Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.13

13.1 Squeezed light from a mechanical resonator (*)

In this exercise you will derive the generation of squeezed light from the interaction of photons with a mechanical oscillator[1]. The interaction between the light and mechanics can be modeled in this system illustrated in Fig. 1 where the cavity resonance frequency is modulated by the mechanical motion of an end mirror.

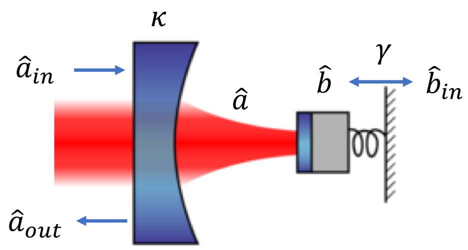

Figure 1: Optomechanical system

The Hamiltonian of such a system of a mechanical oscillator interacting with intracavity photons can be generalized as

$$H = \hbar\omega_c a^\dagger a + \hbar\omega_m b^\dagger b + \hbar g_0 a^\dagger a(b^\dagger + b)$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators for photons and phonons in the system, respectively. Generally the system is driven by a laser at a frequency ω_L close to the cavity frequency ω_c, so that it is more convenient to work in an interaction frame where ω_c is replaced by $\Delta = \omega_c - \omega_L$ in the above Hamiltonian.

1. Assuming that the cavity has a decay rate of κ and the mechanical oscillator is also coupled to a thermal reservoir b_{in} with rate γ, derive the quantum Langevin equations $\frac{d}{dt}\hat{c}(t) = \frac{i}{\hbar}[H,\hat{c}] - \frac{\kappa_c}{2}\hat{c} + \sqrt{\kappa_c}\hat{c}_{\text{in}}$ for operators $\hat{a}(t)$ and $\hat{b}(t)$.

2. At this point, since we are interested in the system fluctuations, we can linearize the derived equations assuming a strong coherent drive field α_0 (for convenience $\alpha_0 = |\alpha_0|$), and displacing the annihilation operator for the photons by making the transformation $\hat{a} \to \alpha_0 + \hat{a}$. This approximation is only valid as long as we are in the vacuum weak coupling regime ($g_0 \ll \kappa$). Introducing $G = g_0\alpha_0$, derive the linearized Langevin equations for operators $\hat{a}(t)$ and $\hat{b}(t)$.

3. The Langevin equations can be easily solved in Fourier domain (note that $(\hat{c}(\omega))^\dagger = \hat{c}^\dagger(-\omega)$), try to derive the following expression for $b(\omega)$,

$$b(\omega) = \frac{\sqrt{\gamma}b_{\text{in}}(\omega)}{i(\omega_m' - \omega) + \gamma'/2} + \frac{iG}{i(\Delta-\omega)+\kappa/2}\frac{-\sqrt{\kappa}a_{\text{in}}(\omega)}{i(\omega_m'-\omega)+\gamma'/2} + \frac{iG}{-i(\Delta-\omega)+\kappa/2}\frac{-\sqrt{\kappa}a_{\text{in}}^\dagger(\omega)}{i(\omega_m'-\omega)+\gamma'/2}$$

   where the renormalized mechanical frequency and mechanical loss rate are $\omega_m' = \omega_m + \delta\omega_m$ and $\gamma' = \gamma + \gamma_{\text{OM}}$. Write down the expressions for $\delta\omega_m$ and γ_{OM}. So you can see that due to the interaction between light and mechanics, the frequency and the damping rate of the mechanical object get modified.

[1] Safavi-Naeini, Amir H., et al. "Squeezed light from a silicon micromechanical resonator." Nature 500.7461 (2013): 185.

4. Now we will try to derive a simplified expression of the squeezing spectrum by making a few reasonable approximations:

   - $\Delta = 0$: The laser is tuned exactly to the optical cavity frequency.
   - $\kappa \gg \omega_m$: The cavity decay rate is much bigger than the mechanical frequency.
   - $\omega \ll \omega_m$: We are only interested in the quasi-static response, so we don't care the feature from the resonance response of the mechanical resonator.

   Under these assumptions, and also introducing $\Gamma_{\text{meas}} \equiv 4|G|^2/\kappa$, derive the simplified expression of the output operator $\hat{a}_{\text{out}}(\omega)$ using the input-ouput theorem $\hat{a}_{\text{out}} + \hat{a}_{\text{in}} = \sqrt{\kappa}\hat{a}$ in terms of $\hat{a}_{\text{in}}^{(\dagger)}$, $\hat{b}_{\text{in}}^{(\dagger)}$ and $\Gamma_{\text{meas}}, \omega_m, \gamma$.

5. Ignore thermal noise by setting $\gamma = 0$, and dropping terms of order $(\Gamma_{\text{meas}}/\omega_m)^2$ (assuming $\Gamma_{\text{meas}} \ll \omega_m$), derive power spectral density $S_{XX}^{\text{out}}(\omega) = \int_{-\infty}^{+\infty} d\omega' \langle \hat{X}_\theta^{\text{out}}(\omega)\hat{X}_\theta^{\text{out}}(\omega')\rangle$ of the output quadrature operator $\hat{X}_\theta^j(t) = \hat{a}_j(t)e^{-i\theta} + \hat{a}_j^\dagger(t)e^{i\theta}$. We assume the input light is quantum limited so that the correlator $\langle \hat{a}_{\text{in}}(\omega)\hat{a}_{\text{in}}^\dagger(\omega')\rangle = \delta(\omega+\omega')$, with all other correlators equal to zero.

   If derived successfully, you will find $S_{XX}^{\text{out}}(\omega) = 1 + \frac{4\Gamma_{\text{meas}}}{\omega_m}\sin 2\theta$. Here, $S_{XX}^{\text{out}}(\omega) = 1$ corresponds to shot noise limit, below which the squeezing is generated at a range of optical quadrature angles ($\sin 2\theta < 0$).

6. Next, we take into account the dynamics of the mechanical resonator while keeping the apprimixation of the bad-cavity limit ($\kappa \gg \omega_m$) and resonant probing ($\Delta = 0$). Here, we will deal with the correlation between the mechanical position and the back-action force. First we define the mechanical position response of the system to a force by its susceptibility

   $$\chi_m(\omega) = \frac{1}{m(\omega_m^2 - \omega^2 - i\gamma\omega_m)}.$$

   We can find the force imparted on the mechanics due to the shot noise of the cavity to be

   $$\hat{F}_{\text{BA}}(t) = \frac{\hbar\sqrt{\Gamma_{\text{meas}}}}{x_{\text{zpf}}}\hat{X}_{\theta=0}^{\text{in}}(t),$$

   with position $\hat{x}/x_{\text{zpf}} = b + b^\dagger$. Derive the position time correlation in the form of

   $$\langle \hat{X}_\theta^{\text{out}}(t)\hat{X}_\theta^{\text{out}}(t')\rangle = \delta(t-t') + 4\Gamma_{\text{meas}}\sin^2\theta\frac{\langle \hat{x}(t)\hat{x}(t')\rangle}{x_{\text{zpf}}^2} + 2\hbar^{-1}\sin\theta\cos\theta\langle \hat{F}_{\text{BA}}(t)\hat{x}(t') + \hat{x}(t)\hat{F}_{\text{BA}}(t')\rangle,$$

   where the terms correspond to shot noise of the light, thermal noise of the mechanics, and the cross-correlation between the back-action noise force and mechanical position fluctuations. Then, further derive the detected spectral density in the form of

   $$S_{XX}^{\text{out}}(\omega) = 1 + \frac{4\Gamma_{\text{meas}}}{x_{\text{zpf}}^2}\left[\overline{S}_{xx}\sin^2\theta + \frac{\hbar}{2}\text{Re}\{\chi_m(\omega)\}\sin 2\theta\right].$$

   From this expression you can see that any reduction of the noise below the vacuum fluctuations can only be caused by the correlations between the back-action noise and the position fluctuations of the system. Also the maximum squeezing will be enhanced around mechanical frequency as $\chi(\omega)$ becomes larger. The other important feature is that the quadrature angles where you will see squeezing have different signs for frequencies below and above the mechanical frequency ω_m, since $\chi_m(\omega)$ changes sign.