Quantum Electrodynamics and Quantum Optics
École Polytechnique Fédérale de Lausanne (EPFL)

Exercise No.10

10.1 Solution: cQED dispersive regime: a QND measurement of photon/phonon numbers

10.1.1 The first approach

1. Note that

$$H|1,n+1\rangle = -\frac{\hbar\omega_{12}}{2}|1,n+1\rangle + \hbar\omega_r(n+1)|1,n+1\rangle + \hbar g\sqrt{n+1}|2,n\rangle$$
$$H|2,n\rangle = \frac{\hbar\omega_{12}}{2}|2,n\rangle + \hbar\omega_r n|2,n\rangle + \hbar g\sqrt{n+1}|1,n+1\rangle$$

Assume that eigenstates of H takes the form of a superposition of $|1,n+1\rangle$ and $|2,n\rangle$, i.e. $a|1,n+1\rangle + b|2,n\rangle$. Thus the Hamiltonian can be rewritten in basis of $(|1,n+1\rangle, |2,n\rangle)$ as

$$H = \begin{pmatrix} -\hbar\omega_{12}/2 + (n+1)\hbar\omega_r & \hbar g\sqrt{n+1} \\ \hbar g\sqrt{n+1} & \hbar\omega_{12}/2 + n\hbar\omega_r \end{pmatrix}$$

The eigenvalues of H are

$$E_{1,2} = \left(n+\frac{1}{2}\right)\hbar\omega_r \pm \frac{\hbar}{2}\sqrt{4g^2(n+1)+\Delta^2} = \left(n+\frac{1}{2}\right)\hbar\omega_r \pm \frac{\hbar}{2}\Omega_n$$

where $\Delta = \omega_r - \omega_{12}$ is the detuning and $\Omega_n = \sqrt{4g^2 n + \Delta^2}$ is the Rabi frequency. To determine the eigenstate, we have the following equations:

$$\begin{pmatrix} -\hbar\omega_{12}/2 + (n+1)\hbar\omega_r - E_1 & \hbar g\sqrt{n+1} \\ \hbar g\sqrt{n+1} & \hbar\omega_{12}/2 + n\hbar\omega_r - E_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0}$$

Finally we find:

$$\begin{aligned} |-,n+1\rangle &= \cos\theta_{n+1}|1,n+1\rangle + \sin\theta_{n+1}|2,n\rangle \\ |+,n+1\rangle &= -\sin\theta_{n+1}|1,n+1\rangle + \cos\theta_{n+1}|2,n\rangle \end{aligned} \tag{1}$$

where $\theta_n = \arctan(\Omega_n - \Delta)/2g\sqrt{n}$.

2. When the detuning is much larger than the coupling rate, i.e. $\Delta \gg 2g\sqrt{n}$, then $\sin\theta_n \approx 0$ and $\cos\theta_n \approx 1$. Hence

$$|+,n\rangle \simeq |2,n\rangle$$
$$|-,n\rangle \simeq |1,n-1\rangle$$

Given that

$$E_{\pm,n} = \left(n-\frac{1}{2}\right)\hbar\omega_r \mp \frac{\hbar}{2}\Omega$$

where

$$\Omega = \left(\Delta^2 + 4g^2 n\right)^{1/2} \simeq \Delta\left(1+\frac{2g^2 n}{\Delta^2}\right)$$

Thus

$$E_{\pm,n} = \left(n-\frac{1}{2}\right)\hbar\omega_r \mp \frac{\hbar}{2}\Delta\left(1+\frac{2g^2 n}{\Delta^2}\right) = \left(n-\frac{1}{2}\right)\hbar\omega_r \mp \frac{\hbar}{2}\Delta \mp \frac{\hbar g^2 n}{\Delta}$$

or explicitly

$$E_{+,n} = \hbar\omega_r(n-1) + \frac{1}{2}\hbar\omega_{12} - \frac{\hbar g^2 n}{\Delta} = E_{\text{cavity},n-1} + E_{\text{atom},2} - \frac{\hbar g^2 n}{\Delta}$$
$$E_{-,n} = n\hbar\omega_r - \frac{1}{2}\hbar\omega_{12} + \frac{\hbar g^2 n}{\Delta} = E_{\text{cavity},n} + E_{\text{atom},1} + \frac{\hbar g^2 n}{\Delta}$$

3. We find that in the dispersive limit, the energy level of the cavity is effectively shifted by an amount of $\pm g^2/\Delta$ depending on the state of the TLS.

10.1.2 The second approach

1. Take a transformation $UHU^\dagger$ for

$$U = \exp\left\{-\frac{g}{\Delta}\left(a\sigma^+ - a^\dagger\sigma_-\right)\right\} = \exp\left\{-\frac{g}{\Delta}X\right\}$$

According to Baker-Campbell-Hausdorff formula, we have

$$UHU^\dagger \simeq H - \frac{g}{\Delta}[X,H] + \frac{g^2}{2\Delta^2}[X,[X,H]]$$

Note that

$$[X, H_{\text{atom}}] = \left[X, \frac{\hbar\omega_{12}}{2}\sigma_z\right] = \frac{\hbar\omega_{12}}{2}\left[a\sigma^+ - a^\dagger\sigma_-, \sigma_z\right] = -\hbar\omega_{12}\left(a\sigma^+ + a^\dagger\sigma^-\right)$$
$$[X, H_{\text{cavity}}] = \left[X, \hbar\omega_r a^\dagger a\right] = \hbar\omega_r\left[a\sigma^+ - a^\dagger\sigma_-, a^\dagger a\right] = \hbar\omega_r\left(a\sigma^+ + a^\dagger\sigma^-\right) \quad (16)$$
$$[X, H_{\text{int}}] = \left[X, \hbar g\left(a\sigma^+ + a^\dagger\sigma^-\right)\right] = \hbar g\left[\left(a\sigma^+ - a^\dagger\sigma^-\right), \left(a\sigma^+ + a^\dagger\sigma^-\right)\right] \simeq 2\hbar g\left(a^\dagger a + \frac{1}{2}\right)\sigma_z$$

So

$$[X,H] = [X, H_{\text{atom}} + H_{\text{cavity}} + H_{\text{int}}] = \hbar\Delta\left(a\sigma^+ + a^\dagger\sigma^-\right) + 2\hbar g\left(a^\dagger a + \frac{1}{2}\right)\sigma_z$$

and naturally we have

$$[X,[X,H]] \simeq \hbar\Delta\left[\left(a\sigma^+ - a^\dagger\sigma_-\right), \left(a\sigma^+ + a^\dagger\sigma_-\right)\right] = 2\hbar\Delta\left(a^\dagger a + \frac{1}{2}\right)\sigma_z$$

Therefore

$$\begin{aligned}
UHU^\dagger \simeq & H - \frac{g}{\Delta}[X,H] + \frac{g^2}{2\Delta^2}[X,[X,H]] \\
\simeq & \frac{\hbar\omega_{12}}{2}\sigma_z + \hbar\omega_r a^\dagger a + \hbar g\left(a\sigma^+ + a^\dagger\sigma_-\right) - \frac{g}{\Delta}\left[\hbar\Delta\left(a\sigma^+ + a^\dagger\sigma^-\right) + 2\hbar g\left(a^\dagger a + \frac{1}{2}\right)\sigma_z\right] \\
& + \frac{g^2}{2\Delta^2}\cdot 2\hbar\Delta\left(a^\dagger a + \frac{1}{2}\right)\sigma_z + O\left(g^3\right) \\
= & \frac{\hbar\omega_{12}}{2}\sigma_z + \hbar\omega_r a^\dagger a - \frac{\hbar g^2}{\Delta}\left(a^\dagger a + \frac{1}{2}\right)\sigma_z \\
= & \hbar\left(\omega_r - \frac{g^2}{\Delta}\sigma_z\right)a^\dagger a + \frac{\hbar}{2}\left(\omega_{12} - \frac{\hbar g^2}{\Delta}\right)\sigma_z
\end{aligned}$$

2. We find that the cavity experiences a state-dependent freqeuency shift $g^2/\Delta\sigma_z$. Meanwhile, we can also interpret the Hamiltonian by regarding the TLS frequency shifted by a AC-Stark shift $2g^2/\Delta\hat{a}^\dagger\hat{a}$, depending on the cavity photon number, together with Lamb shift g^2/Δ, coming from the interaction with vacuum fluctuation.

10.2 Energy participation quantization of Josephson circuits

The Hamiltonian writes

$$H_{\text{full}} = H_{\text{lin}} + H_{\text{nl}} = \hbar\omega_c a_c^\dagger a_c + \hbar\omega_q a_q^\dagger a_q - E_J \left(\cos\phi_J + \frac{\phi_J^2}{2} \right) \tag{2}$$

where the subscripts c and q represent cavity and qubit respectively. And

$$\varphi_J = \varphi_q \left(a_q + a_q^\dagger \right) + \varphi_c \left(a_c + a_c^\dagger \right) \tag{3}$$

is the junction reduced generalized flux. Define energy participation

$$p_m = \frac{\left\langle n_m \left| \frac{1}{2} E_J \varphi_J^2 \right| n_m \right\rangle}{\left\langle n_m \left| \frac{1}{2} H_{\text{lin}} \right| n_m \right\rangle}, \quad m = c, q \tag{4}$$

Note that

$$\varphi_J^2 = \sum_{m=c,q} \varphi_m^2 \left(2a_m^\dagger a_m + a_m^2 + a_m^{\dagger 2} + 1 \right) + \varphi_c \varphi_q \left(a_q + a_q^\dagger \right) \left(a_c + a_c^\phi \right) \tag{5}$$

Here we apply normal ordering for φ_J. The square terms with ladder operators and the last coupling term vabishes when taking expectation value for $|n_m\rangle$. Therefore we have

$$p_c = \frac{\frac{1}{2} E_J \varphi_c^2 \cdot 2n}{\frac{1}{2} \hbar\omega_c n} = \frac{2E_J \varphi_c^2}{\hbar\omega_c} \tag{6}$$

Similarly

$$p_q = \frac{2E_J \varphi_c^2}{\hbar\omega_q} \tag{7}$$

which yield

$$\varphi_c^2 = \frac{p_c \hbar\omega_c}{2E_J} \tag{8}$$

and

$$\varphi_q^2 = \frac{p_q \hbar\omega_q}{2E_J} \tag{9}$$

Expand H_{nl} to the first order of φ

$$\begin{aligned} H_{\text{nl}} &= -E_J \left(1 - \frac{1}{2}\varphi_J^2 + \frac{1}{4!}\varphi_J^4 + \frac{1}{2}\varphi_J^2 + O\left(\varphi_J^6\right) \right) \\ &= -E_J - \frac{1}{24} E_J \varphi_J^4 + O\left(\varphi_J^6\right) \end{aligned} \tag{10}$$

Due to (43) we have

$$\begin{aligned} \varphi_J^4 &= \left[\varphi_c \left(a_c + a_c^\dagger \right) + \varphi_q \left(a_q + a_q^\dagger \right) \right]^4 \\ &= \sum_{k=0}^{4} \binom{4}{k} \varphi_c^k \varphi_q^{4-k} \left(a_c + a_c^\dagger \right)^k \left(a_q + a_q^\dagger \right)^{4-k} \end{aligned} \tag{11}$$

Under the rotating-wave approximation, only excitation-number conserving interactions in H_{nl} (those with an equal number of raising and lowering operators in each mode) contribute. Thus

$$\begin{aligned} \varphi_J^4 =& 6\varphi_c^4 a_c^{\dagger 2} a_c^2 + 6\varphi_q^4 a_q^{\dagger 2} a_q^2 + 24\varphi_c^2 \varphi_q^2 a_q^\dagger a_q a_c^\dagger a_c + 12\varphi_c^4 a_c^\dagger a_c + 12\varphi_q^4 a_q^\dagger a_q \\ =& 6\varphi_c^4 n_c \left(n_c - 1 \right) + 6\varphi_q^4 n_q \left(n_q - 1 \right) + 24\varphi_c^2 \varphi_q^2 n_q n_c \\ &+ \left(4\varphi_c^4 + 8\varphi_c^2 \varphi_q^2 \right) n_c + \left(4\varphi_q^4 + 8\varphi_c^2 \varphi_q^2 \right) n_q \end{aligned} \tag{12}$$

Define $\Delta_m = E_J\varphi_m^4/4\hbar, \chi_{qc} = E_J\varphi_c^2\varphi_q^2/\hbar, \alpha_m = \left(4\varphi_m^4 + 8\varphi_c^2\varphi_q^2\right)E_J/24\hbar, \quad m = c, q$. Hence the effective Hamiltonian can be written as

$$\begin{aligned} H_{\text{eff}} &= H_{\text{lin}} + H_{\text{nl}} \\ &= \hbar\left(\omega_c - \Delta_c\right)n_c + \hbar\left(\omega_q - \Delta_q\right)n_q - \chi_{qc}n_q n_c - \frac{1}{2}\hbar\alpha_c n_c\left(n_c - 1\right) - \frac{1}{2}\hbar\alpha_q n_q\left(n_q - 1\right) \end{aligned} \tag{13}$$