Quantum Electrodynamics and Quantum Optics

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

Exercise No.11

11.1 A quantum switch (*)

In this exercise we illustrate how atoms can play role of an optical switch for a cavity. Furthermore, we show how this switch can hold quantum coherence between its *open* and *closed* states. We follow the method proposed in the following references:

- *Quantum Switches and Nonlocal Microwave Fields*, L. Davidovich, A. Maali, M. Brune, J. M. Raimond, and S. Haroche *Phys. Rev. Lett. 71, 2360*
- *Exploring the Quantum: Atoms, Cavities, and Photons*, Haroche, S. and Raimond, J.M, Page 382

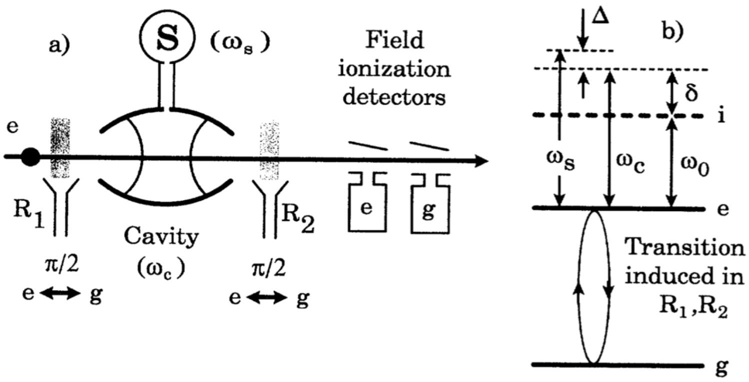

Figure 1: (a) Sketch of a quantum switch experiment. (b) Relevant atomic level scheme (detunings Δ and δ not to scale)

Consider a high-Q optical cavity with superconducting mirrors with resonant frequency of ω_c (Fig 1.a). The cavity is coupled to a monochromatic source at frequency ω_s which is detuned from the cavity resonant by the amount of $\Delta = \omega_s - \omega_c$, much larger than the cavity linewidth ($\Delta \gg \kappa$). In this situation, no field enters the cavity and the cavity is in vacuum state ($|0\rangle_c$).

We start sending single atoms into the cavity. The level scheme of the atom is shown in (Fig 1.b). It has the ground state $|g\rangle_a$, the excited state $|e\rangle_a$, and also an intermediate state $|i\rangle_a$ which mediates the interaction with the cavity. The transition frequency for $e \to i$ transition is ω_0 which is detuned from the cavity by the amount of $\delta = \omega_c - \omega_0$. the cavity is coupled to this transition with the coupling rate g. Assume the parameters being in the regime that the atom-cavity coupling is in the dispersive regime. The cavity is not coupled to any other atomic transition.

1. We inject the cavity with atoms being either in the ground state (g) or in the excited state (e). As the atom *adiabatically* passes through the cavity it causes shift in the cavity resonance frequency (Fig 1.b). What would be the frequency shift of the cavity for each case?

   In a situation where $g^2/\delta = \Delta$ the dispersive shift can bring the cavity on resonant with the source and a coherent field will *fill* the cavity mode and puts it into a coherent state $|\alpha\rangle_c$. Otherwise, the cavity remains in the vacuum state.

2. Consider that the atom is initially prepared in the excited state. Then it passes through the first "*pulse region*", R_1 (Fig 1.a) which it interacts with a classical field, resonant with $e \to g$ transition with the Rabi frequency Ω_0. What is the final state of the atom, $|\psi_1\rangle_a$ if it interact with the field for time $\Delta t = \pi/2\Omega_0$?

3. Next the atom enters the cavity so the initial state of the atom-cavity system is $|\psi_1\rangle_a |0\rangle_c$. What would be the final state of the atom-field system when atom leaves the cavity?
4. After that the atom leaves the cavity, it passes through the second *"pulse region"*, R_2, which is identical to R_1. What would be the final state of the atom-field system after this stage?
5. Finally the atomic state is measured with ionization detectors. What would be the state of the cavity if state e is detected? What if state g is detected?

11.2 Weak continuous measurement with parametrically coupled resonant cavity

In this question we analyze weak continuous measurements in a system consisting of a spin parametrically coupled to a resonant cavity. We will show that this measurement can be made as optimal as allowed by quantum mechanics, as is manifested that the measurement rate of the system can not exceed the decoherence rate due to measurement backaction to the system[1].

The system Hamiltonian is described as

$$H = \frac{1}{2}\hbar\omega_{01}\sigma_z + \hbar\omega_c(1 + A\sigma_z)a^\dagger a + H_{\text{env}}.$$

Here the term H_{env} describes the electromagnetic modes outside the cavity, and their coupling to the cavity. Here you can see that the parametric coupling strength A determines the change in frequency of the cavity as the spin changes direction.

In practice, one terms a measurement QND (quantum non-demolition measurement) if the observable being measured, in this case σ_z, is an eigenstate of the ideal Hamiltonian of the measured system, meaning that once you determined the state of the system, the subsequent QND measurements will not affect the state of the system (but of course the first measurement will). The weak continuous measurement takes the advantage of QND measurement, as for each weak measurement it would not be possible to reliably distinguish the states (in this case the spin induced frequency shift of cavity resonance is small compared to cavity linewidth κ, such that the spin state can't be resolved), but by cascading together a series of such weak measurements we can eventually achieve an unambiguous strong projective measurement (e.g. determine which state the spin is in).

We assume that here the dynamics of the spin is much slower compared to κ the cavity decay rate, so that in this limit the reflected phase shift of the light follows closely the instantaneous value of the spin. We also assume that the coupling A is small enough that the phase shifts are always very small and hence the measurement is weak measurement, meaning that many photons will have to pass through the cavity before much information is gained about the value of the phase shift and hence the value of the spin.

1. We first consider the case of a one-sided cavity where only one of the mirrors is semitransparent, while the other being perfectly reflecting. In this case a light wave incident on the cavity will be perfectly reflected, but with a phase shift θ determined by the cavity and the value of spin σ_z. When the phase shift is small, we can derive its expression as $\theta \approx 4A\omega_c\sigma_z/\kappa$, which in turn allows us to determine the state of the spin σ_z based on the measured θ. However, for a given photon flux $\dot{N}$ incident on the cavity, there will be a phase uncertainty described as $S_{\theta\theta}$ in noise power spectral density. Derive the measurement-imprecision noise of the spin $S_{\sigma_z\sigma_z}$. (when $y = kx$, $S_{yy} = k^2 S_{xx}$.)
2. After integrating up the detected phase signal of the reflected light up to time t, depending on the state of the spin, the mean value of photocurrent I will be $\langle\sigma_z\rangle = \pm 1 \cdot t$, and $\Delta\sigma_z =$

[1] Clerk, Aashish A., et al. "Introduction to quantum noise, measurement, and amplification." Reviews of Modern Physics 82.2 (2010): 1155.

$\sqrt{S_{\sigma_z\sigma_z}t}$. The measurement rate is defined as $\Gamma_{\text{meas}} = \frac{\partial \text{SNR}}{2\partial t}$. Derive the SNR (signal to noise ratio) and the measurement rate. Based on the measurement rate, also write down how much time you need to measure before you can get meaningful information of the state of the spin.

3. Apart from measurement imprecision, there is also backaction from measurement light to the atom. This can be made more clear by rewriting the Hamiltonian as

$$H = \frac{1}{2}\hbar(\omega_{01} + 2A\omega_c a^\dagger a)\sigma_z + \hbar\omega_c a^\dagger a + H_{\text{env}},$$

such that the spin splitting frequency gets shifted depending on how many photon there is in the cavity. While this kind of fluctuation can not cause transitions between the two spin eigenstates, the resulting phase diffusion leads to measurement-induced dephasing of superpositions in the spin state according to

$$\langle e^{-i\phi}\rangle = \left\langle e^{-i\int_0^t d\tau \Delta\omega_{01}(\tau)}\right\rangle \approx e^{-\frac{2}{\hbar^2}S_{FF}t}$$

where $F = -\frac{\partial H}{\partial \sigma_z}$. The dephasing rate is thus defined as $\Gamma_\phi = \frac{2}{\hbar^2}S_{FF}$. Derive the expression of dephasing rate in terms of $S_{\dot{N}\dot{N}}$ (photon flux $\dot{N} = n\kappa/4$, where n is the intra-cavity photon number).

4. Show that $\frac{\Gamma_\phi}{\Gamma_{\text{meas}}} = 4S_{\dot{N}\dot{N}}S_{\theta\theta}$. Since there is relation $S_{\dot{N}\dot{N}}S_{\theta\theta} \geq \frac{1}{4}$, in the ideal case $\Gamma_\phi = \Gamma_{\text{meas}}$. However, since normally we also have excess noise, the dephasing rate is thus bigger than measurement rate. This is a very important takeaway point because now you see that quantum mechanics enforces the constraint that in a QND measurement the best you can possibly do is measure as quickly as you collapse (dephase) the state of the system.