

Super-Resolving Phase Measurement in Quantum Metrology

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Quantum Metrology (1/3): Introduction

Interference in Physics

- ▶ Basis for demanding measurements in various fields:
 - ▶ Ramsey interferometry in atomic spectroscopy (atomic clocks, week 1 QOQI)
 - ▶ X-ray diffraction in crystallography
 - ▶ Optical interferometry in gravitational-wave studies

Entanglement for Super-sensitive Measurements

- ▶ Goal: Create maximally-entangled state

$$|N :: 0\rangle_{a,b} \equiv \frac{1}{\sqrt{2}}(|N, 0\rangle_{a,b} + |0, N\rangle_{a,b})$$

Quantum Metrology (2/3): Phase Estimation using $\hat{J}_z$ Operator

$\hat{J}_z$ Operator in Quantum Metrology

- ▶ Definition: $\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$
- ▶ Measures population difference between modes a and b

Phase Estimation Process

1. State: $|\psi\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + e^{iN\phi}|0, N\rangle)$
2. Expectation value: $\langle \hat{J}_z \rangle = \frac{N}{2} \cos(N\phi)$
3. Variance: $(\Delta \hat{J}_z)^2 = \frac{N^2}{4} - \langle \hat{J}_z \rangle^2$
4. Phase uncertainty: $\Delta\phi = \frac{\Delta \hat{J}_z}{|\partial \langle \hat{J}_z \rangle / \partial \phi|}$

Result

$$\Delta\phi = \frac{1}{N} \text{ (Heisenberg limit)}$$

Quantum Metrology (3/3): N-Photon Interference

N-Photon State and Operator

- ▶ N-photon entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + e^{iN\phi}|0, N\rangle)$$

- ▶ Operator: $A_N \equiv |0, N\rangle\langle N, 0| + |N, 0\rangle\langle 0, N|$

Probability and Phase Sensitivity

- ▶ Probability: $P \propto 1 + \langle A_N \rangle = 1 + \cos(N\Delta\phi)$
- ▶ Phase sensitivity: $\Delta\phi \propto \frac{1}{N}$ (Heisenberg limit)

Implications

- ▶ N-fold increase in fringe frequency

Classical Limit in Phase Estimation (1/2)

Setup and Assumptions

- ▶ Consider N independent particles in a Mach-Zehnder interferometer
- ▶ Each particle has a 50% chance of taking either path
- ▶ Phase difference ϕ between paths

Measurement Process

- ▶ Measure population difference between paths: $M = N_a - N_b$
- ▶ N_a, N_b : number of particles in paths a and b
- ▶ Expected value: $\langle M \rangle = N \cos \phi$

Statistical Analysis

- ▶ Binomial distribution with $p = \frac{1}{2}(1 + \cos \phi)$
- ▶ Variance: $\Delta M^2 = Np(1 - p) = \frac{N}{4}(1 - \cos^2 \phi) = \frac{N}{4} \sin^2 \phi$

Classical Limit in Phase Estimation (2/2)

Error Propagation

- ▶ Use error propagation formula: $\Delta\phi = \frac{\Delta M}{|\frac{d\langle M \rangle}{d\phi}|}$
- ▶ $\frac{d\langle M \rangle}{d\phi} = -N \sin \phi$

Derivation of Classical Limit

$$\Delta\phi = \frac{\Delta M}{|N \sin \phi|} = \frac{\sqrt{\frac{N}{4} \sin^2 \phi}}{N \sin \phi} = \frac{\frac{\sqrt{N}}{2} |\sin \phi|}{N \sin \phi} = \frac{1}{2\sqrt{N}} \quad (1)$$

Result and Interpretation

- ▶ Classical limit: $\Delta\phi \propto \frac{1}{\sqrt{N}}$
- ▶ This is the best precision achievable with classical resources
- ▶ Known as the standard quantum limit or shot noise limit

DC Photon Creation (1/3): Quantitative Description

Key Processes in Spontaneous Parametric Down-Conversion (SPDC)

- ▶ High-energy UV pump through nonlinear crystal
- ▶ Three crucial processes:
 1. Birefringence of the crystal
 2. Energy conservation
 3. Phase matching condition
- ▶ Result: Entangled states and Bell states on demand

Source: Kwiat et al., "New High-Intensity Source of Polarization-Entangled Photon Pairs" (1995)

DC Photon Creation (2/3): Detailed Processes

Birefringence

- ▶ Refractive index depends on polarization and propagation direction
- ▶ Different polarizations travel at different speeds
- ▶ Results in phase difference α between orthogonally polarized light

Energy Conservation and Phase Matching

- ▶ Energy conservation: $\hbar\omega_p = \hbar\omega_1 + \hbar\omega_2$
- ▶ UV (351 nm) $\rightarrow$ Two IR (702 nm) photons
- ▶ Momentum conservation: $\vec{k}_p = \vec{k}_1 + \vec{k}_2$
- ▶ Forms two cones of emitted photons

Phase matching condition:

- ▶ $n_p(\theta_p)\omega_p = n_1(\theta_1)\omega_1 + n_2(\theta_2)\omega_2$

Source: Kwiat et al., "New High-Intensity Source of Polarization-Entangled Photon Pairs" (1995)

DC Photon Creation (3/3): State Evolution

Initial Pump State

$$|\psi_p\rangle = |H\rangle_p$$

Probabilistic Creation of Two Photons

$$|\psi\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|\psi_2\rangle$$

where p is the probability of pair creation

Entangled State at Cone Intersections

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|H_1, V_2\rangle + e^{i\alpha}|V_1, H_2\rangle)$$

where α is the relative phase determined by crystal properties

Source: Kwiat et al., "New High-Intensity Source of Polarization-Entangled Photon Pairs" (1995)

Leveraging N-Photon Entanglement (1/4): Quantitative Description

Goal

Use polarization-entangled (DC) photons to achieve N-photon entanglement

Motivation

- ▶ Exploit $1/N$ scaling in phase sensitivity
- ▶ Signature: Effective de Broglie wavelength

$$\lambda_N = \frac{\lambda_1}{N}$$

Strategy

Entangle as many photons as possible to maximize phase sensitivity

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

Leveraging N-Photon Entanglement (2/4): Double Pass Configuration

Experimental Setup

- ▶ UV pump passes through crystal twice using a mirror
- ▶ Allows for different pair separation probabilities

Resulting State

$$\begin{aligned} |\psi\rangle = & (1 - p)|0\rangle_{a1,a2}|0\rangle_{b1,b2} + \\ & \sqrt{p}(|\Phi^+\rangle_{a1,a2}|0\rangle_{b1,b2} + e^{i\Delta\varphi}|0\rangle_{a1,a2}|\Phi^+\rangle_{b1,b2}) + \\ & p|\Phi^+\rangle_{a1,a2}|\Phi^+\rangle_{b1,b2} \end{aligned}$$

where p is the probability of pair creation

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

Leveraging N-Photon Entanglement (3/4): Interference Cases

Simple Separation (Two-Photon Case)

- ▶ Pair emitted in mode a or b
- ▶ Probability: $P \propto 1 - \cos(2\Delta\phi)$

Four-Photon Cases

1. Double pair emitted once:
 - ▶ Four photons in a1-a2 or b1-b2
 - ▶ Pure four-photon interference
2. Single pair emitted on each pass:
 - ▶ One pair in a1-a2, one in b1-b2
 - ▶ Potential ambiguity in interpretation

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

Leveraging N-Photon Entanglement (4/4): Four-Photon States and Discussion

Explicit Formulas

1. Double pair emission:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|4\rangle_{a1,a2}|0\rangle_{b1,b2} + e^{i4\Delta\varphi}|0\rangle_{a1,a2}|4\rangle_{b1,b2})$$

2. Single pair on each pass:

$$|\psi\rangle = e^{i2\Delta\varphi}(|H\rangle_{a3}|H\rangle_{a4}|H\rangle_{b3}|H\rangle_{b4} + |V\rangle_{a3}|V\rangle_{a4}|V\rangle_{b3}|V\rangle_{b4})$$

Discussion

- ▶ Ambiguity in case (2) due to two-photon interference contributions
- ▶ Extension to $N > 4$ possible with suppression of lower-order interference

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

Actual Paper (1/3): Experimental Setup

- ▶ Uses two down-converted (DC) photons and one local oscillator (LO) photon
- ▶ LO photon: derived from attenuated Ti:sapphire laser beam
- ▶ Key components:
 - ▶ Polarizing beamsplitter (PBS)
 - ▶ Half-wave plate (HWP)
 - ▶ Partial polarizer (PP)
 - ▶ Quarter-wave plate (QWP)
 - ▶ Quartz wedges for phase shifting

Source: Mitchell, M. W., "Super-resolving phase measurements with a multi-photon entangled state" (2005)

Actual Paper (2/3): State Evolution

1. Initial state: $|1\rangle_H |1\rangle_V |1\rangle_V$
2. After PBS: $a_H^\dagger a_V^\dagger |0\rangle \otimes |1\rangle_V$
3. After HWP: $a_{+45}^\dagger a_{-45}^\dagger |0\rangle \otimes |1\rangle_V$
4. After PP: $a_{+60}^\dagger a_{-60}^\dagger |0\rangle \otimes |1\rangle_V$
5. After combining with LO: $a_0^\dagger a_{+60}^\dagger a_{-60}^\dagger |0\rangle$
6. After QWP: $\frac{1}{\sqrt{2}}(|3, 0\rangle_{H,V} - |0, 3\rangle_{H,V})$
7. Final state: $|3 :: 0\rangle_{H,V}^{3\phi} \equiv \frac{1}{\sqrt{2}}(|3, 0\rangle_{H,V} + e^{i3\phi}|0, 3\rangle_{H,V})$

Source: Mitchell, M. W., "Super-resolving phase measurements with a multi-photon entangled state" (2005)

Actual Paper (3/3): Experimental Results and Analysis

Three-Photon Interference

- ▶ Observed probability: $P \propto 1 + \cos(3\phi)$
- ▶ Demonstrates tripling of oscillation frequency

State Analysis

- ▶ $|3, 0\rangle$ and $|0, 3\rangle$ states: Main contributors to 3ϕ dependence
- ▶ $|2, 1\rangle$ and $|1, 2\rangle$ states: Present due to experimental imperfections

Comparison with Classical Limit

- ▶ Achieved phase sensitivity surpasses classical $1/\sqrt{N}$ limit
- ▶ Demonstrates potential for quantum-enhanced metrology

Source: Mitchell, M. W., "Super-resolving phase measurements with a multi-photon entangled state" (2005)

Conclusion (1/3)

How is the NOON state created in experiment?

- ▶ Ti:sapphire laser is:
 - ▶ frequency-doubled to create the UV pump beam for the SPDC
 - ▶ split off and heavily attenuated to create the LO photon
- ▶ Post-selection techniques

How is a single photon state at the input created?

- ▶ Attenuation to single photon level (half wave plate, polarizing beam splitter)
- ▶ "mode matched" with SPDC photons - λ indistinguishable from SPDC

Conclusion (2/3)

What is post-selection and why is it required?

- ▶ Keeping only those measurement outcomes that satisfy certain criteria, discarding the rest
- ▶ Required to:
 - ▶ Filtering out unwanted events: It removes cases where fewer than three photons were detected, or where additional noise photons were present. (proof: detection at "Dark" port, lower-order interference)
 - ▶ State preparation: It effectively prepares the desired three-photon entangled state. (eliminates high order terms for example)

What is meant by super-resolution? What does it compare to in a classical measurement?

- ▶ Definition: Ability to resolve phase shifts smaller than classical limit
- ▶ Comparison:
 - ▶ Classical limit: $\Delta\phi \propto 1/\sqrt{N}$ (Shot Noise)
 - ▶ Quantum limit: $\Delta\phi \propto 1/N$ (Heisenberg limit)

Conclusion (3/3)

What are the experimental signatures of super-resolution and what signal bears these signatures?

- ▶ N-fold increase in fringe frequency
- ▶ Probability: $P \propto 1 + \cos(N\phi)$

Is it possible to scale to large N state?

- ▶ Different set up:
 - ▶ Cascade down-conversion process (double-, triple-pass, ...)
 - ▶ Optical loop configuration
- ▶ Higher-order terms from DC photons:
 - ▶ Probability of creating more than one pair of photons per pump pulse.
 - ▶ BUT probability of these higher-order events decreases rapidly with N