Quantum Electrodynamics and Quantum Optics
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.2

2.1 Evolution of a coherent wavepacket

Hint: Before going through this exercise, we recommend you to carefully read Section 2.3 of M.O. Scully book. Consider a wavefunction that is a wavefunction of a ground state of a harmonic oscillator, displaced in the coordinate space by some arbitrary value x_0 at the initial moment of time $t = 0$ (Note: $m = 1$ here):

$$\psi(x, t = 0) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{\omega}{2\hbar}(x - x_0)^2\right], \tag{1}$$

where ω is the frequency of a harmonic oscillator. **The task is to obtain the probability density function $|\psi(x,t)|^2$ for any moment of time**:

$$|\psi(x,t)|^2 = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \exp\left[-\frac{\omega}{\hbar}(x - x_0 \cos\omega t)^2\right]. \tag{2}$$

To do that,

(a) Prove that the wavefunction (1) is a coherent state.

(b) Find the expansion coefficients a_n of the initial wavefunction at $t = 0$ in the orthonormal Fock state basis, with the known time evolution :

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n \phi_n(x) e^{-iE_n t/\hbar}, \tag{3}$$

where $\phi_n(x) = \langle x|n\rangle$ is a coordinate wavefunction of a Fock state, with eigenenergy $E_n = (n + 1/2)\hbar\omega$.

(c) Now coming to the time evolution, express $|\psi(t)\rangle$ as the product of a phase and a displacement operator acting on the vacuum state $|0\rangle$. Compute the action of this displacement operator onto the wavefunction in the x basis to obtain a convenient expression for $\psi(x,t)$. Finally, write the probability density function.

2.2 Properties of coherent and squeezed states

In this exercise we want to calculate the basic properties of coherent[1] and squeezed states of light. Here we focus on an electromagnetic field mode described by the creation $\hat{a}^\dagger$ and annihilation operators $\hat{a}$.

1. We define generalized field quadratures as:

$$\hat{X}_1(\phi) = \frac{1}{2}\left(\hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi}\right) \tag{4}$$

$$\hat{X}_2(\phi) = \frac{1}{2i}\left(\hat{a}e^{i\phi} - \hat{a}^\dagger e^{-i\phi}\right) \tag{5}$$

Calculate $[\hat{X}_1(\phi), \hat{X}_2(\phi)]$. What do they refer to in case of $\phi = 0$?

2. Calculate the time-dependent evolution of $\hat{X}_1(\phi)$ and $\hat{X}_2(\phi)$ for $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

[1]Glauber, Roy J. "Coherent and incoherent states of the radiation field." Physical Review 131.6 (1963): 2766.

3. A coherent state ($|\alpha\rangle$) is defined as: $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$.
   Show that a coherent state is a minimum uncertainty state for every pair of $\hat{X}_1(\phi)$ and $\hat{X}_2(\phi)$.

4. For simplicity, consider a state $|\psi\rangle$ with vanishing mean for all times: $\langle\psi|\,\hat{a}\,|\psi\rangle = \langle\hat{a}\rangle = \langle\hat{a}^\dagger\rangle = \langle\hat{X}_1\rangle = \langle\hat{X}_2\rangle = 0$

   (a) Express the variance of the field quadratures ($\Delta\hat{X}_i^2$) with respect to $\langle\hat{a}^\dagger\hat{a}\rangle$, $\langle\hat{a}\hat{a}\rangle$ and ϕ.

   (b) Show that the minimal and maximal variances are corresponding to two conjugate quadratures. Give their expressions.

   (c) The state is squeezed if there exists at least a pair of conjugated quadratures for which the uncertainty of one of the quadratures is smaller than $\frac{1}{4}$. Deduce a sufficient and necessary condition to have a squeezed state.

5. In order to obtain a squeezed state, one may apply the squeezing operator with squeezing parameter r:

$$\hat{S}(\epsilon) = e^{\epsilon\hat{a}^{\dagger 2} - \epsilon^*\hat{a}^2},\ \epsilon = \frac{1}{2}re^{-2i\phi} \tag{6}$$

   (a) Derive the transformation of $\hat{a}$ and $\hat{a}^\dagger$ under the squeezing operator. (Heisenberg picture formulation)

   (b) Then deduce the transformation for the quadratures $\hat{X}_1(\phi)$ and $\hat{X}_2(\phi)$

   (c) Apply the squeezing operator on a coherent state $|\alpha\rangle$ and derive the following quantities for the transformed state: $\langle\hat{X}_1\rangle$, $\langle\hat{X}_2\rangle$, $\Delta\hat{X}_1$ and $\Delta\hat{X}_2$.
   The squeezing operator effectively produces a squeezed state from a coherent state. A quadrature (average value and variance) is amplified by the squeezing factor e^r whereas the conjugated quadrature is "de-amplified" by the same factor. In practice, this can be used to enhance the sensitivity in interferometric measurements of electromagnetic field fluctuations[2], such as in *Gravitational Wave detectors*[3]. In fact, squeezed states are now being used in Advanced LIGO for gravitational wave detection[4].

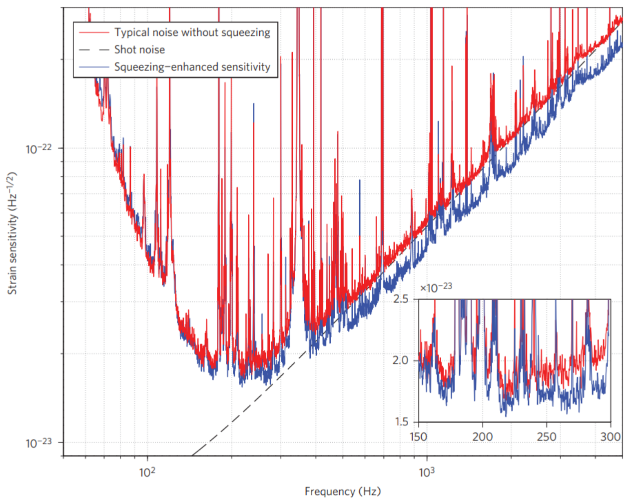

Figure 1: Strain sensitivity of LIGO homodyne detection measured with and without squeezing injection [Aasi, Junaid, et al. *Nature Photonics* (2013)]

   (d) Plot the following states in the quadrature plane ($\hat{X}_1(\phi)$, $\hat{X}_2(\phi)$):

[2] Caves, Carlton M. "Quantum-mechanical noise in an interferometer." Physical Review D 23.8 (1981): 1693.

[3] Abadie, J., et al. "A gravitational wave observatory operating beyond the quantum shot-noise limit." Nature Physics 7.12 (2011): 962.

[4] Aasi, Junaid, et al. "Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light." Nature Photonics 7.8 (2013): 613.

i. A coherent state $|\alpha\rangle$.
ii. A squeezed state $\hat{S}(\epsilon)|\alpha\rangle$ for $r = 2$.

(e) Show that the average amplitude $|\beta|^2 = \langle \hat{X}_1 \rangle^2 + \langle \hat{X}_2 \rangle^2$ for the squeezed state satisfies:

$$|\beta|^2 = |\alpha|^2 \cosh(2r) + \mathrm{Re}\left(\alpha^2 e^{2i\phi} \sinh(2r)\right) \tag{7}$$

(f) From the result of the previous part justify the names "amplitude squeezed state" and "phase squeezed state".

6. Next we can calculate the effects of the squeezing operator on the photon numbers $\hat{n}$:

(a) Give the average value $\langle \hat{n} \rangle$ of the photon number for a squeezed state. Express the result with respect to the average amplitude $|\beta|^2$ of the squeezed field. Comment in particular the case of $\beta = 0$.

(b) A similar derivation gives the variance of the photon number $\langle \Delta \hat{n}^2 \rangle$:

$$\langle \Delta \hat{n}^2 \rangle = |\beta|^2 \left(e^{-2r} \cos^2(\theta - \phi) + e^{2r} \sin^2(\theta - \phi)\right) + \frac{\sinh^2(2r)}{2} \tag{8}$$

i. Simplify this expression when squeezing is weak with respect to the average value of the field($|\beta|^2 \gg e^{2r}$).
ii. Deduce the conditions for which the distribution is *subpoissonnian*, *poissonnian* or *superpoissonnian*.
iii. Use your result from part (i) and show that the squeezed vacuum has counter intuitively non-zero photon number fluctuation (in contrast to $|0\rangle$ i.e. vacuum).

2.3 Two-photon coherent states (*)[5]

An investigation of the possible ways to generate squeezed states leads to a broad class of radiation states that are called "two-photon coherent states" (TCS's). They include the minimum-uncertainty states of which the coherent state is a special case.

In general, a two-photon coherent state differs from a coherent state in several ways: they are generated by different photon processes, they have different quantum statistical properties, and they have different coherence properties. Coherent states are generated from ideal one-photon stimulated processes, whereas TCS's are obtained from ideal stimulated two-photon processes for two photons of the same mode. The quantum noise properties of TCS's are basically the same as those of squeezed states.[6]. These states are also used to define squeezed states in an alternative but equivalent way than you have seen in the previous exercise. In this exercise we look at the definition and properties of these states.

1. For a fixed radiation mode for photon anihilation operator $\hat{a}$ we define:

$$\hat{b} = \mu \hat{a} + \nu \hat{a}^\dagger, \; |\mu|^2 - |\nu|^2 = 1 \tag{9}$$

Show that $[\hat{b}, \hat{b}^\dagger] = 1$.

2. Any transformation that keeps the commutation invariant is called *a canonical transformation*. Therefore, the transformation from $\hat{a}$ to $\hat{b}$ in Eq. (9) is a linear canonical transformation. A theory of Von Neumann states that every canonical transformation can be formulated as a unitary transformation such that:

$$\hat{b} = \hat{U} \hat{a} \hat{U}^\dagger, \; \hat{U}\hat{U}^\dagger = \hat{\mathbb{1}} \tag{10}$$

[5] Graded exercise
[6] Yuen, Horace P. "Two-photon coherent states of the radiation field." Physical Review A 13.6 (1976): 2226.

where $\hat{\mathbb{1}}$ is the unity operator (identity).

We call the eigen-states of $\hat{b}$ two-photon coherent states $|\beta\rangle_g$ with β being the eigen-value ($\hat{b}\,|\beta\rangle_g = \beta\,|\beta\rangle_g$). They are also know as Bogoliubov modes. Express these eigen-states using coherent states $|\beta\rangle$

3. Using the canonical transformation, show that TCS's eigen states can be expressed in the following form:

$$|\beta\rangle_g = \hat{D}_g(\beta)\,|0\rangle_g \tag{11}$$

where $\hat{D}_g(\beta)$ is the displacement operator for TCS's and $|0\rangle_g$ is the vacuum TCS state. Using the displacement operator of a coherent state, express $\hat{D}_g(\beta)$ in explicit form using $\hat{b}$ and $\hat{b}^\dagger$. How is the $|0\rangle_g$ related to the vacuum state $|0\rangle$?

4. Let us now consider the case of $\mu = \cosh r$ and $\nu = e^{i2\phi}\sinh r$.

   (a) Show that there exists a unitary transformation $\hat{U} \equiv \hat{S}(\epsilon)$, where $\epsilon = \frac{r}{2}e^{i2\phi}$, that relates $\hat{b}$ to $\hat{a}$ when $\hat{b}$ is defined as Eq. (9). Here $\hat{S}(\epsilon)$ is a squeezing operator with squeezing parameter ϵ.

   (b) Use the result from part 3 and show that the two-photon coherent state can be expressed as:

$$|\beta\rangle_g = \hat{S}(\epsilon)\hat{D}(\beta)\,|0\rangle \tag{12}$$

   (c) Rewrite the TCS displacement operator $\hat{D}_g$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ and show:

$$|\beta\rangle_g = \hat{D}(\alpha)\hat{S}(\epsilon)\,|0\rangle\,,\ \alpha = \mu\beta - \nu\beta^* \tag{13}$$

   Hence we have found the equivalent squeezed state for the given two-photon coherent state. Thus the two-photon coherent state is generated by first displacing the vacuum state, then applying the squeezing operator. This is the *opposite* procedure to that which defines the squeezed state. The two procedures yield the same state if the displacement parameters α and β are related as in Eq. (13).