rangle_2 && (32)\\
&= \frac{1}{2}(\hat{a}^\dagger_{3,H} + i\hat{a}^\dagger_{4,H})(i\hat{a}^\dagger_{3,H} + \hat{a}^\dagger_{4,H})|0\rangle_1|0\rangle_2 && (33)\\
&= \frac{1}{2}(i\hat{a}^\dagger_{3,H}\hat{a}^\dagger_{3,H} + \hat{a}^\dagger_{3,H}\hat{a}^\dagger_{4,H} - \hat{a}^\dagger_{4,H}\hat{a}^\dagger_{3,H} + i\hat{a}^\dagger_{4,H}\hat{a}^\dagger_{4,H})|0\rangle_1|0\rangle_2 && (34)\\
&= \frac{1}{2}(i\sqrt{2}|2,H\rangle_3 + |1,H\rangle_3|1,H\rangle_4 - |1,H\rangle_3|1,H\rangle_4 + i\sqrt{2}|2,H\rangle_4) && (35)\\
&= \frac{i}{\sqrt{2}}(|2,H\rangle_3|0\rangle_4 + |0,H\rangle_3|2,H\rangle_4) && (36)
\end{aligned}
$$

is NOON state with $N = 2$. (b) The probability of detecting one photon in each output port is obviously 0. (c) The probability of detecting photon pairs in port H is

$$|(\langle 2,H|_3\langle 0|_4)|\psi_{out}\rangle|^2 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} \tag{37}$$

Similarly, the probability of detecting photon pairs in port V is

$$|(\langle 0|_3\langle 2,H|_4)|\psi_{out}\rangle|^2 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} \tag{38}$$

3. For figure a, the probability of detecting one photon in each output port is $11/20 \approx \frac{1}{2}$, the probability of detecting photo pairs in port H is $5/20 = 1/4$,probability of detecting photo pairs in port V is $4/20 \approx 1/4$. For figure b, probability of detecting photo pairs in port H is $10/20 = 1/2$,probability of detecting photo pairs in port V is $9/20 \approx 1/2$. The experimental result follows what we predicted.