

Quantum Electrodynamics and Quantum Optics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to Exercise No.2

Solution: Evolution of a coherent wavepacket

Consider the wavefunction expansion and the initial displaced ground state (or vacuum state) expression given by

$$\begin{aligned}\psi(x, t) &= \sum_{n=0}^{\infty} a_n \phi_n(x) e^{-iE_n t/\hbar} = \langle x | \psi(t) \rangle \\ \psi(x, t=0) &= \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{\omega}{2\hbar}(x-x_0)^2\right] = \frac{1}{\sqrt{\sigma}\sqrt{\pi}} \exp\left[-\frac{1}{2\sigma^2}(x-x_0)^2\right],\end{aligned}\quad (1)$$

where $\sigma^2 = \hbar/\omega$.

- (a) First, we try to give some interpretation on the initial condition by considering the action of $\hat{a}$ onto it. Indeed recall that the destruction operator has the form (recall here $m = 1$)

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega\hat{x} + i\hat{p}) \quad (2)$$

Now computing the action of $\hat{a}$ onto the initial wavefunction in the position basis yields

$$\begin{aligned}\langle x | \hat{a} | \psi(t=0) \rangle &= \frac{1}{\sqrt{2\hbar\omega}} \langle x | (\omega\hat{x} + i\hat{p}) | \psi(t=0) \rangle \\ &= x \sqrt{\frac{\omega}{2\hbar}} \langle x | \psi(t=0) \rangle + \frac{i}{\sqrt{2\hbar\omega}} \left(-i\hbar \frac{d}{dx}\right) \langle x | \psi(t=0) \rangle \\ &= x \sqrt{\frac{\omega}{2\hbar}} \langle x | \psi(t=0) \rangle + \frac{\hbar}{\sqrt{2\hbar\omega}} \frac{d}{dx} \langle x | \psi(t=0) \rangle \\ &= x \sqrt{\frac{\omega}{2\hbar}} \langle x | \psi(t=0) \rangle - \frac{\hbar}{\sqrt{2\hbar\omega}} \frac{x-x_0}{\sigma^2} \langle x | \psi(t=0) \rangle \\ &= \sqrt{\frac{\omega}{2\hbar}} x_0 \langle x | \psi(t=0) \rangle = \frac{x_0}{\sigma\sqrt{2}} \langle x | \psi(t=0) \rangle.\end{aligned}\quad (3)$$

Therefore $|\psi(t=0)\rangle$ is an eigenstate of $\hat{a}$ with eigenvalue $\alpha = \frac{x_0}{\sigma\sqrt{2}}$. It is a coherent state! In what follows, we denote $|\psi(t=0)\rangle = |\alpha\rangle$.

- (b) Using the Baker-Campbell-Hausdorff formula (or properties of coherent states), we rewrite

$$\begin{aligned}|\alpha\rangle &= e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} |0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} |0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} |0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (\hat{a}^\dagger)^n |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\end{aligned}\quad (4)$$

which gives the decomposition of the initial wavefunction in the orthogonal Fock state basis. Together with the orthogonality condition, we find the amplitude as

$$a_n = \langle n | \alpha \rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} = e^{-(x_0/2\sigma)^2} \frac{(x_0/\sqrt{2})^n}{\sigma^n \sqrt{n!}} \quad (5)$$

- (c) From the superposition principle and with the energies $E_n = \hbar\omega(n + 1/2)$, the time evolution of the coherent state is given by

$$\begin{aligned}
 |\psi(t)\rangle &= \sum_{n=0}^{\infty} a_n e^{-iE_n t/\hbar} |n\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} e^{-iE_n t/\hbar} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
 &= \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} e^{-in\omega t} e^{-i\omega t/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
 &= e^{-|\alpha|^2/2} e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle \\
 &= e^{-|\alpha|^2/2} e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t} \hat{a}^\dagger)^n}{n!} |0\rangle \\
 &= e^{-i\omega t/2} e^{-|\alpha e^{-i\omega t}|^2/2} e^{\alpha e^{-i\omega t} \hat{a}^\dagger} |0\rangle
 \end{aligned} \tag{6}$$

We define the time-dependent evolution of the coherent amplitude as $\alpha(t) = \alpha e^{-i\omega t}$, giving

$$\begin{aligned}
 |\psi(t)\rangle &= e^{-i\omega t/2} e^{-|\alpha(t)|^2/2} e^{\alpha(t) \hat{a}^\dagger} |0\rangle \\
 &= e^{-i\omega t/2} e^{-|\alpha(t)|^2/2} e^{\alpha(t) \hat{a}^\dagger} e^{-\alpha^*(t) \hat{a}} |0\rangle \\
 &= e^{-i\omega t/2} e^{\alpha(t) \hat{a}^\dagger - \alpha^*(t) \hat{a}} |0\rangle
 \end{aligned} \tag{7}$$

The complex phase will disappear in the norm of the wavefunction, so will concentrate on the operator part $e^{\hat{D}}$, where

$$\begin{aligned}
 \hat{D} &= \alpha(t) \hat{a}^\dagger - \alpha^*(t) \hat{a} = \frac{\alpha(t)}{\sqrt{2}} \left(\frac{\hat{x}}{\sigma} - \sigma \frac{\partial}{\partial x} \right) - \frac{\alpha^*(t)}{\sqrt{2}} \left(\frac{\hat{x}}{\sigma} + \sigma \frac{\partial}{\partial x} \right) \\
 &= \sqrt{2}i \operatorname{Im}(\alpha(t)) \frac{\hat{x}}{\sigma} - \sqrt{2} \operatorname{Re}(\alpha(t)) \sigma \frac{\partial}{\partial x} \\
 &= \sqrt{2} \left(i \operatorname{Im}(\alpha(t)) \hat{y} - \operatorname{Re}(\alpha(t)) \frac{\partial}{\partial y} \right).
 \end{aligned} \tag{8}$$

We introduced $\hat{y} = \hat{x}/\sigma$ to simplify the notation. Now we expand the exponential operator to orders of $\hat{D}$ and compute their action onto the wavefunction in the y basis : $e^{\hat{D}} \langle y|0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{D}^n \langle y|0\rangle$. We will use $\langle y|0\rangle = \phi_0(y) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}} e^{-y^2/2}$

$$\begin{aligned}
 \underline{n=0} \quad & \phi_0(y)
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 \underline{n=1} \quad & \hat{D}\phi_0(y) = \sqrt{2} (i \operatorname{Im}(\alpha(t)) y + \operatorname{Re}(\alpha(t)) y) \phi_0(y) \\
 & = \sqrt{2} \alpha(t) y \phi_0(y)
 \end{aligned} \tag{10}$$

$$\begin{aligned}
 \underline{n=2} \quad & \frac{\hat{D}^2}{2!} \phi_0(y) = \frac{\sqrt{2}\alpha(t)}{2!} \hat{D}(y\phi_0(y)) = \alpha(t) \left(i \operatorname{Im}(\alpha(t)) \hat{y} - \operatorname{Re}(\alpha(t)) \frac{\partial}{\partial y} \right) (y\phi_0(y)) \\
 & = \alpha(t) (\alpha(t) y^2 - \operatorname{Re}(\alpha(t))) \phi_0(y)
 \end{aligned} \tag{11}$$

$$\begin{aligned}
 \underline{n=3} \quad & \frac{\hat{D}^3}{3!} \phi_0(y) = \frac{\alpha(t)}{3} \hat{D} ((\alpha(t) y^2 - \operatorname{Re}(\alpha(t))) \phi_0(y)) = \frac{\alpha(t)}{3} (\alpha(t) \hat{D} (y^2 \phi_0(y)) - \operatorname{Re}(\alpha(t)) \hat{D} (\phi_0(y))) \\
 & = \dots = \frac{\sqrt{2}\alpha^2(t)}{3} (\alpha(t) y^3 - 3 \operatorname{Re}(\alpha(t)) y) \phi_0(y)
 \end{aligned} \tag{12}$$

$n = 4$

$$\frac{\hat{D}^4}{4!} \phi_0(y) = \dots = \frac{\alpha^2(t)}{6} \left(\alpha^2(t) y^4 - 6\alpha(t) \operatorname{Re}(\alpha(t)) y^2 + 3 \operatorname{Re}^2(\alpha(t)) \right) \phi_0(y) \quad (13)$$

...

Compiling the results, we obtain

$$\begin{aligned} e^{\hat{D}} \phi_0(y) &= \left[1 + \alpha(t) \left(\sqrt{2}y - \operatorname{Re}(\alpha(t)) \right) + \alpha^2(t) \left(y^2 - \sqrt{2} \operatorname{Re}(\alpha(t)) y + \frac{1}{2} \operatorname{Re}^2(\alpha(t)) \right) + \dots \right] \phi_0(y) \\ &= \left[1 + \alpha(t) \left(\sqrt{2}y - \operatorname{Re}(\alpha(t)) \right) + \frac{\alpha^2(t)}{2!} \left(\sqrt{2}y - \operatorname{Re}(\alpha(t)) \right)^2 + \dots \right] \phi_0(y) \\ &= \exp \left\{ \alpha(t) \left(\sqrt{2}y - \operatorname{Re}(\alpha(t)) \right) \right\} \phi_0(y). \end{aligned} \quad (14)$$

Finally, we can derive the probability density function through

$$\begin{aligned} |\psi(y, t)|^2 &= \langle \psi(t) | y \rangle \langle y | \psi(t) \rangle = \langle 0 | (e^{\hat{D}})^\dagger | y \rangle \langle y | e^{\hat{D}} | 0 \rangle \\ &= \exp \left\{ \alpha(t)^* \left(\sqrt{2}y - \operatorname{Re}(\alpha(t)) \right) \right\} \exp \left\{ \alpha(t) \left(\sqrt{2}y - \operatorname{Re}(\alpha(t)) \right) \right\} |\phi_0(y)|^2 \\ &= \frac{1}{\sigma \sqrt{\pi}} \exp \left\{ -y^2 + 2\sqrt{2}y \operatorname{Re}(\alpha(t)) - 2 \operatorname{Re}^2(\alpha(t)) \right\} \\ &= \frac{1}{\sigma \sqrt{\pi}} \exp \left\{ - \left(y - \sqrt{2} \operatorname{Re}(\alpha(t)) \right)^2 \right\} \end{aligned} \quad (15)$$

Returning to the original variable $x = \sigma y$, using $\sigma^2 = \hbar/\omega$ and plugging in the value of $\operatorname{Re}(\alpha(t)) = \frac{x_0}{\sigma \sqrt{2}} \cos(\omega t)$, we get the final result

$$\begin{aligned} |\psi(x, t)|^2 &= \frac{1}{\sigma \sqrt{\pi}} \exp \left\{ - \frac{(x - x_0 \cos(\omega t))^2}{\sigma^2} \right\} \\ &= \sqrt{\frac{\omega}{\pi \hbar}} \exp \left\{ - \frac{\omega}{\hbar} (x - x_0 \cos \omega t)^2 \right\}. \end{aligned} \quad (16)$$

2.1 Properties of coherent and squeezed states

In this exercise we want to calculate the basic properties of coherent¹ and squeezed states of light. Here we focus on an electromagnetic field mode described by the creation $\hat{a}^\dagger$ and annihilation operators $\hat{a}$.

1. Calculate $[\hat{X}_1(\phi), \hat{X}_2(\phi)]$. What do they refer to in case of $\phi = 0$?

In case $\phi = 0$:

$$\hat{X}_1(\phi) = \frac{1}{2} \left(\hat{a} e^{i\phi} + \hat{a}^\dagger e^{-i\phi} \right) \Big|_{\phi=0} = \frac{1}{\sqrt{2}} \frac{\hat{x}}{x_{\text{ZPF}}} \quad (17)$$

$$\hat{X}_2(\phi) = \frac{1}{2i} \left(\hat{a} e^{i\phi} - \hat{a}^\dagger e^{-i\phi} \right) \Big|_{\phi=0} = \frac{1}{\sqrt{2}} \frac{\hat{p}}{p_{\text{ZPF}}} \quad (18)$$

The commutator is calculated using the linearity

$$[\hat{X}_1(\phi), \hat{X}_2(\phi)] = -\frac{i}{4} [\hat{a}, \hat{a}^\dagger] + \frac{i}{4} [\hat{a}^\dagger, \hat{a}] = -\frac{i}{2} \quad (19)$$

¹Glauber, Roy J. "Coherent and incoherent states of the radiation field." Physical Review 131.6 (1963): 2766.

2. The time evolution of an operator in Heisenberg picture is determined by the Heisenberg equation

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}] + \frac{\partial \hat{O}}{\partial t} \quad (20)$$

Applying it to the quadratures (17, 18) we obtain

$$\dot{\hat{X}}_1 = -\frac{i}{\hbar} \frac{\hbar\omega}{2} [\hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi}, \hat{a}^\dagger \hat{a}] = -\frac{i\omega}{2} (\hat{a}e^{i\phi} - \hat{a}^\dagger e^{-i\phi}) = -\omega \hat{X}_2 \quad (21)$$

and similarly

$$\dot{\hat{X}}_2 = \dots = \omega \hat{X}_1 \quad (22)$$

Solving these coupled differential equations we derive that

$$\hat{X}_1(\phi, t) = \hat{X}_1(\phi + \omega t, 0) \quad (23)$$

$$\hat{X}_2(\phi, t) = \hat{X}_2(\phi + \omega t, 0) \quad (24)$$

Another way to arrive to this solution is to first derive the creation and annihilation operators in the Heisenberg picture and then substitute them in equations (17, 18). Note that later in the course we will extensively use the Heisenberg picture, without always mentioning it explicitly.

3. A coherent state ($|\alpha\rangle$) is defined as: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

Show that a coherent state is a minimum uncertainty state for every pair of $\hat{X}_1(\phi)$ and $\hat{X}_2(\phi)$.

Uncertainty for a hermitian operator is defined as $\Delta\hat{A} = \sqrt{\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$. Minimal uncertainty implies that for a pair of non-commuting operators the Heisenberg uncertainty inequality $\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$ is saturated². A straightforward calculation of the respective expectation values for the quadratures yields the following

$$\langle\alpha|\hat{X}_1(\phi)|\alpha\rangle = \text{Re}(\alpha e^{i\phi}) \quad (25)$$

$$\langle\alpha|\hat{X}_2(\phi)|\alpha\rangle = \text{Im}(\alpha e^{i\phi}) \quad (26)$$

$$\langle\alpha|\hat{X}_1^2(\phi)|\alpha\rangle = \frac{1}{4}(\alpha^2 e^{i2\phi} + \alpha^{*2} e^{-i2\phi} + 2|\alpha|^2 + 1) \quad (27)$$

$$\langle\alpha|\hat{X}_1^2(\phi)|\alpha\rangle = -\frac{1}{4}(\alpha^2 e^{i2\phi} + \alpha^{*2} e^{-i2\phi} - 2|\alpha|^2 - 1) \quad (28)$$

and finally

$$(\Delta\hat{X}_1(\phi))^2 = \frac{1}{4} \quad (29)$$

$$(\Delta\hat{X}_2(\phi))^2 = \frac{1}{4} \quad (30)$$

which confirms that

$$\Delta\hat{X}_1\Delta\hat{X}_2 = \frac{1}{2}|\langle[\hat{X}_1, \hat{X}_2]\rangle| \quad (31)$$

4. For simplicity, consider a state $|\psi\rangle$ with vanishing mean for all times: $\langle\psi|\hat{a}|\psi\rangle = \langle\hat{a}\rangle = \langle\hat{a}^\dagger\rangle = \langle\hat{X}_1\rangle = \langle\hat{X}_2\rangle = 0$

(a) Express the variance of the field quadratures ($\Delta\hat{X}_i^2$) with respect to $\langle\hat{a}^\dagger\hat{a}\rangle$, $\langle\hat{a}\hat{a}\rangle$ and ϕ .

$$(\Delta\hat{X}_1)^2 = \langle\hat{X}_1^2\rangle_{|\psi\rangle} = \frac{1}{4} \left(\langle\hat{a}^2\rangle e^{i2\phi} + \langle(\hat{a}^\dagger)^2\rangle e^{-i2\phi} + 2\langle\hat{a}^\dagger\hat{a}\rangle + 1 \right) \quad (32)$$

$$(\Delta\hat{X}_2)^2 = \langle\hat{X}_2^2\rangle_{|\psi\rangle} = -\frac{1}{4} \left(\langle\hat{a}^2\rangle e^{i2\phi} + \langle(\hat{a}^\dagger)^2\rangle e^{-i2\phi} - 2\langle\hat{a}^\dagger\hat{a}\rangle - 1 \right) \quad (33)$$

²Note that there are many inequalities that express uncertainty principle, refer e.g. to [wikipedia.org/wiki/Uncertainty_principle](https://en.wikipedia.org/wiki/Uncertainty_principle)

combining these two expressions we obtain

$$(\Delta \hat{X}_j)^2 = \frac{(-1)^{j+1}}{2} (\text{Re} \langle \hat{a}^2 \rangle \cos 2\phi - \text{Im} \langle \hat{a}^2 \rangle \sin 2\phi) + \frac{1}{4} (2 \langle \hat{a}^\dagger \hat{a} \rangle + 1) \quad (34)$$

where $j = 1, 2$.

- (b) *Show that the minimal and maximal variances are corresponding to two conjugate quadratures. Give their expressions.*

To analyse the extrema of the quadratures we find the derivatives

$$\frac{\partial}{\partial \phi} \{(\Delta \hat{X}_j)^2\} = (-1)^j (\text{Re} \langle \hat{a}^2 \rangle \sin 2\phi + \text{Im} \langle \hat{a}^2 \rangle \cos 2\phi) \quad (35)$$

$$\frac{\partial^2}{\partial \phi^2} \{(\Delta \hat{X}_j)^2\} = (-1)^j \cdot 2 (\text{Re} \langle \hat{a}^2 \rangle \cos 2\phi - \text{Im} \langle \hat{a}^2 \rangle \sin 2\phi) \quad (36)$$

The extrema of the $\Delta \hat{X}_j$ are reached when

$$\frac{\partial}{\partial \phi} \{(\Delta \hat{X}_j)^2\} = 0 \Rightarrow \frac{\sin 2\phi}{\cos 2\phi} = -\frac{\text{Im} \langle \hat{a}^2 \rangle}{\text{Re} \langle \hat{a}^2 \rangle} \quad (37)$$

and the sign of the second derivative determines whether it is a maximum or a minimum

$$\left. \frac{\partial^2}{\partial \phi^2} \{(\Delta \hat{X}_j)^2\} \right|_{\text{extremum}} = (-1)^j \cdot 2 \frac{|\langle \hat{a}^2 \rangle|^2}{\text{Re} \langle \hat{a}^2 \rangle} \cos 2\phi \quad (38)$$

Note that the factor of $(-1)^j$ guarantees that when one of the quadratures has a minimum the other has a maximum and vice versa.

- (c) *The state is squeezed if there exists at least a pair of conjugated quadratures for which the uncertainty of one of the quadratures is smaller than $\frac{1}{4}$. Deduce a sufficient and necessary condition to have a squeezed state.*

Let us concentrate on the first quadrature. By definition a squeezed state is such that

$$(\Delta \hat{X}_1)^2 = \frac{1}{4} (\text{Re} \langle \hat{a}^2 \rangle \cos 2\phi - \text{Im} \langle \hat{a}^2 \rangle \sin 2\phi + 2 \langle \hat{a}^\dagger \hat{a} \rangle + 1) < \frac{1}{4} \quad (39)$$

Which is equivalent to

$$\cos \left(2\phi + \text{atan} \left(\frac{\text{Im} \langle \hat{a}^2 \rangle}{\text{Re} \langle \hat{a}^2 \rangle} \right) \right) < -\frac{\langle \hat{a}^\dagger \hat{a} \rangle}{|\langle \hat{a}^2 \rangle|} \quad (40)$$

5. *In order to obtain a squeezed state, one may apply the squeezing operator with squeezing parameter r :*

$$\hat{S}(\epsilon) = e^{\epsilon \hat{a}^{\dagger 2} - \epsilon^* \hat{a}^2}, \quad \epsilon = \frac{1}{2} r e^{-2i\phi} \quad (41)$$

- (a) *Derive the transformation of $\hat{a}$ and $\hat{a}^\dagger$ under the squeezing operator. (Heisenberg picture formulation) Here we use Baker-Campbell-Hausdorff formula for any operators $\hat{A}$ and $\hat{B}$:*

$$e^{-\hat{A}} \hat{B} e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \underbrace{[\hat{A}, [\hat{A}, \dots [\hat{A}, \hat{B}]]]}_{n \text{ brackets}} \quad (42)$$

Let $\hat{A} = \epsilon \hat{a}^{\dagger 2} - \epsilon^* \hat{a}^2$ and $\hat{B} = \hat{a}$, we find

$$\begin{aligned} [\hat{A}, \hat{B}] &= [\epsilon \hat{a}^{\dagger 2} - \epsilon^* \hat{a}^2, \hat{a}] = -2\epsilon \hat{a}^\dagger = -r e^{-2i\phi} \hat{a}^\dagger \\ [\hat{A}, [\hat{A}, \hat{B}]] &= [\epsilon \hat{a}^{\dagger 2} - \epsilon^* \hat{a}^2, -2\epsilon \hat{a}^\dagger] = 4\epsilon \epsilon^* \hat{a}^\dagger = r^2 \hat{a}^\dagger \\ [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] &= [\epsilon \hat{a}^{\dagger 2} - \epsilon^* \hat{a}^2, 4|\epsilon|^2 \hat{a}^\dagger] = r^3 e^{-2i\phi} \hat{a}^\dagger \end{aligned} \quad (43)$$

Therefore, we have

$$\begin{aligned}
\hat{S}^\dagger(\epsilon)\hat{a}\hat{S}(\epsilon) &= e^{-\epsilon\hat{a}^\dagger + \epsilon^*\hat{a}^2}\hat{a}e^{\epsilon\hat{a}^\dagger - \epsilon^*\hat{a}^2} \\
&= \hat{a} - re^{-2i\phi}\hat{a}^\dagger + \frac{1}{2!}r^2\hat{a} - \frac{1}{3!}r^3e^{-2i\phi}\hat{a}^\dagger + \dots \\
&= \hat{a}\left(1 + \frac{1}{2!}r^2 + \dots\right) - \hat{a}^\dagger e^{-2i\phi}\left(1 + \frac{1}{3!}r^3 + \dots\right) \\
&= \hat{a}\cosh(r) - \hat{a}^\dagger e^{-2i\phi}\sinh(r)
\end{aligned} \tag{44}$$

Treating the same for $\hat{a}^\dagger$, we have

$$\begin{aligned}
\hat{S}^\dagger(\epsilon)\hat{a}^\dagger\hat{S}(\epsilon) &= e^{-\epsilon\hat{a}^\dagger + \epsilon^*\hat{a}^2}\hat{a}^\dagger e^{\epsilon\hat{a}^\dagger - \epsilon^*\hat{a}^2} \\
&= \hat{a}^\dagger - re^{2i\phi}\hat{a} + \frac{1}{2!}r^2\hat{a}^\dagger - \frac{1}{3!}r^3e^{2i\phi}\hat{a} + \dots \\
&= \hat{a}^\dagger\left(1 + \frac{1}{2!}r^2 + \dots\right) - \hat{a}e^{2i\phi}\left(1 + \frac{1}{3!}r^3 + \dots\right) \\
&= \hat{a}^\dagger\cosh(r) - \hat{a}e^{2i\phi}\sinh(r)
\end{aligned} \tag{45}$$

(b) Then deduce the transformation for the quadratures $\hat{X}_1(\phi)$ and $\hat{X}_2(\phi)$

From the definition shown in Eq. 17, Eq. 18, we have

$$\begin{aligned}
S^\dagger(\epsilon)X_1S(\epsilon) &= \frac{1}{2}e^{i\phi}S^\dagger(\epsilon)aS(\epsilon) + \frac{1}{2}e^{-i\phi}S^\dagger(\epsilon)a^\dagger S(\epsilon) \\
&= \frac{1}{2}\left(e^{i\phi}a\cosh(r) - e^{-i\phi}a^\dagger\sinh(r)\right) + \frac{1}{2}\left(e^{-i\phi}a^\dagger\cosh(r) - e^{i\phi}a\sinh(r)\right) \\
&= \frac{e^{i\phi}a + e^{-i\phi}a^\dagger}{2}(\cosh(r) - \sinh(r)) \\
&= X_1e^{-r}
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
S^\dagger(\epsilon)X_2S(\epsilon) &= \frac{i}{2}e^{i\phi}S^\dagger(\epsilon)aS(\epsilon) - \frac{i}{2}e^{-i\phi}S^\dagger(\epsilon)a^\dagger S(\epsilon) \\
&= \frac{i}{2}\left(e^{i\phi}a\cosh(r) - e^{-i\phi}a^\dagger\sinh(r)\right) - \frac{i}{2}\left(e^{-i\phi}a^\dagger\cosh(r) - e^{i\phi}a\sinh(r)\right) \\
&= i\frac{e^{i\phi}a - e^{-i\phi}a^\dagger}{2}(\cosh(r) + \sinh(r)) \\
&= X_2e^r
\end{aligned} \tag{47}$$

Therefore, we find one of the quadrature is amplified by e^r while the other one is shrunk by e^r .

(c) Apply the squeezing operator on a coherent state $|\alpha\rangle$ and derive the following quantities for the transformed state: $\langle\hat{X}_1\rangle$, $\langle\hat{X}_2\rangle$, $\Delta\hat{X}_1$ and $\Delta\hat{X}_2$.

For a squeezed coherent state $\hat{S}|\alpha\rangle$, the expectation values of $\hat{X}_1$ and $\hat{X}_2$ directly follow from equations 46, 47:

$$\langle\alpha|\hat{S}^\dagger\hat{X}_1\hat{S}|\alpha\rangle = e^r\langle\alpha|\hat{X}_1|\alpha\rangle = e^{-r}\text{Re}\{\alpha e^{i\phi}\} \tag{48}$$

$$\langle\alpha|\hat{S}^\dagger\hat{X}_2\hat{S}|\alpha\rangle = e^r\langle\alpha|\hat{X}_2|\alpha\rangle = e^r\text{Im}\{\alpha e^{i\phi}\} \tag{49}$$

To calculate the standard deviation of $\hat{X}_1$ and $\hat{X}_2$, we first calculate the expectation value of $\hat{X}_1^2$ and $\hat{X}_2^2$:

$$\begin{aligned}\langle \alpha | \hat{S}^\dagger \hat{X}_1^2 \hat{S} | \alpha \rangle &= \langle \alpha | \hat{S}^\dagger \hat{X}_1 \hat{S} \hat{S}^\dagger \hat{X}_1 \hat{S} | \alpha \rangle \\ &= e^{-2r} \langle \alpha | X_1^2 | \alpha \rangle \\ &= \frac{e^{-2r}}{4} (\alpha^2 e^{i2\phi} + \alpha^{*2} e^{-i2\phi} + 2|\alpha|^2 + 1)\end{aligned}\quad (50)$$

$$\begin{aligned}\langle \alpha | \hat{S}^\dagger \hat{X}_2^2 \hat{S} | \alpha \rangle &= \langle \alpha | \hat{S}^\dagger \hat{X}_2 \hat{S} \hat{S}^\dagger \hat{X}_2 \hat{S} | \alpha \rangle \\ &= e^{2r} \langle \alpha | X_2^2 | \alpha \rangle \\ &= -\frac{e^{2r}}{4} (\alpha^2 e^{i2\phi} + \alpha^{*2} e^{-i2\phi} - 2|\alpha|^2 - 1)\end{aligned}\quad (51)$$

Finally, we have

$$\Delta X_1 = \sqrt{\langle X_1^2 \rangle - \langle X_1 \rangle^2} = \frac{e^{-r}}{2} \quad (52)$$

$$\Delta X_2 = \sqrt{\langle X_2^2 \rangle - \langle X_2 \rangle^2} = \frac{e^r}{2} \quad (53)$$

The squeezing operator effectively produces a squeezed state from a coherent state. A quadrature (average value and variance) is amplified by the squeezing factor e^r whereas the conjugated quadrature is “de-amplified” by the same factor. In practice, this can be used to enhance the sensitivity in interferometric measurements of electromagnetic field fluctuations³, such as in *Gravitational Wave detectors*⁴. In fact, squeezed states are now being used in Advanced LIGO for gravitational wave detection⁵.

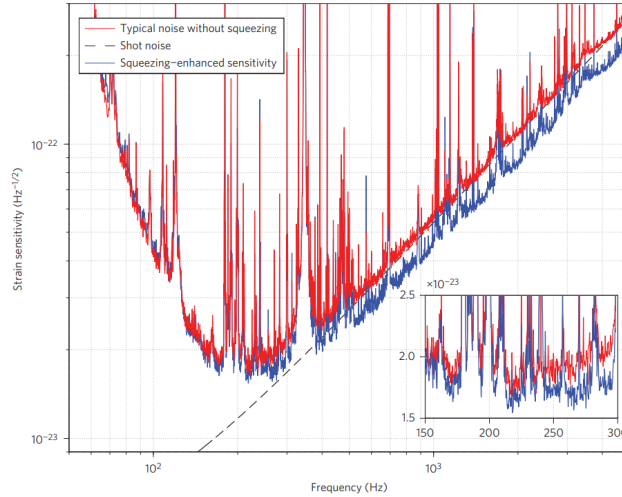


Figure 1: Strain sensitivity of LIGO homodyne detection measured with and without squeezing injection [Aasi, Junaid, et al. *Nature Photonics* (2013)]

(d) Plot the following states in the quadrature plane ($\hat{X}_1(\phi)$, $\hat{X}_2(\phi)$):

- i. A coherent state $|\alpha\rangle$.
- ii. A squeezed state $\hat{S}(\epsilon)|\alpha\rangle$ for $r = 2$.

See FIG. 2

³Caves, Carlton M. “Quantum-mechanical noise in an interferometer.” *Physical Review D* 23.8 (1981): 1693.

⁴Abadie, J., et al. “A gravitational wave observatory operating beyond the quantum shot-noise limit.” *Nature Physics* 7.12 (2011): 962.

⁵Aasi, Junaid, et al. “Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light.” *Nature Photonics* 7.8 (2013): 613.

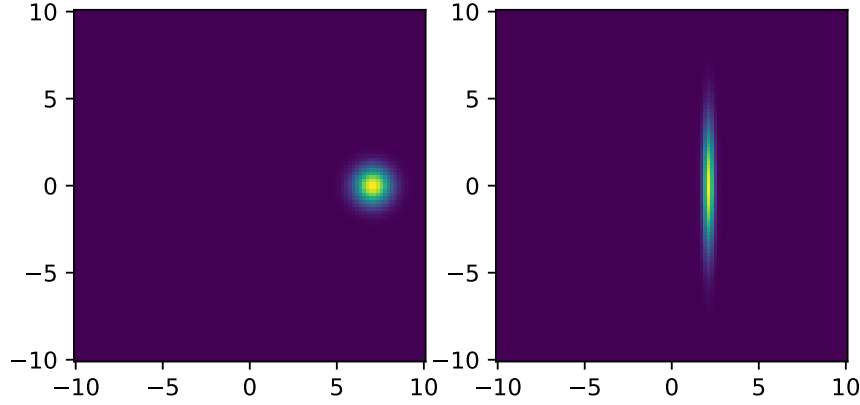


Figure 2: Left: coherent state; Right: squeeze coherent state. Parameters are chosen as: $\alpha = 5$, $\epsilon = 1.2$. Plotted by QuTip.

- (e) Show that the average amplitude $|\beta|^2 = \langle \hat{X}_1 \rangle^2 + \langle \hat{X}_2 \rangle^2$ for the squeezed state satisfies:

$$|\beta|^2 = |\alpha|^2 \cosh(2r) + \text{Re} \left(\alpha^2 e^{2i\phi} \sinh(2r) \right) \quad (54)$$

By summing Eq. 50 and Eq. 51, we can directly obtain the result.

- (f) From the result of the previous part justify the names “amplitude squeezed state” and “phase squeezed state”.

By analogy with a complex number we can think of $|\beta|$ as the amplitude of a squeezed state, while X_1 and X_2 as its real and imaginary parts. Also in 5b we established that operator $\hat{S}(\epsilon = \frac{1}{2}re^{-2i\phi})$ squeezes the coherent state along the X_1 and X_2 quadratures (see FIG. 2). In a special case when in the initial coherent state $\langle \hat{X}_1 \rangle_{|\alpha\rangle} \neq 0$ and $\langle \hat{X}_2 \rangle_{|\alpha\rangle} = 0$ (i.e. $\text{Im } \alpha e^{i\phi} = 0$), X_1 is called the amplitude quadrature and X_2 is called the phase quadrature. Then a squeezed state with $\Delta X_1 < 1/2$ is called amplitude squeezed state, while the one with $\Delta X_1 > 1/2$ is called phase squeezed state.

6. Next we can calculate the effects of the squeezing operator on the photon numbers $\hat{n}$:

- (a) Give the average value $\langle \hat{n} \rangle$ of the photon number for a squeezed state. Express the result with respect to the average amplitude $|\beta|^2$ of the squeezed field. Comment in particular the case of $\beta = 0$.

From Eq. 44 and Eq. 45, the averaged photon number of a squeezed state can be calculated as

$$\langle \hat{n} \rangle = \langle \alpha | \hat{S}^\dagger \hat{a}^\dagger \hat{a} \hat{S} | \alpha \rangle = |\alpha|^2 \cosh(2r) + \sinh^2(r) + \text{Re} \left(\alpha^2 e^{2i\phi} \sinh(2r) \right) = |\beta|^2 + \sinh^2(r) \quad (55)$$

Specifically, if $|\beta| = 0$, we have

$$\langle \hat{n} \rangle = \sinh^2(r) \quad (56)$$

- (b) A similar derivation gives the variance of the photon number $\langle \Delta \hat{n}^2 \rangle$:

$$\langle \Delta \hat{n}^2 \rangle = |\beta|^2 (e^{-2r} \cos^2(\theta - \phi) + e^{2r} \sin^2(\theta - \phi)) + \frac{\sinh^2(2r)}{2} \quad (57)$$

- i. Simplify this expression when squeezing is weak with respect to the average value of the field ($|\beta|^2 \gg e^{2r}$).

When the squeezing is weak, we can assume $r \approx 0$, thus $e^{-2r} \approx e^{2r} \approx 1$. So, we find

$$\langle \Delta \hat{n}^2 \rangle \approx |\beta|^2 + \frac{\sinh^2(2r)}{2} \quad (58)$$

- ii. Deduce the conditions for which the distribution is subpoissonnian, poissonnian or superpoissonnian.

For poissonnian distribution, $\langle \hat{n} \rangle = \langle \Delta \hat{n}^2 \rangle$, which yields

$$\sinh^2(r) = \frac{\sinh^2(2r)}{2} \quad (59)$$

Similarly, $\sinh^2(r) > \frac{\sinh^2(2r)}{2}$ for superpoissonnian and $\sinh^2(r) < \frac{\sinh^2(2r)}{2}$ for subpoissonnian.

- iii. Use your result from part (i) and show that the squeezed vacuum has counter intuitively non-zero photon number fluctuation (in contrast to $|0\rangle$ i.e. vacuum).

For a squeezed vacuum $\alpha = 0$, we naturally have $|\beta| = 0$ and

$$\langle \Delta \hat{n}^2 \rangle = \frac{\sinh^2(2r)}{2} \quad (60)$$

which is non-zero for $r \neq 0$.