Quantum Electrodynamics and Quantum Optics
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Solutions to Exercise No.1

1.1 Solution: Classical electromagnetic field modes density in free space and field quantization

a) According to Maxwell equation, for electromagnetic wave we have

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

Introduce potential vector $\boldsymbol{A}$, thus

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$
$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}$$

Substitute (1) by (2) we obtain

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{\mu_0}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$$

Since we choose Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, so

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot A) - \nabla^2 \boldsymbol{A} = -\nabla^2 \boldsymbol{A}$$

So

$$\boldsymbol{\nabla} \times \boldsymbol{H} = -\frac{1}{\mu_0}\nabla^2 \boldsymbol{A} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = -\epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2}$$

Here we have the wave equation for $\boldsymbol{A}$ by using $\mu_0\epsilon_0 = c^{-2}$

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

We can separate the vector potential into two complex terms

$$\boldsymbol{A} = \boldsymbol{A}^+ + \boldsymbol{A}^-$$

where + , - respectively represents the amplitudes vary by $e^{-i\omega t}$ and $e^{i\omega t}$. For one particular polarization, we can rewrite the equation into

$$\sum_{i=1,2,3} \frac{\partial^2}{\partial x_i} A_j^{(\pm)} - \frac{1}{c^2}\frac{\partial^2 A_j^{(+\pm)}}{\partial t^2} = 0, \quad j = 1,2,3$$

where the subscript $1,2,3$ indicate the components on x, y, z direction. $+$ indicates different polarization.

b) To solve the equation in a cube of side length L, we suppose that the vector potential can be expressed as a superposition of different modes. We use + for example

$$A_j^{(\lambda)} = \sum_{k_j} c_j^{(+)} u_{k_j}(x_j)\, e^{-i\omega t}$$

Thus, (5) can be rewritten as an equation of $u_{k_j}(x_j)$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) u = 0$$

The solution takes the form of

$$u_{k_j}\left(x_j\right) = De^{ik_j x_j}$$

where D is the normalized coefficient. Apply the period boundary condition $u\left(k_j L\right) = u(0)$. Then we obtain

$$k_j = \frac{2\pi n_j}{L}, n_j \in \mathbb{Z}$$

c) In order to calculate the mode density, we have to replace the discrete summation by integral of k.

$$\sum_k \to 2\left(\frac{L}{2\pi}\right)^3 \int \mathrm{d}^3 k$$

where 2 indicates 2 ways of polarization. We calculate the integral in polar coordinates, so that

$$\mathrm{d}^3 k = k^2\, \mathrm{d}k \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

So

$$N = 2\left(\frac{L}{2\pi}\right)^3 \int_{[0,k]^3} \mathrm{d}^3 k = 2\left(\frac{L}{2\pi}\right)^3 \int_0^k k^2\, \mathrm{d}k \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = \frac{L^3 k^3}{3\pi^2}$$

If we let $L \to \infty$ and $L^3 \to V$ be the volume of the space, then the mode density should be

$$\frac{\mathrm{d}N}{V\, \mathrm{d}k} = \frac{k^2}{\pi^2}$$

d) In order to express the mode density in terms of frequency, we rewrite the integral of k into integral of ω by using $k = \omega/c$

$$\mathrm{d}^3 k = \frac{\omega^2}{c^3}\, \mathrm{d}\omega \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

So

$$\begin{aligned} N &= 2\left(\frac{L}{2\pi}\right)^3 \int_{[0,\omega]^3} \mathrm{d}^3\omega \\ &= 2\left(\frac{L}{2\pi}\right)^3 \int_0^\omega \frac{\omega^2}{c^3}\, \mathrm{d}k \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = \frac{V\omega^3}{3\pi^2 c^3} \end{aligned}$$

Since $\omega = 2\pi\nu$, so

$$N = \frac{8\pi V \nu^3}{3c^3}$$

Hence

$$\rho(\nu) = \frac{\mathrm{d}N}{V\, \mathrm{d}\nu} = \frac{8\pi\nu^2}{c^3}$$

e) The Hamiltonian of a 1D harmonics oscillator can be expressed as

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega x^2$$

According to equipartition theorem, every quadratic term of p or x contributes $1/2k_BT$ to the energy, so the total energy should be $\epsilon = k_BT$. Hence, we can give Rayleigh-Jeans law

$$\frac{\mathrm{d}E}{V\,\mathrm{d}\nu} = \epsilon\rho(\nu) = \frac{8\pi\nu^2 k_B T}{c^3}$$

f) By replacing $u(\boldsymbol{r}) = \alpha(\boldsymbol{r},t) = \alpha e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}$, we can rewrite the vector potential can be expressed as a superposition of $\alpha_{\boldsymbol{k}}$ and $\alpha^*_{\boldsymbol{k}}$

$$\boldsymbol{A} = \sum_{\boldsymbol{k},\lambda} C\left[\hat{\epsilon}_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} + \hat{\epsilon}^*_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}}e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

where C is a constant and $\hat{\epsilon}$ is a direction vector and $\lambda = \pm 1$ indicates the polarization. Due to relation (2) we obatin

$$\boldsymbol{E} = \sum_{\boldsymbol{k},\lambda} i\omega C\left[\hat{\epsilon}_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}^*_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}}e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$
$$\boldsymbol{B} = \sum_{\boldsymbol{k},\lambda} iC\left[\left(\boldsymbol{k}\times\hat{\epsilon}_{\boldsymbol{k}}\right)\alpha_{\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \left(\boldsymbol{k}\times\hat{\epsilon}^*_{\boldsymbol{k}}\right)\alpha^*_{\boldsymbol{k}}e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

Here we use $\boldsymbol{\nabla}\times e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{k}\times e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$. So

$$\begin{aligned}
|\boldsymbol{E}|^2 =& \boldsymbol{E}\cdot\boldsymbol{E}^* \\
=& \sum_{\boldsymbol{k},\lambda} i\omega C\left[\hat{\epsilon}_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}^*_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}}e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
& \sum_{\boldsymbol{k},\lambda}\cdot(-i\omega)C^*\left[\hat{\epsilon}_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}}e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}^*_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
=& \sum_{\boldsymbol{k},\lambda}\omega^2|C|^2\left(\alpha_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}} + \alpha^*_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}\right)
\end{aligned}$$

and

$$\begin{aligned}
|\boldsymbol{B}|^2 =& \boldsymbol{B}\cdot\boldsymbol{B}^* \\
=& i\sum_{\boldsymbol{k},\lambda} C\left[\left(\boldsymbol{k}\times\hat{\epsilon}_{\boldsymbol{k}}\right)\alpha_{\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \left(\boldsymbol{k}\times\hat{\epsilon}^*_{\boldsymbol{k}}\right)\alpha^*_{\boldsymbol{k}}e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
& \cdot(-i)\sum_{\boldsymbol{k}} C^*\left[\left(\boldsymbol{k}\times\hat{\epsilon}_{\boldsymbol{k}}\right)\alpha^*_{\boldsymbol{k}}e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}} - \left(\boldsymbol{k}\times\hat{\epsilon}^*_{\boldsymbol{k}}\right)\alpha_{\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \\
=& \sum_{\boldsymbol{k},\lambda}|\boldsymbol{k}|^2|C|^2\left(\alpha_{\boldsymbol{k}}\alpha^*_{\boldsymbol{k}} + \alpha^*_{\boldsymbol{k}}\alpha_{\boldsymbol{k}}\right)
\end{aligned}$$

Then the Hamiltonian writes by using (67) and (23)
$H = \frac{1}{2}\int_V\left(\epsilon_0|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right)\mathrm{d}V = \frac{1}{2}\epsilon_0\int_V\left(|\boldsymbol{E}|^2 + c^2|\boldsymbol{B}|^2\right)\mathrm{d}V = \sum_{\boldsymbol{k},\lambda}|C|^2V\epsilon_0\omega^2\left(\alpha\alpha^* + \alpha^*\alpha\right)$

In order to make the Hamiltonian have the unit of energy, we take $C = \sqrt{\frac{\hbar}{2\omega V\epsilon_0}}$. So, finally we obtain

$$H = \sum_k\sum_{\lambda=\pm1}\hbar\omega\left(\alpha_k\alpha^*_k + \alpha^*_k\alpha_k\right)$$

g) In order to follow the symmetric postulate we keep the form of $\alpha\alpha* + \alpha^*\alpha$. By replacing α and α^* by operators a and $a^\dagger$, the Hamiltonian becomes

$$H = \sum_k\sum_{\lambda=\pm1}\hbar\omega\left(a_k a^\dagger_k + a^\dagger_k a_k\right)$$

For electric field operator and magnetic field operator

$$\boldsymbol{E} = \sum_{k,\lambda} i\omega\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[\hat{\epsilon}_k a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^* a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} = \sum_{k,\lambda}\frac{i}{\mu_0}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[(\boldsymbol{k}\times\hat{\epsilon}_k)\, a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - (\boldsymbol{k}\times\hat{\epsilon}_k^*)\, a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]$$

h) Due to (27) and (28) and the facts that

$$[a_k, a_l] = \left[a_k^\dagger, a_l^\dagger\right] = 0$$
$$\left[a_k, a_l^\dagger\right] = \delta_{k,l}$$

the commutation relationship of different components of $\boldsymbol{E}$ and $\boldsymbol{B}$, i.e $\left[\boldsymbol{E}_{ki}, \boldsymbol{B}_{lj}\right]$ should be

$$\begin{aligned}\left[\boldsymbol{E}_{ki}(\boldsymbol{r}), \boldsymbol{B}_{lj}\left(\boldsymbol{r}'\right)\right] &= -\frac{\hbar}{2V\epsilon_0}\sum_{k,l,\lambda}\left[\epsilon_k^{(\lambda)}\otimes\left(\boldsymbol{l}\times\epsilon_l^{(\lambda)}\right)\left[a_k, a_l^\dagger\right] - \epsilon_k^{(\lambda)}\otimes\left(\boldsymbol{l}\times\epsilon_l^{(\lambda)}\right)\left[a_k^\dagger, a_l\right]\right] \\ &\times\left[e^{i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}')} - e^{-i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}'}\right)\Big] \\ &= -\frac{\hbar}{2V\epsilon_0}k_z\left[e^{i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}')} - e^{-i(\boldsymbol{k}-\boldsymbol{l})\cdot(\boldsymbol{r}-\boldsymbol{r}'}\right)\Big]\end{aligned}$$

Replace the sum of k by integral we will have

$$\left[\boldsymbol{E}_{ki}(\boldsymbol{r}), \boldsymbol{B}_{lj}\left(\boldsymbol{r}'\right)\right] = \delta_{k,l}\delta_{r,r} = -i\frac{\hbar}{\epsilon_0}\delta_{k,k'}\frac{\partial}{\partial k}\delta_{r,r'}\epsilon_{ijk}$$

i) For vacuum state

$$\langle E\rangle = \langle 0|\boldsymbol{E}|0\rangle = \left\langle 0\left| i\omega\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[\hat{\epsilon}_k a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^* a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]\right| 0\right\rangle = 0$$

Since $|\boldsymbol{E}|^2 = \sum_k \frac{\hbar\omega}{2V\epsilon_0}(n_k + 1/2)$, so

$$\left\langle (E-\langle E\rangle)^2\right\rangle = \left\langle 0\left|(E-\langle E\rangle)^2\right| 0\right\rangle = \left\langle n_k\left|E^2\right| n_k\right\rangle = \frac{\hbar\omega}{4V\epsilon_0}$$

For a higher state $|n_k\rangle$,

$$\langle E\rangle = \langle n_k|\boldsymbol{E}|n_k\rangle = \left\langle n_k\left| i\omega\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\left[\hat{\epsilon}_k a_k e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \hat{\epsilon}_k^* a_k^\dagger e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right]\right| n_k\right\rangle = 0$$

and

$$\left\langle (E-\langle E\rangle)^2\right\rangle = \left\langle n_k\left|(E-\langle E\rangle)^2\right| n_k\right\rangle = \left\langle n_k\left|E^2\right| n_k\right\rangle = \frac{\hbar\omega}{2V\epsilon_0}(n_k + 1/2)$$

1.2 Solution: Review on commutation relations and operators

The first two commutators are proven by induction and the last three build upon the previously derived commutation relations.

1. Let us demonstrate that $\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = n\left(\hat{a}^\dagger\right)^{n-1}$ by induction,
   For $n = 0: [\hat{a}, 1] = 0$
   For $n = 1: \left[\hat{a}, \hat{a}^\dagger\right] = 1$
   For $n = 2: \left[\hat{a}, \left(\hat{a}^\dagger\right)^2\right] = \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger = 2\hat{a}^\dagger$

For $n=3:\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{3}\right]=\ldots=3\left(\hat{a}^{\dagger}\right)^{2}$
We can formulate a hypothesis for the n-th step as : $\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n}\right]=n\left(\hat{a}^{\dagger}\right)^{n-1}$,
applying the induction step for the $n+1$ case yields

$$\begin{aligned}\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n+1}\right]&=\left[\hat{a},\hat{a}^{\dagger}\left(\hat{a}^{\dagger}\right)^{n}\right]=\hat{a}^{\dagger}\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n}\right]+\left[\hat{a},\hat{a}^{\dagger}\right]\left(\hat{a}^{\dagger}\right)^{n}\\&=\hat{a}^{\dagger}n\left(\hat{a}^{\dagger}\right)^{n-1}+\left(\hat{a}^{\dagger}\right)^{n}=(n+1)\left(\hat{a}^{\dagger}\right)^{n},\end{aligned}\tag{1}$$

verifying the hypothesis.

2. We will follow the same strategy and use a proof by induction to obtain $\left[\hat{a}^{\dagger},(\hat{a})^{n}\right]=-n\hat{a}^{n-1}$,
For $n=0:\left[\hat{a}^{\dagger},1\right]=0$
For $n=1:\left[\hat{a}^{\dagger},\hat{a}\right]=-1$
For $n=2:\left[\hat{a}^{\dagger},\hat{a}^{2}\right]=\hat{a}[\hat{a}^{\dagger},\hat{a}]+[\hat{a}^{\dagger},\hat{a}]\hat{a}=-2\hat{a}$
For $n=3:\left[\hat{a}^{\dagger},\hat{a}^{3}\right]=\ldots=-3\hat{a}^{2}$
Similarly, the hypothesis for the n-th step is : $\left[\hat{a}^{\dagger},\hat{a}^{n}\right]=-n\hat{a}^{n-1}$,
and applying the induction step for the $n+1$ case yields

$$\begin{aligned}\left[\hat{a}^{\dagger},\hat{a}^{n+1}\right]&=\left[\hat{a}^{\dagger},\hat{a}\hat{a}^{n}\right]=\hat{a}\left[\hat{a}^{\dagger},\hat{a}^{n}\right]+\left[\hat{a}^{\dagger},\hat{a}\right]\hat{a}^{n}\\&=-\hat{a}n\hat{a}^{n-1}-\hat{a}^{n}=-(n+1)\hat{a}^{n}.\end{aligned}\tag{2}$$

3. Now let $f(x)=\sum_{n}f_{n}x^{n}$ be a well defined function for any value of x. We can develop f, use the bilinearity of the commutator and obtain

$$\left[\hat{a},f\left(\hat{a}^{\dagger}\right)\right]=\left[\hat{a},\sum_{n}f_{n}\left(\hat{a}^{\dagger}\right)^{n}\right]=\sum_{n}f_{n}\left[\hat{a},\left(\hat{a}^{\dagger}\right)^{n}\right]=\sum_{n}f_{n}\cdot n\left(\hat{a}^{\dagger}\right)^{n-1}=\frac{\partial f\left(\hat{a}^{\dagger}\right)}{\partial\hat{a}^{\dagger}}\tag{3}$$

4. Similarly

$$\left[\hat{a}^{\dagger},f\left(\hat{a}\right)\right]=\left[\hat{a}^{\dagger},\sum_{n}f_{n}\hat{a}^{n}\right]=\sum_{n}f_{n}\left[\hat{a}^{\dagger},\hat{a}^{n}\right]=\sum_{n}f_{n}\cdot(-n)\hat{a}^{n-1}=-\frac{\partial f\left(\hat{a}\right)}{\partial\hat{a}}\tag{4}$$

5. We define $f(\alpha)=e^{-\alpha\hat{A}}\hat{B}e^{\alpha\hat{A}}$ and we perform a Taylor expansion of $f(\alpha)$ near $\alpha=0$ using the Taylor expansion of the exponentials. Recall the Taylor expansion of a function g as

$$g(\alpha)\cong g(0)+g'(0)\alpha+\frac{1}{2!}g''(0)\alpha^{2}+\frac{1}{3!}g'''(0)\alpha^{3}+\ldots$$

Expanding the exponentials and collecting the terms gives

$$\begin{aligned}f(\alpha)&\cong\left(1-\alpha\hat{A}+\frac{\alpha^{2}}{2!}\hat{A}^{2}-\frac{\alpha^{3}}{3!}\hat{A}^{3}+\ldots\right)\hat{B}\left(1+\alpha\hat{A}+\frac{\alpha^{2}}{2!}\hat{A}^{2}+\frac{\alpha^{3}}{3!}\hat{A}^{3}+\ldots\right)\\&\cong\hat{B}-\alpha\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)+\frac{\alpha^{2}}{2!}\left(\hat{A}^{2}\hat{B}-\hat{A}\hat{B}\hat{A}+\hat{B}\hat{A}^{2}-\hat{A}\hat{B}\hat{A}\right)+\mathcal{O}(\alpha^{3})\\&\cong\hat{B}-\alpha\left[\hat{A},\hat{B}\right]+\frac{\alpha^{2}}{2!}\left([\hat{A},\hat{A}\hat{B}]+[\hat{B}\hat{A},\hat{A}]\right)+\mathcal{O}(\alpha^{3})\\&\cong\hat{B}-\alpha\left[\hat{A},\hat{B}\right]+\frac{\alpha^{2}}{2!}\left[\hat{A},\hat{A}\hat{B}-\hat{B}\hat{A}\right]+\mathcal{O}(\alpha^{3})\\&\cong\hat{B}-\alpha\left[\hat{A},\hat{B}\right]+\frac{\alpha^{2}}{2!}\left[\hat{A},[\hat{A},\hat{B}]\right]+\mathcal{O}(\alpha^{3})\end{aligned}\tag{5}$$

6. Let us define $\hat{A} = -\hat{a}^\dagger \hat{a}$, we have $\left[\hat{A}, \hat{a}\right] = \hat{a}$ and using the result from 5, we obtain

$$\begin{aligned}
\left[\hat{a}, e^{-\alpha \hat{a}^\dagger \hat{a}}\right] &= \left[\hat{a}, e^{\alpha \hat{A}}\right] = \hat{a} e^{\alpha \hat{A}} - e^{\alpha \hat{A}} \hat{a} \\
&= e^{\alpha \hat{A}} \left(e^{-\alpha \hat{A}} \hat{a} e^{\alpha \hat{A}} - \hat{a}\right) \\
&= e^{\alpha \hat{A}} \left(\hat{a} - \alpha \left[\hat{A}, \hat{a}\right] + \frac{\alpha^2}{2!} \left[\hat{A}, \left[\hat{A}, \hat{a}\right]\right] + \ldots - \hat{a}\right) \\
&= e^{\alpha \hat{A}} \left(\hat{a} - \alpha \hat{a} + \frac{\alpha^2}{2!} \hat{a} + \ldots - \hat{a}\right) = e^{\alpha \hat{A}} \left(1 - \alpha + \frac{\alpha^2}{2!} + \ldots - 1\right) \hat{a} \\
&= e^{\alpha \hat{A}} \left(e^{-\alpha} - 1\right) \hat{a} = (e^{-\alpha} - 1) e^{-\alpha \hat{a}^\dagger \hat{a}} \hat{a}
\end{aligned} \tag{6}$$

1.3 Solution: Quantized field properties: linear momentum

Let's consider a plane transverse EM wave of frequency ω with a wave-vector $\boldsymbol{k}$ and linear polarization $\boldsymbol{\epsilon}$ and write down the expressions for the quantized EM field

$$\hat{\mathbf{E}} = i \left(\frac{\hbar \omega}{2\epsilon_0 L^3}\right)^{1/2} \left[\hat{a} \boldsymbol{\epsilon} e^{i(\boldsymbol{kr} - \omega t)} - \hat{a}^\dagger \boldsymbol{\epsilon} e^{-i(\boldsymbol{kr} - \omega t)}\right] \tag{7}$$

and

$$\hat{\boldsymbol{H}} = i \left(\frac{\hbar |\boldsymbol{k}|}{2\epsilon_0 c L^3}\right)^{1/2} \left[\hat{a} \frac{\boldsymbol{k} \times \boldsymbol{\epsilon}}{|\boldsymbol{k}|} e^{i(\boldsymbol{kr} - \omega t)} - \hat{a}^\dagger \frac{\boldsymbol{k} \times \boldsymbol{\epsilon}}{|\boldsymbol{k}|} e^{i(\boldsymbol{kr} - \omega t)}\right] \tag{8}$$

Now we substitute these expessions into the expression for the Pointing vector and note the followin

- terms containing a spacial exponent of the form $e^{in\boldsymbol{kr}}$ where $n \neq 0$ will average out in the integration over $d^3\boldsymbol{r}$
- for a linearly polarized plane wave $[\boldsymbol{\epsilon} \times [\boldsymbol{k} \times \boldsymbol{\epsilon}]] = \boldsymbol{k}$
- $\omega = c|\boldsymbol{k}|$

and thus we are left with

$$\mathbf{P} = \epsilon_0 \int d^3 r \, \mathbf{E} \times \mathbf{B} = -\int d^3 r \frac{\hbar \boldsymbol{k}}{2L^3} \left[-\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}\right] = \hbar \boldsymbol{k} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \tag{9}$$

Note, that to include the rest of modes into consideration, it is sufficient to summ the individual contributions over $\boldsymbol{k}$ as any cross terms average out during the spacial integration. The field thus carries no zero point momentum, as for each pair of $\boldsymbol{k}$ and $-\boldsymbol{k}$ the $\frac{1}{2}$ term cancel out.

1.4 Solution: Quantization of an electrical LC circuit

(a) A conventional electric LC circuit consists of a capacitor and an inductor, and the total energy includes the energy in the capacitor (electric energy) and the energy in the inductor (magnetic energy). Hence, classical Hamiltonian can be written as

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

It is possible to rewrite this Hamiltonian in terms of E and B (electric and magnetic field in the capacitor and inductor) using standard relations and definitions of capacitance and inductance:

$$Q = CV = CdE$$

$$\Phi = IL = ANB$$

$$H = \frac{Cd^2E^2}{2} + \frac{A^2N^2B^2}{2L}$$

(b) For quantization, one can introduce the following set of canonically conjugate variables and their proper commutation relations:

$$\hat{Q} = -iQ_{zpf}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{\Phi} = \Phi_{zpf}(\hat{a} + \hat{a}^\dagger)$$

Note that this is not a unique choice of variables. ZPF (zero point fluctuation) coefficients are introduced here for the normalization, which also can be defined in different ways. For these variables to be canonically conjugate, the following commutation relations should be requested:

$$[\hat{a}, \hat{a}^\dagger] = 1$$

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$

Now using the definitions of variables one can compute the Hamiltonian, canceling the terms aa and $a^\dagger a^\dagger$, and expand the second commutator. The final Hamiltonian is

$$\hat{H} = \hbar\frac{1}{\sqrt{LC}}(a^\dagger a + \frac{1}{2})$$

And ZPF coefficients are

$$Q_{zpf} = \sqrt{\frac{\hbar}{2Z_0}}$$

$$\Phi_{zpf} = \sqrt{\frac{\hbar Z_0}{2}}$$

Here, $Z_0 = \sqrt{\frac{L}{C}}$ is a value that is called characteristic impedance.

The Hamiltonian expressed via electric and magnetic fields undergoes the same procedure since it has the same mathematical form, but just different symbols, so there are just different variables in the final expressions.

(c) By straightforward expansion of the expression, one obtains the zero point fluctuation coefficients derived above, and immediately gets a proof that Heisenberg uncertainty is satisfied.

(d) It is equal to Q_{zpf} coefficient.