Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.3

3.1 Beamsplitter scattering matrix

A beamsplitter is an optical device which is used for splitting beams of light into two beams. Figure 2 shows a schematic of a beamsplitter. Ports 1 and 2 are considered to be the input modes and ports 3 and 4 to be the output modes of the beamsplitter.

1. Consider the beamsplitter is subjected to two classical beams in the modes 1 and 2 with electric field phasors E_1 and E_2. Consider the electric fields to be polarised in the vertical direction. The Fresnel coefficients for transmission and reflection are t and r for mode 1 and t' and r' for mode 2. Write down the phasors of the electric fields of the 3 and 4 modes (E_3 and E_4). The scattering matrix, S, is defined as the matrix which relates the output fields to the input fields:
$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = S \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \tag{1}$$
Write down the scattering matrix for this beamsplitter. For a lossless beamsplitter what constraints does the conservation of energy impose on the coefficients?

2. Following the standard quantization of the fields, the relation between annihilation operators of the modes is given by the same scattering matrix:
$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = S \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{2}$$
Given that $\hat{a}_1$ and $\hat{a}_2$ satisfy the following bosonic commutation relations:
$$[\hat{a}_i, \hat{a}_j] = 0, \;\; [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \;\; [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \quad (i, j = 1, 2) \tag{3}$$
what relations t, r, t' and r' should hold such that $\hat{a}_3$ and $\hat{a}_4$ also satisfy similar bosonic commutation relations?

3. Next we want to see how quantum states of light propagate through the beamsplitter. Consider a 50:50 beamsplitter with Fresnel coefficients of
$$t = t' = \frac{1}{\sqrt{2}}$$
$$r = r' = \frac{i}{\sqrt{2}}$$
Now consider there is one photon in mode 1 and 0 in mode 2 (mode 2 is in vacuum state). We denote the input state of the beamsplitter as $|\psi\rangle_{\text{in}} = |1\rangle_1 |0\rangle_2$. In Heisenberg picture formulation the states are kept fixed and the operators evolve. In our case the annihilation operators $\hat{a}_1$ and $\hat{a}_2$, evolve to $\hat{a}_3$ and $\hat{a}_4$ according to equation (2). But we are interested to see what is the output state in the basis of the output modes. The input state can be written as
$$|\psi\rangle_{\text{in}} = |1\rangle_1 |0\rangle_2 = \hat{a}_1^\dagger |0\rangle_1 |0\rangle_2 \tag{4}$$
Now use equation (2) to rewrite $\hat{a}_1^\dagger$ in terms of output operators. Use this trick[1] and also the fact that a vacuum state in the input modes is also a vacuum state in the output modes to compute the state of the output light $|\psi\rangle_{\text{out}}$.

[1] In fact this trick is general since any arbitrary input state can be written as a superposition of Fock states and Fock states can be written as powers of creation operator times vacuum state.

3.2 Beamsplitter treatment in phase space representation[2]

(a) Use the method explained in the exercise **Beamsplitter scattering matrix** to show if a coherent state hits a 50:50 beamsplitter, it will split into two coherent states:

$$|\alpha\rangle_1 |0\rangle_2 \xrightarrow{\text{BS}} |\frac{\alpha}{\sqrt{2}}\rangle_3 |\frac{i\alpha}{\sqrt{2}}\rangle_4 \tag{5}$$

Hint: use the definition $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$

(b) Use the result of the previous section to show that if a field with the P-representation (Glauber-Sudarshan phase space function) $P_{\text{in}}(\alpha)$ is incident on a 50:50 beamsplitter, the output field has a P-representation given by $P_{\text{out}}(\alpha) = 2P_{\text{in}}(\sqrt{2}\alpha)$.

3.3 Phase measurements with multiphoton entangled states

In this exercise we demonstrate how the use of multiphoton entangled states of light can improve the phase sensitivity in interferometry experiments (i.e. demonstrating the principle of *Quantum Metrology*). Figure (4) shows a scheme of a Mach-Zehnder interferometer. The input light is split into two paths (1 and 2) on the first beamsplitter (BS1) and then recombined on the second beamsplitter (BS2) and then detected on the photodetectors (PD1 and PD2). The interferometer arms have the lengths of L_1 and L_2 with difference of ΔL. For simplicity assume that the beamsplitters are 50:50. The input light has angular frequency of ω and propagates in the interferometer arms with speed of c.

1. Consider we send one photon through input port one (in1) and zero through port 2 (in2). The input state of the interferometer would be

$$|\psi_{\text{in}}\rangle = |1\rangle_{\text{in1}} \otimes |0\rangle_{\text{in2}} \equiv |1\rangle_{\text{in1}} |0\rangle_{\text{in2}} \tag{6}$$

   What is the joint state of the light in paths 1 and 2 right after the first beamsplitter? You may use the trick explained in the exercise **Beamsplitter scattering matrix**. Denote this joint state as kets of the form $|\rangle_1 |\rangle_2$.

2. The path length difference will cause the photons propagating in different paths to acquire a phase difference. Compute the joint state after propagation along the interferometer arms (which is at the input of the second beamsplitter).

3. Compute the state of the light at the output ports of the interferometer (Use the notation $|\rangle_{\text{out1}} |\rangle_{\text{out2}}$).

4. What is the probability of detecting 1 photon in PD1? What is the probability of detecting 1 photon in PD2? Sketch these probabilities as functions of $\phi = \frac{\omega \Delta L}{c}$.

5. Next we use a different input state for the interferometry. We send 1 photon in each of the input ports, meaning the input state would be

$$|\psi_{\text{in}}\rangle = |1\rangle_{\text{in1}} |1\rangle_{\text{in2}} \tag{7}$$

   repeat the steps (a) to (c) and compute the probability of detecting 2 photons in PD1[3]. Sketch this probability as a function of ϕ.

[2] This exercise is adapted from the book Quantum optics by D. Walls and G. Milburn, chapter 4

[3] Note: Photodetectors normally do not resolve photon numbers as assumed here. As it will be discussed later a conventional detector (i.e. photoelectric effect) produces a current $I \propto \langle \hat{a}^\dagger \hat{a} \rangle$

6. Now consider a case that we have managed to create a *NOON* state (as defined below) right after the first beamsplitter (before propagation)

$$|\psi_{12}\rangle = |N :: 0\rangle \equiv \frac{1}{\sqrt{2}}(|N\rangle_1 |0\rangle_2 + |0\rangle_1 |N\rangle_2) \tag{8}$$

repeat the steps (b) and (c) to calculate the probability of detecting N photons in PD1 [3]. Sketch this probability as a function of ϕ. Show that this is equivalent to interferometry with light with wavelength of λ/N (i.e. the so called *photonic de Broglie wavelength*).

The improvement in the phase sensitivity in the interferometers has been experimentally demonstrated with 3 photon NOON states [4] and also four photon NOON states [5]. The biggest challenge though, for going for higher order NOON states is the generation process. However there are theoretical proposals for systematic methods to generate large photon number entangled states [6].

3.4 Introduction to QuTiP: Wigner quasi-probability distribution(*)[7]

QuTiP[8] is a Python simulation package for simulating open quantum systems. Using this tool arbitrary Hamiltonians, including time-dependent systems, may be built up from operators and states defined by a quantum object class, and then passed on to a choice of master equation or Monte Carlo solvers. This open-source software is very popular in the field of quantum optics and is used in several recent papers in this field. In this exercise we will use this powerful tool to investigate Wigner function visualization of quasi-probability distribution of some quantum states that has been introduced in the lecture so far. There exists several tutorials and documentations for QuTiP toolbox, however we promote the original documentation by the software developers[9]. We advise you to follow the steps and install this useful software once for this course (and hopefully your future scientific career).

1. Write down a general function using the built-in *plot_wigner* function to visualize the Wigner function in 2D and 3D for a given state $|\psi\rangle$.

2. The Fock state $|n\rangle$ is defined using the built-in function *basis(N,n)* for a Fock space of length N. In this exercise we assume a Fock space of length $N = 20$.

3. Plot the Wigner function for the first four Fock states ($n = 0, 1, 2, 3$) and comment on the sign of the Wigner function for different Fock states. From your results, show the origin of "quasi-probability" nature of the Wigner representation.

4. The coherent state $|\alpha\rangle$ is defined using the built-in function *coherent(N,α)*, where N is similar to part (2). Plot the Wigner function of a coherent state $|\alpha\rangle$ for $\alpha = 2$ and -2. From your results, verify that a coherent state is a displaced vacuum state by $\hat{D}(\alpha)$.

5. In quantum mechanics a *cat state* is a quantum state that is a superposition of two diametrically opposed states at the same time, such as the possibilities that a cat be alive and dead at the same time (Schrödinger's cat). In quantum optics a Schrödinger cat state is composed of two coherent states with opposite phases ($|\alpha\rangle$ and $|-\alpha\rangle$):

$$|\text{cat}\rangle = \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2}} \tag{9}$$

[4] Mitchell et.al. "Super-resolving phase measurements with a multiphoton entangled state", Nature 429, 161-164 (2004)

[5] Walther et.al. "De Broglie wavelength of a non-local four-photon state", Nature 429, 158-161 (2004)

[6] Kok et.al. "Creation of large-photon-number path entanglement conditioned on photodetection", Phys. Rev. A 65, 052104

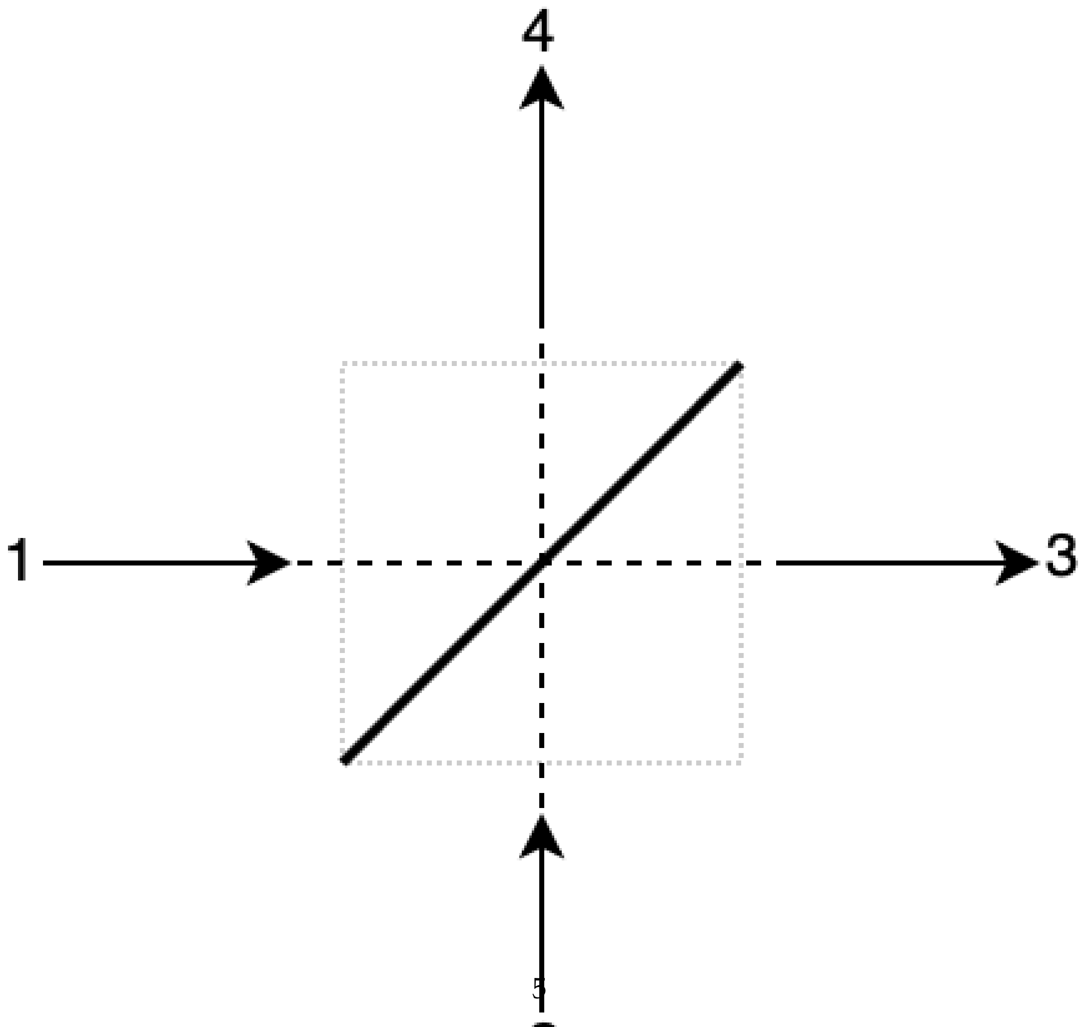
[7] Graded exercise

[8] http://qutip.org/

[9] http://qutip.org/documentation.html

where $|\alpha|$ is the amplitude of the coherent state. Plot the Wigner function of a Schrödinger cat state for $\alpha = 2$. Compare your results using your plots for individual coherent states $|\alpha\rangle$ and $|-\alpha\rangle$.

Figure 1: A 50:50 beamsplitter.



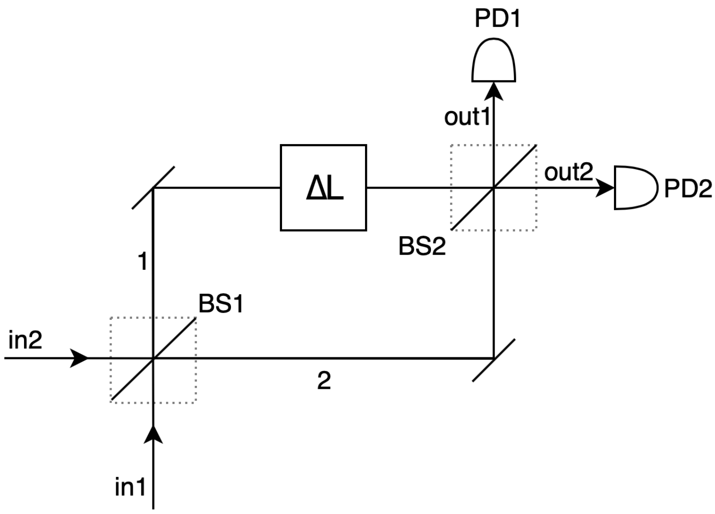


Figure 4: A Mach-Zehnder interferometer. BS: Beamsplitter, PD: Photodetector