Quantum Electrodynamics and Quantum Optics

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to Exercise No.3

3.1 Beamsplitter scattering matrix

Since we know the reflection and transmission coefficients are (r, t) and (r', t') for mode 1 and 2 respectively, we have

$$\begin{aligned} E_3 &= tE_1 + r'E_2 \\ E_4 &= t'E_2 + rE_1 \end{aligned} \tag{1}$$

or in the form of matrix multiplication

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = S \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{2}$$

For a lossless beam splitter, we have a contraint for conservation of energy

$$|E_3|^2 + |E_4|^2 = |E_1|^2 + |E_2|^2 \tag{3}$$

which yields

$$\begin{aligned} |t|^2 + |r|^2 &= |t'|^2 + |r'|^2 = 1 \\ tr^* + t^*r' &= t'r^* + t'^*r = 0 \end{aligned} \tag{4}$$

3.1.1

Since

$$\begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = S \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{5}$$

We have

$$\begin{aligned} a_3 &= ta_1 + r'a_2 \\ a_4 &= ra_1 + t'a_2 \\ a_3^\dagger &= t^*a_1^\dagger + r'^*a_2^\dagger \\ a_4^\dagger &= r^*a_1^\dagger + t'^*a_2^\dagger \end{aligned} \tag{6}$$

In order to make $\left(a_3, a_3^\dagger, a_4, a_4^\dagger\right)$ satisfy the same bosonic commutation relationship, then we can derive the following constraints

$$\begin{aligned} \left[a_3, a_3^\dagger\right] &= \left[a_4, a_4^\dagger\right] = |t|^2 + |r|^2 = 1 \\ \left[a_3, a_4^\dagger\right] &= \left[a_4, a_3^\dagger\right] = tr^* + t'^*r = t^*r + t'r^* = 0 \end{aligned} \tag{7}$$

3.1.2

From the above equation of a_3, a_4 we obtain the operator evolved after 50/50 beam splitter

$$a_1^\dagger = \frac{1}{\sqrt{2}}\left(a_3^\dagger + ia_4^\dagger\right) \tag{8}$$

So, the output state should be

$$|\psi\rangle_{\text{output}} = \frac{1}{\sqrt{2}}\left(a_3^\dagger + ia_4^\dagger\right)|0\rangle_3|0\rangle_4 = \frac{1}{\sqrt{2}}|1\rangle_3|0\rangle_4 + \frac{i}{\sqrt{2}}|0\rangle_3|1\rangle_4 \tag{9}$$

3.2 Solution: Beamsplitter treatment in phase space representation

(a) Let $|\psi\rangle_{\text{in}} = |\alpha\rangle_1 |0\rangle_2 = \hat{D}_1(\alpha) |0\rangle_1 |0\rangle_2$ with $\hat{D}_1(\alpha) = \exp\left(\alpha \hat{a}_1^\dagger - \alpha^* \hat{a}_1\right)$. Using the technique explained in the exercise **Beamsplitter scattering matrix**, we write the raising operator $\hat{a}_1^\dagger$ as $\hat{a}_1^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_3^\dagger + i\hat{a}_4^\dagger\right)$. Then we can express the displacement operator $\hat{D}_1(\alpha)$ as

$$\begin{aligned}
\hat{D}_1(\alpha) = e^{\alpha \hat{a}_1^\dagger - \alpha^* \hat{a}_1} &= \exp\left(\frac{\alpha}{\sqrt{2}}\left(\hat{a}_3^\dagger + i\hat{a}_4^\dagger\right) - \frac{\alpha^*}{\sqrt{2}}\left(\hat{a}_3 - i\hat{a}_4\right)\right) \\
&= \exp\left(\left(\frac{\alpha}{\sqrt{2}}\hat{a}_3^\dagger - \frac{\alpha^*}{\sqrt{2}}\hat{a}_3\right) + \left(\frac{i\alpha}{\sqrt{2}}\hat{a}_4^\dagger - \frac{(i\alpha)^*}{\sqrt{2}}\hat{a}_4\right)\right) \\
&= \exp\left(\frac{\alpha}{\sqrt{2}}\hat{a}_3^\dagger - \frac{\alpha^*}{\sqrt{2}}\hat{a}_3\right)\exp\left(\frac{i\alpha}{\sqrt{2}}\hat{a}_4^\dagger - \frac{(i\alpha)^*}{\sqrt{2}}\hat{a}_4\right) \\
&= \hat{D}_3\left(\frac{\alpha}{\sqrt{2}}\right)\hat{D}_4\left(\frac{i\alpha}{\sqrt{2}}\right)
\end{aligned} \tag{10}$$

where in the lasts steps we used the fact that operators on system 3 commute with operators on system 4 and the definition of the displacement operator. This means that the final state is therefore

$$|\psi\rangle_{\text{out}} = \hat{D}_3\left(\frac{\alpha}{\sqrt{2}}\right)\hat{D}_4\left(\frac{i\alpha}{\sqrt{2}}\right)|0\rangle_3 |0\rangle_4 = \left|\frac{\alpha}{\sqrt{2}}\right\rangle_3 \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_4 \tag{11}$$

(b) In the Heisenberg picture, the density operator is constant, because the time dependency rests onto the operators themselves. Therefore, starting from the P-representation of the coherent state

$$\begin{aligned}
\hat{\rho}_{\text{in}} &= \hat{\rho}_{\text{in},1}(\alpha) \otimes \hat{\rho}_{\text{in},2}(0) = \int \mathrm{d}^2\alpha\, P_{\text{in}}(\alpha)\, |\alpha\rangle_{1\,1}\langle\alpha| \otimes \hat{\rho}_{\text{in},2}(0) \\
&= \int \mathrm{d}^2\alpha\, P_{\text{in}}(\alpha)\hat{D}_1(\alpha)\,|0\rangle_{1\,1}\langle 0|\hat{D}_1^\dagger(\alpha) \otimes |0\rangle_{2\,2}\langle 0| = \int \mathrm{d}^2\alpha\, P_{\text{in}}(\alpha)\hat{D}_1(\alpha)\left(|0\rangle_{1\,1}\langle 0| \otimes |0\rangle_{2\,2}\langle 0|\right)\hat{D}_1^\dagger(\alpha) \\
&= \int \mathrm{d}^2\alpha\, P_{\text{in}}(\alpha)\hat{D}_3\left(\frac{\alpha}{\sqrt{2}}\right)\hat{D}_4\left(\frac{i\alpha}{\sqrt{2}}\right)\left(|0\rangle_3 |0\rangle_{4\,4}\langle 0|\,{}_3\langle 0|\right)\hat{D}_4^\dagger\left(\frac{i\alpha}{\sqrt{2}}\right)\hat{D}_3^\dagger\left(\frac{\alpha}{\sqrt{2}}\right) \\
&= \int \mathrm{d}^2\alpha\, P_{\text{in}}(\alpha)\left(\left|\frac{\alpha}{\sqrt{2}}\right\rangle_3 \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_{4\,4}\left\langle\frac{i\alpha}{\sqrt{2}}\right|\,{}_3\left\langle\frac{\alpha}{\sqrt{2}}\right|\right) \\
&= \int \mathrm{d}^2\beta\, (\sqrt{2})^2 P_{\text{in}}(\sqrt{2}\beta)\,|\beta\rangle_3 |i\beta\rangle_{4\,4}\langle i\beta|\,{}_3\langle\beta| = \int \mathrm{d}^2\beta\, 2P_{\text{in}}(\sqrt{2}\beta)\,|\beta\rangle_{34\,34}\langle\beta| \\
&= \int \mathrm{d}^2\beta\, P_{\text{out}}(\beta)\,|\beta\rangle_{34\,34}\langle\beta| = \hat{\rho}_{\text{out}}
\end{aligned} \tag{12}$$

where we performed the change $\alpha = \sqrt{2}\beta$ to obtain the P-representation of the output field $P_{\text{out}}(\beta) = 2P_{\text{in}}(\sqrt{2}\beta)$ and $|\beta\rangle_{34} = |\beta\rangle_3 |i\beta\rangle_4$.

3.3 Phase measurements with multiphoton entangled states

In this exercise we demonstrate how the use of multiphoton entangled states of light can improve the phase sensitivity in interferometry experiments (i.e. demonstrating the principle of *Quantum Metrology*). Figure (1) shows a scheme of a Mach-Zehnder interferometer. The input light is split into two paths (1 and 2) on the first beamsplitter (BS1) and then recombined on the second beamsplitter (BS2) and then detected on the photodetectors (PD1 and PD2). The interferometer arms have the lengths of L_1 and L_2 with difference of ΔL. For simplicity assume that the beamsplitters are 50:50. The input light has angular frequency of ω and propagates in the interferometer arms with speed of c.

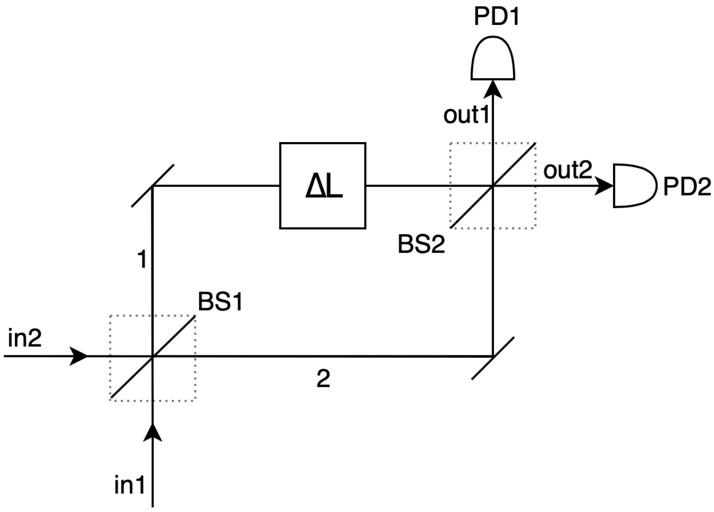

Figure 1: A Mach-Zehnder interferometer. BS: Beamsplitter, PD: Photodetector

1. *Consider we send one photon through input port one (in1) and zero through port 2 (in2). The input state of the interferometer would be*

$$|\psi_{in}\rangle = |1\rangle_{in1} \otimes |0\rangle_{in2} \equiv |1\rangle_{in1} |0\rangle_{in2} \tag{13}$$

What is the joint state of the light in paths 1 and 2 right after the first beamsplitter? You may use the trick explained in the exercise ***Beamsplitter scattering matrix****. Denote this joint state as kets of the form* $|\rangle_1 |\rangle_2$.
As we have learned from the **Beamsplitter scattering matrix** exercise, the scattering matrix that transforms the creation and annihilation operators of the two input modes into the two output modes of a beamsplitter are the same as the scattering matrices for the electromagnetic field. For the first beamsplitter BS1 we have

$$\begin{pmatrix} \hat{a}_{\text{in1}} \\ \hat{a}_{\text{in2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{14}$$

We now have to express the input state in terms of $\hat{a}_{\text{in1}}, \hat{a}_{\text{in2}}, \hat{a}^\dagger_{\text{in1}}, \hat{a}^\dagger_{\text{in2}}$ and apply the unitary transformation described by the scattering matrix

$$|1\rangle_{\text{in1}} |0\rangle_{\text{in2}} = \hat{a}^\dagger_{\text{in1}} |0\rangle = \frac{1}{\sqrt{2}}(\hat{a}^\dagger_1 + i\hat{a}^\dagger_2) |0\rangle = \frac{1}{\sqrt{2}}(|1\rangle_1 + i\,|1\rangle_2) \tag{15}$$

2. *The path length difference will cause the photons propagating in different paths to acquire a phase difference. Compute the joint state after propagation along the interferometer arms (which is at the input of the second beamsplitter).*
Approach is the same, first identify the scattering matrx

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_1 \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}'_1 \\ \hat{a}'_2 \end{pmatrix}, \tag{16}$$

then decompose the state in terms of creation and annihilation operators and apply the transformation

$$|1\rangle_{\text{in1}} |0\rangle_{\text{in2}} = \cdots = \frac{1}{\sqrt{2}}(e^{i\phi}\hat{a}'^\dagger_1 + i\hat{a}'^\dagger_2) |0\rangle = \frac{1}{\sqrt{2}}(e^{i\phi}\,|1\rangle'_1 + i\,|1\rangle'_2) \tag{17}$$

3. *Compute the state of the light at the output ports of the interferometer (Use the notation* $|\,\rangle_{out1}\,|\,\rangle_{out2}$*).*

$$|1\rangle_{\text{in1}}|0\rangle_{\text{in2}} = \dots = \frac{1}{\sqrt{2}}\left(e^{i\phi}\frac{1}{\sqrt{2}}(\hat{a}^\dagger_{\text{out1}} + i\hat{a}^\dagger_{\text{out2}}) + i\frac{1}{\sqrt{2}}(i\hat{a}^\dagger_{\text{out1}} + \hat{a}^\dagger_{\text{out2}})\right)|0\rangle = \\ = \frac{1}{2}((e^{i\phi}-1)|1\rangle_{\text{out1}} + i(e^{i\phi}+1)|1\rangle_{\text{out2}}) \tag{18}$$

4. *What is the probability of detecting 1 photon in PD1? What is the probability of detecting 1 photon in PD2? Sketch these probabilities as functions of* $\phi = \frac{\omega\Delta L}{c}$.
For now we will just compute the probability of the detecting 1 photon in PD1 as absolute value squared of the overlap of the $|1\rangle_{\text{out1}}$ state with the output state computed above. Note however, that to properly describe a real photodetector used in experiments the signal from the detector (photocurrent) should be computed as the expectation of a normally ordered operator corresponding to the PD [1].

So in our simplified treatment

$$|\langle 1_{\text{out1}}|\psi_{in}\rangle|^2 = \frac{1}{4}(e^{i\phi}-1)(e^{-i\phi}-1) = \frac{1}{2}(1-\cos\phi) \tag{19}$$
$$|\langle 1_{\text{out2}}|\psi_{in}\rangle|^2 = \frac{1}{4}(e^{i\phi}+1)(e^{-i\phi}+1) = \frac{1}{2}(1+\cos\phi) \tag{20}$$

Note, the probabilities add to 1 and they oscillate with a 2π period as they would for a classical interfereometer

5. *Next we use a different input state for the interferometry. We send 1 photon in each of the input ports, meaning the input state would be*

$$|\psi_{in}\rangle = |1\rangle_{in1}|1\rangle_{in2} \tag{21}$$

repeat the steps (a) to (c) and compute the probability of detecting 2 photons in PD1[2]*. Sketch this probability as a function of* ϕ.
Following the same algorithm outlined above

$$|1\rangle_{\text{in1}}|1\rangle_{\text{in2}} = \hat{a}^\dagger_{\text{in1}}\hat{a}^\dagger_{\text{in2}}|0\rangle = \tag{22}$$
$$= \frac{1}{2}(\hat{a}^\dagger_1 + i\hat{a}^\dagger_2)(i\hat{a}^\dagger_1 + \hat{a}^\dagger_2)|0\rangle = \frac{i}{2}\left(\left(\hat{a}^\dagger_{\text{in1}}\right)^2 + \left(\hat{a}^\dagger_{\text{in1}}\right)^2\right)|0\rangle = \tag{23}$$
$$= \frac{i}{2}\left(e^{i\phi}\left(\hat{a}'^\dagger_{\text{in1}}\right)^2 + \left(\hat{a}'^\dagger_{\text{in1}}\right)^2\right)|0\rangle = \frac{i}{2}\left(e^{i2\phi}\left(\hat{a}'^\dagger_{\text{in1}}\right)^2 + \left(\hat{a}'^\dagger_{\text{in1}}\right)^2\right)|0\rangle = \tag{24}$$
$$= \frac{i}{4}\left(e^{i2\phi}\left(\hat{a}^\dagger_{\text{out1}} + i\hat{a}^\dagger_{\text{out2}}\right)^2 + \left(i\hat{a}^\dagger_{\text{out1}} + \hat{a}^\dagger_{\text{out2}}\right)^2\right)|0\rangle = \tag{25}$$
$$= \frac{i}{4}\left((e^{i2\phi}-1)\left(\hat{a}^\dagger_{\text{out1}}\right)^2 + (e^{i2\phi}+1)\left(\hat{a}^\dagger_{\text{out2}}\right)^2 + 2i(e^{i2\phi}+1)\hat{a}^\dagger_{\text{out1}}\hat{a}^\dagger_{\text{out2}}\right)|0\rangle = \tag{26}$$
$$= \frac{e^{i2\phi}-1}{4}|2\rangle_{\text{out1}} + \frac{e^{i2\phi}+1}{4}|2\rangle_{\text{out2}} + i\frac{e^{i2\phi}+1}{2}|1\rangle_{\text{out1}}|1\rangle_{\text{out2}} \tag{27}$$

Before computing the probabilities of two-photon detection, note two things First, the state after the first beamsplitter is exactly a two photon NOON state discussed below. Second, the computation even with two photons becoum notoriously complex. Now let us compute the

[1] "Coherent and Incoherent States of the Radiation Field", Roy J. Glauber, Phys. Rev. 1963

[2] Note: Photodetectors normally do not resolve photon numbers as assumed here. As it will be discussed later a conventional detector (i.e. photoelectric effect) produces a current $I \propto \langle \hat{a}^\dagger \hat{a}\rangle$

two-photon detection probabilities for PD1 and PD2

$$|\langle 2_{\text{out1}}|\psi_{in}\rangle|^2 = \frac{1}{8}(1 - \cos 2\phi) \tag{28}$$

$$|\langle 2_{\text{out2}}|\psi_{in}\rangle|^2 = \frac{1}{8}(1 + \cos 2\phi) \tag{29}$$

They oscillate with a peroud of π, providing a 2-times enhancement of the phase sensitivity compared to a single photon case.

6. *Now consider a case that we have managed to create a* NOON *state (as defined below) right after the first beamsplitter (before propagation)*

$$|\psi_{12}\rangle = |N :: 0\rangle \equiv \frac{1}{\sqrt{2}}(|N\rangle_1 |0\rangle_2 + |0\rangle_1 |N\rangle_2) \tag{30}$$

repeat the steps (b) and (c) to calculate the probability of detecting N photons in PD1 [2]*. Sketch this probability as a function of ϕ. Show that this is equivalent to interferometry with light with wavelength of λ/N (i.e. the so called* photonic de Broglie wavelength*).*

As we noted in the previous question computation even for the 2 photon state is difficult to keep track of. Solution is easy, istead of propagating the input state through the interferometer, we can do the inverse and compute the probabilities in the Fock space of the input modes

$$|\langle N_{\text{out1}}|\psi_{in}\rangle|^2 = |\langle 0| (\hat{a}_{\text{out1}})^N |\psi_{in}\rangle|^2| = \tag{31}$$

$$= \left| \langle 0| \left(\frac{e^{i\phi}\hat{a}_1 - i\hat{a}_2}{\sqrt{2}} \right)^N \frac{|N\rangle_1 |0\rangle_2 + |0\rangle_1 |N\rangle_2}{\sqrt{2}} \right|^2 = \left| \frac{e^{iN\phi} + (-i)^N}{2^{N/2+1}} \right|^2 \tag{32}$$

Without going into any further calculation we note that for any N the detection probability will have terms that oscillate as $\cos N\phi$, providing N-times enhancement of the phase sesitivity, which could be achieved by using classical light with an N-times shorter wavelenght.

The improvement in the phase sensitivity in the interferometers has been experimentally demonstrated with 3 photon NOON states [3] *and also four photon NOON states* [4]*. The biggest challenge though, for going for higher order NOON states is the generation process. However there are theoretical proposals for systematic methods to generate large photon number entangled states* [5]*.*

[3] Mitchell et.al. ”Super-resolving phase measurements with a multiphoton entangled state”, Nature 429, 161-164 (2004)

[4] Walther et.al. ”De Broglie wavelength of a non-local four-photon state”, Nature 429, 158-161 (2004)

[5] Kok et.al. ”Creation of large-photon-number path entanglement conditioned on photodetection”, Phys. Rev. A 65, 052104