

## Quantum Electrodynamics and Quantum Optics

### ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE (EPFL)

#### Exercise No.10

### 10.1 cQED dispersive regime: a QND measurement of photon/phonon numbers

In this exercise we introduce a quantum non-demolition measurement (QND) of the photon numbers of a number state  $|n\rangle$  in a high-Q cavity. This method is based on the detection of the dispersive phase shift produced by the cavity field on the wave function of non-resonant atoms crossing the cavity. In this QND scheme, the probe is no longer a field but a beam of atoms (TLS) which interact nonlinearly and non-resonantly with the signal field.

We consider a TLS with  $|1\rangle$  and  $|2\rangle$  as the ground and excited states interacting with a cavity. The transition frequency of the TLS is  $\omega_{12}$  and the cavity resonance frequency is  $\omega_r$  with a detuning  $\Delta = \omega_{12} - \omega_r$ . The Hamiltonian describing the system is:

$$\hat{H} = \hbar \frac{\omega_{12}}{2} \hat{\sigma}_z + \hbar \omega_r \hat{a}^\dagger \hat{a} + \hbar g (\hat{\sigma}^+ \hat{a} + \hat{\sigma}^- \hat{a}^\dagger) \quad (1)$$

where  $\hat{\sigma}_z = (|2\rangle\langle 2| - |1\rangle\langle 1|)$ ,  $\hat{\sigma}^+ = |2\rangle\langle 1|$  and  $\hat{\sigma}^- = |1\rangle\langle 2|$ .

In order to understand the behavior of this system in the dispersive regime, we work on two different approaches. The first one makes the approximation on the eigenstates and eigenvalues of the original Hamiltonian<sup>1</sup>, while the second approach transforms the Hamiltonian using a unitary transformation and looks at the result in this limit<sup>2</sup>.

1. In the first approach, we look at the original Hamiltonian:

(a) Show that the Hamiltonian can be diagonalized to obtain TLS-field dressed states:

$$|+, n\rangle = \cos \theta_n |2, n-1\rangle - \sin \theta_n |1, n\rangle \quad (2)$$

$$|-, n\rangle = \sin \theta_n |2, n-1\rangle + \cos \theta_n |1, n\rangle \quad (3)$$

with the corresponding eigenvalues:

$$E_{+,n} = \hbar \left[ (n-1)\omega_r + \frac{\omega_{12}}{2} \right] - \frac{\hbar}{2} (\Omega_n - \Delta) \quad (4)$$

$$E_{-,n} = \hbar \left[ n\omega_r - \frac{\omega_{12}}{2} \right] + \frac{\hbar}{2} (\Omega_n - \Delta) \quad (5)$$

where:

$$\Omega_n = \sqrt{\Delta^2 + 4g^2n} \quad (6)$$

$$\theta_n = \tan^{-1} \left( \frac{\Omega_n - \Delta}{2g\sqrt{n}} \right) \quad (7)$$

(b) Assume that there is a large detuning between the cavity and the TLS, such that  $\Delta \gg 2g\sqrt{n}$ . Show that in this limit  $\theta_n \approx 0$  and recalculate the eigenvalues and eigenstates in this so-called “dispersive limit”.

(c) Compare your results with the first part and identify the change in the photon number and the energy of  $|-, n\rangle$  state in the dispersive limit.

2. In the second approach, we transform the Hamiltonian using a unitary transformation:

<sup>1</sup>Quantum Optics, Scully, Chapter 19.3

<sup>2</sup>Blais, Alexandre, et al. “Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation.” *Physical Review A* 69.6 (2004): 062320.

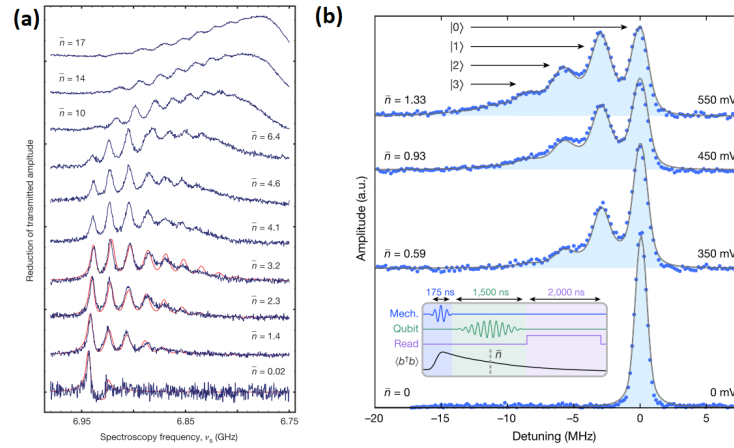


Figure 1: Photon (a) [Schuster et al. 2007] and phonon (b) [Arrangoiz-Arriola et al. 2019] number splitting of a qubit in the dispersive regime.

(a) Apply the unitary transformation  $\hat{U}\hat{H}\hat{U}^\dagger$  for:

$$\hat{U} = \exp \left[ \frac{g}{\Delta} (\hat{a}\hat{\sigma}^+ - \hat{a}^\dagger\hat{\sigma}^-) \right] \quad (8)$$

and show that by expanding to the second order in  $g$  we obtain:

$$\hat{U}\hat{H}\hat{U}^\dagger \approx \hbar \left[ \omega_r + \frac{g^2}{\Delta} \hat{\sigma}_z \right] \hat{a}^\dagger \hat{a} + \frac{\hbar}{2} \left[ \omega_{12} + \frac{g^2}{\Delta} \right] \hat{\sigma}_z \quad (9)$$

*Hint: Use BCH relation*

(b) Interpret the transformed Hamiltonian in Eq. 9. What are the shifts experienced by the TLS and cavity? How much is the TLS pulling the cavity frequency?

This scheme has been realized experimentally both in cavity- and circuit-QED using Rydberg atoms<sup>3</sup> and superconducting qubits<sup>4</sup> for an optical harmonic oscillator (cavity). Recently, this scheme has been used to resolve the number of “phonons” in a mechanical oscillator coupled to a TLS<sup>5</sup>.

## 10.2 Rydberg atoms in microwave cavities: Testbed for the Jaynes-Cummings model(\*)<sup>6</sup>

Over the last decade, several experiments have been carried out in various groups (e.g. Prof. Serge Haroche (LKB Paris)<sup>7,8</sup>, Prof. Jeff Kimble (Caltech)<sup>9,10</sup> and Prof. Gerhard Rempe (MPQ Munich)<sup>11</sup>, in which atomic systems are coupled to electromagnetic modes as modeled by the Jaynes-Cummings Hamiltonian, to perform tasks that pave the way for quantum information processing in the future. In recent years, the Jaynes-Cummings model has also been realized using artificial man-made resonant systems, with electrical resonators coupled to Josephson junctions, most

<sup>3</sup>Brune, M., et al. “Quantum nondemolition measurement of small photon numbers by Rydberg-atom phase-sensitive detection.” *Physical Review Letters* 65.8 (1990): 976.

<sup>4</sup>Schuster, D. I., et al. “Resolving photon number states in a superconducting circuit.” *Nature* 445.7127 (2007): 515.

<sup>5</sup>Arrangoiz-Arriola, Patricio, et al. “Resolving the energy levels of a nanomechanical oscillator.” *Nature* 571.7766 (2019): 537-540.

<sup>6</sup>Graded exercise

<sup>7</sup>S. Haroche and J-M. Raimond, *Exploring the Quantum*, Oxford (2006)

<sup>8</sup>C. Guerlin et al., *Nature* 448, 889 (2007)

<sup>9</sup>J. McKeever et al., *Science* 303, 1992 (2004)

<sup>10</sup>J. McKeever et al., *Nature* 425, 268 (2004)

<sup>11</sup>P.W.H Pinkse et al., *Nature* 404, 365 (2000)

prominently in the groups of Prof. Rob Schoelkopf (Yale)<sup>12</sup>, Prof. John Martinis (UCSB/Google)<sup>13</sup> and Prof. Andreas Wallraff (ETHZ)<sup>14</sup>.

In this exercise, we consider the experimental features of atomic systems coupled to electromagnetic modes, as modeled by the Jaynes-Cummings Hamiltonian. These systems pave the way for quantum information processing. Serge Haroche group is one of the leaders in this field, and here we focus on their experimental realization<sup>15</sup>. They send rubidium atoms, prepared in Rydberg states, across a microwave 3D cavity. We will first describe both systems individually and then how they interact with each other.

### 10.2.1 Rydberg atoms

Rydberg atoms are atoms whose valence electron has been excited to an orbit with a very high principal quantum number. From the Bohr model of an atom, the radius of the orbit is given by  $r = r_0 n^2$ , where  $r_0$  is the Bohr radius ( $\approx 53$  pm). For a typical Rydberg atom  $n \approx 50$ , so large that the outermost electron has a hydrogen-like spectrum. The ‘nucleus’ consists of the core nucleus plus all the other electrons in lower orbits.

1. What is the typical size of a Rydberg atom, with  $n = 50$  and  $n = 100$ ? Why do you think a Rydberg atom is used to realize Jaynes-Cummings model?
2. Compute the typical transition frequency between the adjacent levels of a Rydberg atom (for both  $n = 50$  and  $n = 100$ ). Compute the functional dependence of this on  $n$ .

*Hint:* The energies of the hydrogen atom are  $E_n = -R_H/n^2$ , where  $R_H$  is the Rydberg constant for hydrogen.  $R_H$  can be expressed as  $R_H = Ry/(1 + m_e/m_p)$  where  $m_e$  and  $m_p$  are the masses of the electron and the proton, and  $Ry = hcR_\infty$  with  $R_\infty$  in terms of universal constants:

$$R_\infty = \frac{m_e e^4}{8\epsilon_0^2 h^3 c} \simeq 1.1 \times 10^7 \text{ m}^{-1} \quad (10)$$

### 10.2.2 Microwave cavity

The microwave resonant cavity at  $\approx 50$  GHz is an open cavity with spherical mirrors as shown in Fig. 2, which is operated on its  $q = 9$  longitudinal fundamental Gaussian mode  $TEM_{00q}$  with a waist size  $w_0 \simeq \lambda$ .

1. Estimate the mode volume of the cavity field. You can approximately estimate the mode volume by calculating the half volume of a cylinder with height same as the distance between the mirrors and diameter of  $\lambda/\sqrt{2}$ <sup>16</sup>
2. Compute the electric field per photon, defined to be  $E_{\omega_\tau} = \sqrt{2\hbar\omega_\tau/(\epsilon_0 V)}$ .
3. Compute the vacuum Rabi frequency and compare it with the typical cavity damping time of  $\sim 0.1$  s and the excited state lifetime of  $\approx 30$  ms. Comment on these results.

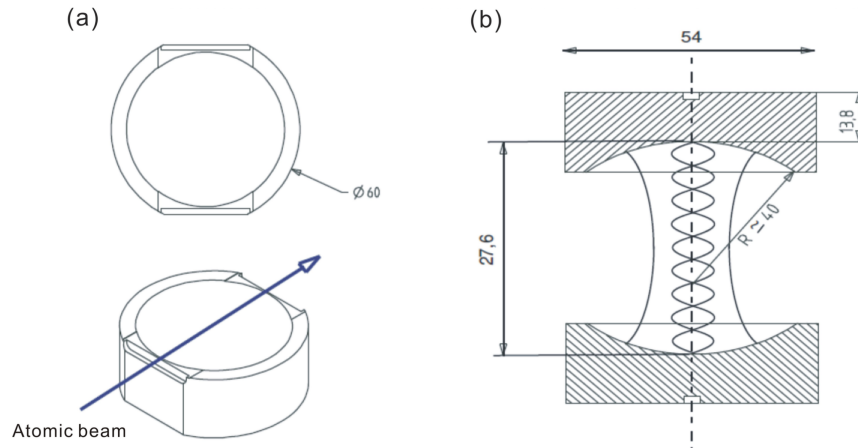


Figure I.11. Geometrical parameters of the cavity and mirrors. (a) Top and side view of a mirror. (b) Cavity scheme and the  $TEM_{900}$  mode of the microwave field. The values in the figure are in unit of millimeter.

Figure 2: Microwave cavity used in the Haroche group (LKB, Paris)

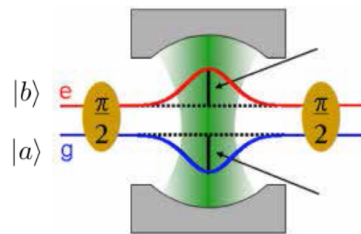


Figure 3: Rydberg atom shot laterally in microwave cavity. The letters 'g' and 'e' stand respectively for the ground state  $|a\rangle$  and the excited state  $|b\rangle$  in our notation.

### 10.2.3 Dispersive atom-field interaction

The experiment starts as follows: a single atom is shot laterally into the microwave cavity (Fig.3), which it crosses slowly at an almost uniform velocity. The incoming atom is in some arbitrary superposition of  $|a\rangle$  and  $|b\rangle$ , while the field is prepared in the Fock state  $|n\rangle$ .

1. What subspaces of the Hilbert space are necessary to describe the subsequent interaction?
2. We assume that the coupling is established adiabatically (slowly enough that the system can follow its eigenstates). What is the evolution of the eigenstates? (in the two cases where the atom enters in  $|a\rangle$  or  $|b\rangle$  state). Is there any difference between the cases of negative and positive detunings ( $\Delta = \omega - \omega_\tau$ )?

*Hint:* Remember the form of the dressed states (similar to what you saw in the exercise 'cQED dispersive regime: a QND measurement of photon numbers').

3. Show that the presence of the cavity light field induces a modified phase shift between an atomic state in a superposition of  $|a\rangle$  and  $|b\rangle$  (called here "light shift"), which, in the dispersive limit ( $\Delta \gg 4\Omega_R^2 n$ ) is given by

$$\Delta\phi = \Delta E \frac{\delta t}{\hbar} \approx \frac{2g^2}{\Delta} \left( n + \frac{1}{2} \right) \frac{\delta t}{\hbar} \quad (11)$$

where  $\delta t$  is the effective time that the atom interacts with the EM field as it crosses the cavity. What could be inferred by measuring this light shift? What is the state of the cavity field after the passage of the atom in the dispersive limit?

NB: It is possible to measure this phase shift of the atomic state in practice by performing Ramsey interferometry.

### 10.3 Energy-participation quantization of Josephson circuits

In this exercise we do again the quantization of a transmon qubit coupled to a cavity. We will use the energy-participation quantization method proposed in this paper.<sup>17</sup> The quantum zero point fluctuations and Kerr-couplings appeared in the Hamiltonian are expressed in terms of the "Energy participation"  $p_m$ , defined as

$$p_m \equiv \frac{\text{Energy stored in the junction}}{\text{Inductive energy stored in mode } m}, \text{ or } p_m = \frac{\langle \psi_m | \frac{1}{2} E_J \hat{\phi}_J^2 | \psi_m \rangle}{\langle \psi_m | \frac{1}{2} \hat{H}_{\text{lin}} | \psi_m \rangle},$$

where  $|\psi_m\rangle$  denotes a coherent state or a Fock excitation of mode  $m$ .

We start from the Hamiltonian and conceptually separate it into the linear and the non-linear part:

$$\begin{aligned} \hat{H}_{\text{full}} &= \hat{H}_{\text{lin}} + \hat{H}_{\text{nl}}, \\ \hat{H}_{\text{lin}} &= \hbar\omega_c \hat{a}_c^\dagger \hat{a}_c + \hbar\omega_q \hat{a}_q^\dagger \hat{a}_q, \\ \hat{H}_{\text{nl}} &= -E_J \left[ \cos(\hat{\phi}_J) + \hat{\phi}_J^2/2 \right], \\ \hat{\phi}_J &= \varphi_q (\hat{a}_q + \hat{a}_q^\dagger) + \varphi_c (\hat{a}_c + \hat{a}_c^\dagger), \end{aligned}$$

<sup>12</sup>D.I. Schuster et al., Nature 445, 515 (2007)

<sup>13</sup>M. Hofheinz et al., Nature 454, 310 (2008)

<sup>14</sup>J.M. Fink et al., Nature 454, 315 (2008)

<sup>15</sup>Gleyzes, Sebastien, et al. "Quantum jumps of light recording the birth and death of a photon in a cavity" Nature 446.7133 (2007)

<sup>16</sup>For an electromagnetic mode with electric field profile  $\vec{E}(\vec{r})$  the mode volume is defined as  $V = \int |\vec{E}(\vec{r})|^2 d^3\vec{r} / |\vec{E}(\vec{r})|_{\text{max}}^2$ . Here due to the symmetry of the mode and approximately flat curvature of the mirrors compared to the beam waist we can approximate the volume by the half of the physical volume.

<sup>17</sup>Mineev Z K, Leghtas Z, Mundhada S O, et al. Energy-participation quantization of Josephson circuits[J]. arXiv preprint arXiv:2010.00620, 2020.

where  $\omega_c$  and  $\omega_q$  are the angular frequencies of the cavity and qubit eigenmodes,  $\hat{a}_c$  and  $\hat{a}_q$  are their annihilation operators, respectively. To complete the quantization, we need to determine the unknown quantities in the Hamiltonian, i.e.  $\omega_c, \omega_q, \varphi_c, \varphi_q$ , where  $\omega_c, \omega_q$  are obtained by solving the eigenmodes and the eigen-frequencies.

1. Show that  $\varphi_c^2 = p_c \frac{\hbar\omega_c}{2E_J}$  and  $\varphi_q^2 = p_q \frac{\hbar\omega_q}{2E_J}$  for Fock excitation  $|\psi_m\rangle = |n\rangle_m$ . Now the Hamiltonian is fully determined.
2. Now we assume the nonlinearity is weak and the system is in dispersive regime. Expand the nonlinear Hamiltonian to the fourth order of  $\hat{\varphi}_J$  and use rotation wave approximation to obtain the effective excitation-number-conserving Hamiltonian:

$$\hat{H}_{\text{eff}} = (\omega_q - \Delta_q) \hat{n}_q + (\omega_c - \Delta_c) \hat{n}_c - \chi_{qc} \hat{n}_q \hat{n}_c - \frac{1}{2} \alpha_q \hat{n}_q (\hat{n}_q - \hat{1}) - \frac{1}{2} \alpha_c \hat{n}_c (\hat{n}_c - \hat{1}).$$

Finally, calculate the self-Kerr couplings  $\alpha_q, \alpha_c$  and the cross-Kerr coupling  $\chi_{qc}$  in terms of energy participation  $p_q$  and  $p_c$ .