Quantum Electrodynamics and Quantum Optics
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.1

1.1 Classical Electromagnetic Field Modes Density and Field Quantization

In this exercise, we want to derive the electromagnetic field modes density in the free space and also go through the procedure for field quantization once again.

1.1.1 Classical Field

We solve Maxwell's equations in a volume V and then count the number of modes. The mode density is defined by the number dN of modes in the frequency interval $[\nu, \nu + d\nu]$ divided by $d\nu$, divided by the volume V.

1. Starting from the Maxwell's equations in vacuum (zero charges and current terms), where electric and magnetic field ($\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$) follow from the vector potential $\mathbf{A}(\mathbf{r}, t)$ (assuming the Columb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$), show that the vector potential satisfies the 3D wave equation:

$$\boldsymbol{\nabla}^2 \mathbf{A}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} = 0. \tag{1}$$

   Given the fact that this equation has two independent solutions for two polarization, rewrite this equation as a scalar equation for the vector potential amplitude for one of the polarizations.

2. Solve the scalar equation in a cube of edge length L (one can assume either a box with periodic boundary conditions or a metallic box). Use separation of variables method. Then, determine the wave vectors of the discrete independent solutions (modes) $k_n = (k_x, k_y, k_z)$.

3. In order to calculate the mode density for free space, we take the limit of infinitely large box ($L \to \infty$). In reciprocal space (space constructed by k_x, k_y, and k_z), count the number of modes dN with wave vector module within the interval $[k, k + dk]$. (This is the definition of mode density, which is modes per volume per frequency interval). Note for each wave vector there are two solutions for different polarizations.

4. Express the wave vector module k in terms of the frequency ν. The 3D mode density in free space is then finally obtained from

$$\rho(\nu) = \frac{dN}{V d\nu}.$$

   What is the characteristic frequency dependence?

5. The **Rayleigh-Jeans** law is the classical electromagnetic energy density of the field at low frequency for a field in a state of thermal equilibrium. Derive this law by using Boltzmann's energy equipartition principle (for harmonic oscillators) and the mode density previously derived.

1.1.2 Quantum Field

At this stage, we are ready to proceed to the field quantization.

1. Among the solutions of Eq. 1 that you previously derived, pick one and assume its time evolution to be proportional to a dimensionless complex function $\alpha(t)$. Use the relations between electric field and magnetic field and vector potential to express the electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$ in terms of the complex function $\alpha(t)$. Then, write the classical electromagnetic Hamiltonian, as written below, in terms of $\alpha(t)$ and $\alpha^*(t)$:

$$\hat{H} = \frac{1}{2}\int_V \left(\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2\right) dV.$$

2. Now, quantize the Hamiltonian by making use of the operator correspondence ($\alpha \to \hat{a}$, $\alpha^* \to \hat{a}^\dagger$), proper energy normalization and the symmetrization postulate. Obtain the operators for $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$.

3. Compute the commutator of the electrical and magnetic field components $[\hat{E}_k, \hat{B}_l]$ for two box modes k and l.

4. Compute the mean and variance of the electrical field ($\langle\hat{\mathbf{E}}\rangle$ and $\langle(\hat{\mathbf{E}} - \langle\hat{\mathbf{E}}\rangle)^2\rangle$) of one of the box modes for the vacuum state $|0\rangle$ (called vacuum fluctuations), and for a higher order Fock state $|n\rangle$.

1.2 Review on commutation relations and operators

Let $\hat{a}$ and $\hat{a}^\dagger$ be operators satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ and $f(x)$ a function such that $f(x) = \sum_n f_n x^n$ is well defined for any value of x.

Calculate the following:

1. $\left[\hat{a}, (\hat{a}^\dagger)^n\right]$

2. $\left[\hat{a}^\dagger, (\hat{a})^n\right]$

3. $\left[\hat{a}, f\left(\hat{a}^\dagger\right)\right]$

4. $\left[\hat{a}^\dagger, f\left(\hat{a}\right)\right]$

5. Show that: $e^{-\alpha\hat{A}}\hat{B}e^{\alpha\hat{A}} = \hat{B} - \alpha[\hat{A},\hat{B}] + \frac{\alpha^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \ldots$, where $\hat{A}$ and $\hat{B}$ are two noncommuting operators and α is a parameter.

6. Show that: $\left[\hat{a}, e^{-\alpha\hat{a}^\dagger\hat{a}}\right] = (e^{-\alpha} - 1)e^{-\alpha\hat{a}^\dagger\hat{a}}\hat{a}$, where α is a parameter.

<u>Hints</u>: For steps 1 and 2 derive sequentially the cases $n = 1$, 2, and 3, using the commutation relation. Guess the general form and prove it by induction. Use the obtained result for steps 3 and 4. For step 5 define a function $f(\alpha)$ for the LHS and do Taylor expansion around $\alpha = 0$. You can then use a power series to expand the exponential terms. For step 6 use the result from step 5.

1.3 Quantized field properties: linear momentum

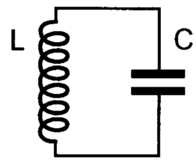
The linear momentum of an electromagnetic field can be expressed using the Poynting vector as the following:

$$\mathbf{P} = \epsilon_0 \int d^3r\ \mathbf{E} \times \mathbf{B} \tag{2}$$

Prove that the momentum of a plane wave is a multiple of $\hbar\mathbf{k}$ - where $\mathbf{k}$ is the wave vector and $\hbar$ is the reduced Planck constant. To do so, start with the classical Poynting vector. Use the operator representation of the electric and magnetic fields and apply the orthogonality conditions to simplify the overall sums.

1.4 Quantization of an electrical LC circuit

We will quantize the excitations of a parallel inductor-capacitor (LC) circuit. A capacitor is a circuit element which stores a charge $Q(t) = CV(t)$ proportional to the voltage across its ports (C is the capacitance). The capacitor is formed by two parallel plates at a distance d. The inductor is a circuit element which develops a voltage when the current $I(t)$ is changing through it: $V_L(t) = L\partial_t I(t)$ where L is the inductance. The inductor has a cross section of A and N number of turns. One can also define the magnetic flux $\Phi(t) = \int_0^t V(t')dt'$. The inductor stores a flux proportional to the current through it: $I(t) = \Phi(t)/L$.



(a) Write down the classical Hamiltonian (total energy) of the system in terms of the charge Q and the magnetic flux Φ. Also rewrite the Hamiltonian in terms of E and B (electric and magnetic field in the capacitor and inductor).

(b) For quantization, introduce two sets of canonically conjugate variables and their proper commutation relations. Express the Hamiltonian in both sets and arrange your expressions similar to the case of a harmonic oscillator.

(c) Find the zero-point vacuum fluctuations for the conjugate variables (X), defined as:

$$\Delta X_{\text{zpf}} = \sqrt{\langle 0|\hat{X}^2|0\rangle - \langle 0|\hat{X}|0\rangle^2} \tag{3}$$

with $|0\rangle$ the ground state of the harmonic oscillator) and show that they satisfy the Heisenberg uncertainty relation.

(d) For the LC oscillator in the quantum ground state (i.e. vacuum state $|0\rangle$) find the charge fluctuations on the capacitor.

1.5 The Casimir effect(*)[1]

The Casimir effect[2] predicted in 1948, is one of the physical phenomena arising due to *vacuum fluctuations*. In the simplest version, is the occurrence of a force between two parallel perfectly conducting plates owing to a change in the zero point energy (ZPE) resulting from the boundary conditions on the plates. In this exercise we go through two treatments of Casimir force. First a simplified treatment, known as 1D Casimir effect is presented and then we go through the general treatment of the problem.

Consider two square shape perfectly conducting plates with edge length of L and at the distance of a from each other (Figure 1).

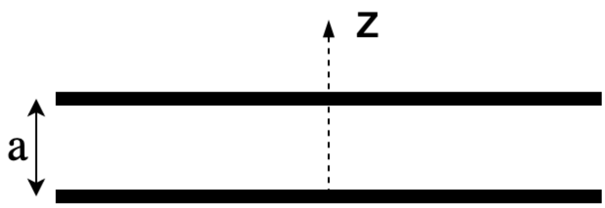


Figure 1: Two perfectly conducting plates

The Hamiltonian for this system can be written as

$$\hat{H} = \sum_k \hbar\omega_k(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}) \tag{4}$$

[1]Graded exercise
[2]Casimir, H. B. G. (1948). "On the attraction between two perfectly conducting plates"

where the sum is taken over all the electromagnetic modes of the system. In the simplest case, we are interested to study the effects arising from the vacuum fluctuations and we assume all the modes to be in their vacuum state. Then the energy of the system (as a function of L and a) is written as

$$E_{vac}(L,a) = \sum_k \frac{1}{2}\hbar\omega_k \tag{5}$$

The potential energy is given by energy difference from infinity

$$U(L,a) = E_{vac}(L,a) - E_{vac}(L,\infty) \tag{6}$$

I. Simplified treatment:

(a) Assuming the dimensions of the plates to be much larger than the distance between them ($L \gg a$), the electromagnetic modes in the space between the plates have the wave vectors $\vec{k} = (\frac{\pi}{L}n_x, \frac{\pi}{L}n_y, \frac{\pi}{a}n_z)$, where n_x, n_y, n_z are positive integers. Write down $E_{vac}(L,a)$ as a summation on n_x, n_y, n_z. Note the two possible polarizations for each wave vector (except for n_x or n_y or $n_z = 0$).

(b) In the limit of $L \to \infty$ the wavelengths in the lateral direction are very long and wave vectors are small. Neglect the lateral terms and simplify the expression for $E_{vac}(L,a)$.

(c) For $E_{vac}(L,\infty)$, transform the sum you obtained in previous part into an integral. Show that the potential energy is written as

$$U(a) = \frac{\hbar\pi c}{a}[\sum_{n=1}^{\infty} n - \frac{a^2}{\pi^2}\int_0^\infty udu] \tag{7}$$

(d) The expression calculated in the previous part contains divergent terms. In order to obtain a finite result from it we should include a *regularization* term in the integral by multiplying the integrand by $e^{-\alpha u}$ and for the summation by multiplying the terms in the series by $e^{-\alpha\frac{\pi}{a}n}$. The physical reason is due to the fact that for any conductor there exists a cut-off frequency which the metal is no longer a good conductor above it. Hence, for very short wavelengths (e.g. UV wavelengths) our plates cease to hold electromagnetic standing waves. Simplify the expression for $U(a)$ and write it as function of a and α.

(e) Take the limit of $\alpha \to 0$ and expand $U(a)$ as a power series of α. What is the first non zero term?

(f) The Casimir force is given by $-\frac{\partial U(a)}{\partial a}$. What is the sign of this force? What is the dependence on a?

II. General treatment:

(a) Given the fact that $L \gg a$ transform the sum on n_x and n_y, to an integral on the $x-y$ plane. Then transform this integral into a polar integral. Using a similar trick, write $E_{vac}(L,\infty)$ as an integral.

(b) Using proper change of variables, show that $U(L,a)$ can be written as:

$$U(L,a) = U_0[\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n) - \int_0^\infty d\zeta f(\zeta)] \tag{8}$$

with

$$f(\zeta) = \int_0^\infty dx\sqrt{x+\zeta^2} \tag{9}$$

compute the prefactor U_0.

The expression given in Eq. (8) is subtraction of two diverging terms. In order to obtain a finite result it is necessary to multiply the integrand in $f(\zeta)$ by a function $g(\sqrt{x+\zeta^2}/\zeta_m)$ which is unity for $\sqrt{x+\zeta^2} \ll \zeta_m$ but tends to zero sufficiently rapid for $\sqrt{x+\zeta^2}/\zeta_m \to \infty$. The physical meaning is obvious: for very short waves (which here means large $\sqrt{x+\zeta^2}$) (X-rays e.g.) our plate is hardly an obstacle at all and therefore the zero point energy of these waves will not be influenced by the position of this plate. Then $f(\zeta)$ can be rewritten as

$$F(\zeta,\zeta_m) = \int_0^{\infty} dx \sqrt{x+\zeta^2}\, g(\sqrt{x+\zeta^2}/\zeta_m) \tag{10}$$

(c) Euler-Maclaurin formula is given as:

$$\sum_0^n f(i) = \int_0^n f(x)dx + \frac{1}{2}(f(0)+f(n)) + \frac{1}{12}(f'(n)-f'(0)) - \frac{1}{720}(f'''(n)-f'''(0)) + ... \tag{11}$$

Use the Euler-Maclaurin formula and also the properties of g function to simplify Eq. (8). Take the limit of $\zeta_m \to \infty$.

(d) The Casimir pressure is given by $-\frac{1}{L^2}\frac{\partial U(L,a)}{\partial a}$. What is the sign of this pressure? What is the dependence on a?