Quantum Electrodynamics and Quantum Optics

École Polytechnique Fédérale de Lausanne (EPFL)

Exercise No.4

4.1 Quasiprobability distributions: Wigner-Weyl distribution[1]

One way to define Wigner quasiprobability distribution is through the so called characteristic function $C^{(s)}(\beta,\beta^*) = \mathrm{Tr}\left(e^{i\beta\hat{a}^\dagger + i\beta^*\hat{a}}\rho\right)$. The Wigner distribution is simply a Fourier transform of the characteristic function:

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\beta e^{-i\beta\alpha^* - i\beta^*\alpha} C^{(s)}(\beta,\beta^*) \tag{1}$$

Expand the operator exponent in the definition of the characteristic function $C^{(s)}(\beta,\beta^*)$ in a power series with respect to β and β^* and comment on the ordering of the operators in each of the terms of the expansion. Use these observations to prove that:

$$\frac{1}{2}\langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\rangle = \int W(\alpha,\alpha^*)|\alpha|^2 d^2\alpha. \tag{2}$$

(Hint: recall the mathematical meaning of the coefficients of the Taylor expansion)

Make a general conclusion on how the Wigner function can be used to compute the expectation values of symmetrically ordered polynomials in $\hat{a}$ and $\hat{a}^\dagger$.

4.2 Hanbury Brown–Twiss effect for thermal and laser light

In 1956, Robert Hanbury Brown and Richard Q. Twiss published "A test of a new type of stellar interferometer on Sirius"[2], in which two photomultiplier tubes (PMTs) separated by about 6 meters were aimed at the star Sirius. The correlation between two photocurrents was recorded depending on the separation between the two PMTs, revealing interference fringes, *despite the fact that no phase information can be collected by direct photodetection*. Such photocurrent correlation is proportional to the so called second order intensity correlation function[3] of the field. In this exercise we aim to compute the intensity correlation function for two different light fields: thermal light and the radiation of a laser.

Familiar to you from the first homework, the quantized electric field can be expressed as a sum of positive and negative frequency components each containing respectively annihilation and creation operators only:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^{(+)}(\mathbf{r},t) + \mathbf{E}^{(-)}(\mathbf{r},t), \tag{3}$$

where each component is a sum of plane waves:

$$\begin{aligned} \mathbf{E}^{(+)}(\mathbf{r},t) &= \sum_k \epsilon_k E_k^{\mathrm{vac}} \hat{a}_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}}, \\ \mathbf{E}^{(-)}(\mathbf{r},t) &= \sum_k \epsilon_k E_k^{\mathrm{vac}} \hat{a}_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{r}} \end{aligned} \tag{4}$$

The second order correlation function is defined in terms of the positive and negative frequency components as follows:

$$G^{(2)}(\mathbf{r}_1,\mathbf{r}_2;t,t) = \langle E^{(-)}(\mathbf{r}_1,t)E^{(-)}(\mathbf{r}_2,t)E^{(+)}(\mathbf{r}_2,t)E^{(+)}(\mathbf{r}_1,t)\rangle \tag{5}$$

[1] This exercise is adapted from Exercise 3.1 Quantum Optics, Scully M. O., Zubairy M. S.

[2] Brown, R. Hanbury, and Richard Q. Twiss. "A test of a new type of stellar interferometer on Sirius." Nature 178.4541 (1956): 1046-1048.

[3] Also called the fourth order field correlation, due to the number of the field operators in the definition

We now consider the light emitted by to independent sources S and S' into the modes with wave vectors $\mathbf{k}$ and $\mathbf{k}'$ respectively (in the case of a remote star you can think of these sources as diametrically opposite parts of the star, seen from the Earth at slightly different angles corresponding to $\mathbf{k}$ and $\mathbf{k}'$, and being almost plane waves on the relevant scale). We can describe such field as:

$$E^{(+)}(\mathbf{r}_i, t) = E_k^{\text{vac}} e^{-i\omega t} \left(\hat{a}_k e^{i\mathbf{k}\cdot\mathbf{r}_i} + \hat{a}_{k'} e^{i\mathbf{k}'\cdot\mathbf{r}_i} \right). \tag{6}$$

List all the implicit assumptions that we make when reducing the general form of the field of Eq. 7 to the simplified Eq.6.

Calculate the second order correlation function in terms of the average photon number considering it is the same for both modes $\langle n \rangle = \langle n_{\mathbf{k}} \rangle = \langle n_{\mathbf{k}'} \rangle$ for:

1. A thermal light given by the density matrix $\hat{\rho} = \sum\limits_n \frac{\langle n \rangle^n}{(1+\langle n \rangle)^{n+1}} |n\rangle \langle n|$
2. A laser source high above threshold (i.e. a phase-diffused coherent state) given by the density matrix $\hat{\rho} = e^{-\langle n \rangle} \sum\limits_n \frac{\langle n \rangle^n}{n!} |n\rangle \langle n|$

4.3 $g^{(2)}(\tau)$, second order intensity autocorrelation: Measuring quantum statistics and effects of light

One of the applications of the second order correlation function is to distinguish between different light sources, particularly classical from non-classical light. The normalized second order correlation function is defined as:

$$g^{(2)}(\tau) = \frac{\langle \hat{E}^{(-)}(t) \hat{E}^{(-)}(t+\tau) \hat{E}^{(+)}(t+\tau) \hat{E}^{(+)}(t) \rangle}{\langle \hat{E}^{(-)}(t) \hat{E}^{(+)}(t) \rangle^2} = \frac{\langle \hat{a}^\dagger(t) \hat{a}^\dagger(t+\tau) \hat{a}(t+\tau) \hat{a}(t) \rangle}{\langle \hat{a}^\dagger(t) \hat{a}(t) \rangle^2} \tag{7}$$

1. Show that $g^{(2)}(\tau \to 0)$ has the following form:

$$g^{(2)}(0) = 1 + \frac{\Delta \hat{n}^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} \tag{8}$$

   where $\hat{n}$ is the number operator with $\Delta \hat{n}^2$, being the variance in photon number.

2. Re-calculate Eq. (8) for:
   (a) a coherent state $|\alpha\rangle$
   (b) a Fock state $|n\rangle$
   (c) a state $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$, where $\alpha_i (i = 0, 1)$ are complex coefficients of vacuum and single photon states.
   (d) a squeezed vacuum state $|\epsilon\rangle$

   Using your results, determine if these states are classical or non-classical, knowing the fact that for a non-classical state $g^{(2)}(0) < 1$.

4.4 Hong-Ou-Mandel effect(*)

The Hong–Ou–Mandel effect[4] is a two-photon interference effect in quantum optics which was demonstrated by Chung Ki Hong, Zhe Yu Ou and Leonard Mandel in 1987. The effect occurs when two identical single-photon (i.e. same frequency ω and same spatial mode $\vec{k}$) waves enter a 50:50 beam splitter, one in each input port. Here we look at case of propagation of two single photon states from two ports of a 50:50 beam splitter. In this exercise we shall see the final state differs in case of distinguishable and indistinguishable photons.

[4] Chung Ki Hong, Zhe-Yu Ou, and Leonard Mandel. "Measurement of subpicosecond time intervals between two photons by interference." Physical review letters 59.18 (1987): 2044.

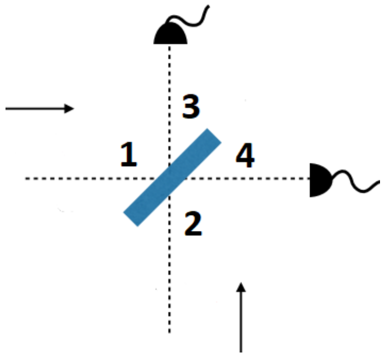

Figure 1: Schematic of a beamsplitter. 1 and 2 modes are input modes and 3 and 4 are output.

1. Consider the case where two single photon states with orthogonal polarization (H,V) enter the beamsplitter from port 1 and 2 individually ($|1, H\rangle_1 |1, V\rangle_2$).
   (a) What will be the output state?
   (b) What is the probability of detecting one photon in each output port?
   (c) What is the probability of detecting photon pairs in each output port (NOON states)?
2. Consider the case where two single photon states with the same polarization (indistinguishable photons) enter the beamsplitter from port 1 and 2 individually ($|1, H\rangle_1 |1, H\rangle_2$ or $|1, V\rangle_1 |1, V\rangle_2$).
   (a) What will be the output state?
   (b) What is the probability of detecting one photon in each output port?
   (c) What is the probability of detecting photon pairs in each output port (NOON states)?
3. These two cases (polarization distinguished photons) can be realized and imaged experimentally using high sensitivity CMOS cameras[5]. In Figure 2, panel (a) you can see the beamsplitter output states for two distinguishable photons entering the beamsplitter while in panel (b) the photons are indistinguishable. Compare your results to the Figure 2.

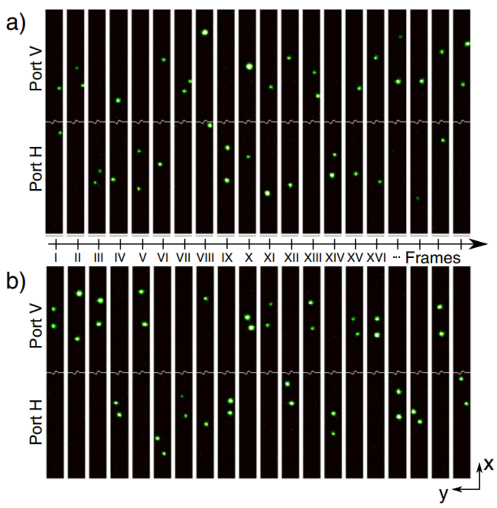

Figure 2: 20 frames from coincidences on different ports of a beamsplitter for (a) distinguishable and (b) indistinguishable photons [Micha et al. *Optics Letters (2015)*]

[5]Jachura, Michał, and Radosław Chrapkiewicz. "Shot-by-shot imaging of Hong–Ou–Mandel interference with an intensified sCMOS camera." Optics letters 40.7 (2015): 1540-1543.