Super-Resolving Phase Measurement in Quantum Metrology

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- Practical Implementation:
  - DC Photon Creation
  - 4-photon entanglement
  - 3-photon entanglement (Paper case)
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Quantum Metrology (1/3): Introduction

Interference in Physics

- Basis for demanding measurements in various fields:
  - Ramsey interferometry in atomic spectroscopy (atomic clocks, week 1 QOQI)
  - X-ray diffraction in crystallography
  - Optical interferometry in gravitational-wave studies

Quantum Phenomena: Entanglement

- NOON states

$$|N :: 0\rangle_{a,b} \equiv \frac{1}{\sqrt{2}}(|N, 0\rangle_{a,b} + |0, N\rangle_{a,b})$$

Quantum Metrology (2/3): N-Photon Interference

Phase measurements with multiphoton entangled states

- NOON state through an interferometer:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|N,0\rangle + e^{iN\phi}|0,N\rangle)$$

Probability and Phase Sensitivity

- Probability: $P \propto 1 + \cos(N\Delta\phi)$
- Phase sensitivity: $\Delta\phi \propto \frac{1}{N}$ (Heisenberg limit)

Implications

- N-fold increase in fringe frequency

$$|2 :: 0\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle)$$

- After phase shift:

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + e^{2i\phi}|0,2\rangle)$$

- Beamsplitter transformation:

$$|2,0\rangle \to \frac{1}{2}(|2,0\rangle + i\sqrt{2}|1,1\rangle - |0,2\rangle)$$
$$|0,2\rangle \to \frac{1}{2}(-|2,0\rangle + i\sqrt{2}|1,1\rangle + |0,2\rangle)$$

- Final state:

$$|\psi_{\text{final}}\rangle = \frac{1}{2\sqrt{2}}[(1 - e^{2i\phi})|2,0\rangle + i\sqrt{2}(1 + e^{2i\phi})|1,1\rangle + (-1 + e^{2i\phi})|0,2\rangle]$$

- Probability of detecting both photons in port 1:

$$\begin{aligned} P(2,0) &= |\frac{1}{2\sqrt{2}}(1 - e^{2i\phi})|^2 \\ &= \frac{1}{8}|1 - e^{2i\phi}|^2 \\ &= \frac{1}{8}(1 - e^{2i\phi} - e^{-2i\phi} + 1) \\ &= \frac{1}{4}(1 - \cos(2\phi)) \end{aligned}$$

Quantum Metrology (3/3): Phase Estimation using $\hat{J}_z$ Operator

$\hat{J}_z$ Operator in Quantum Metrology

- Definition: $\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$
- Measures population difference between modes a and b

Phase Estimation Process

1. State: $|\psi\rangle = \frac{1}{\sqrt{2}}(|N, 0\rangle + e^{iN\phi}|0, N\rangle)$
2. Expectation value: $\langle \hat{J}_z \rangle = \frac{N}{2}\cos(N\phi)$
3. Variance: $(\Delta \hat{J}_z)^2 = \frac{N^2}{4} - \langle \hat{J}_z \rangle^2$
4. Phase uncertainty: $\Delta\phi = \frac{\Delta \hat{J}_z}{|\partial\langle \hat{J}_z\rangle/\partial\phi|}$

Result

$$\Delta\phi = \frac{1}{N} \text{ (Heisenberg limit)}$$

Classical Limit in Phase Estimation (1/2)

Setup and Assumptions

- Consider N independent particles in a Mach-Zehnder interferometer
- Each particle has a 50% chance of taking either path
- Phase difference ϕ between paths

Measurement Process

- Measure population difference between paths: $M = N_a - N_b$
- N_a, N_b: number of particles in paths a and b
- Expected value: $\langle M \rangle = N \cos\phi$

Statistical Analysis

- Binomial distribution with $p = \frac{1}{2}(1 + \cos\phi)$
- Variance: $\Delta M^2 = Np(1-p) = \frac{N}{4}(1 - \cos^2\phi) = \frac{N}{4}\sin^2\phi$

Classical Limit in Phase Estimation (2/2)

Error Propagation

- Use error propagation formula: $\Delta\phi = \frac{\Delta M}{\left|\frac{d\langle M\rangle}{d\phi}\right|}$
- $\frac{d\langle M\rangle}{d\phi} = -N\sin\phi$

Derivation of Classical Limit

$$\Delta\phi = \frac{\Delta M}{|N\sin\phi|} = \frac{\sqrt{\frac{N}{4}\sin^2\phi}}{N\sin\phi} = \frac{\frac{\sqrt{N}}{2}|\sin\phi|}{N\sin\phi} = \frac{1}{2\sqrt{N}} \tag{1}$$

Result and Interpretation

- Classical limit:$\Delta\phi \propto \frac{1}{\sqrt{N}}$
- This is the best precision achievable with classical resources
- Known as the standard quantum limit or shot noise limit

Practical Implementation

DC Photon Creation (1/3): Qualitative Description

- Goal: Creation of (Polarized-)Entangled Photons

Recipe

- High-energy UV pump through nonlinear crystal - Emission of photon pairs (ordinarily/extraordinarily polarized photons)
- Three crucial concepts:
  1. Birefringence of the crystal
  2. Energy conservation
  3. Phase matching condition
- Result: Entangled states and Bell states on demand

Source: Kwiat et al., "New High-Intensity Source of Polarization-Entangled Photon Pairs" (1995)

DC Photon Creation (2/3): Detailed Processes

Birefringence

- Refractive index depends on polarization and propagation direction
- Different polarizations travel at different speeds
- Results in phase difference α between orthogonally polarized light

Energy Conservation and Phase Matching

- Energy conservation: $\hbar\omega_p = \hbar\omega_o + \hbar\omega_e$
- UV (351 nm) $\rightarrow$ Two IR (702 nm) photons
- Momentum conservation: $\vec{k}_p = \vec{k}_o + \vec{k}_e$

Phase matching condition:

from $|\overrightarrow{k}| = \frac{n\omega}{c}$

- $n_p(\theta_p)\omega_p = n_o(\theta_o)\omega_o + n_e(\theta_e)\omega_e$

Source: Kwiat et al., "New High-Intensity Source of Polarization-Entangled Photon Pairs" (1995)

DC Photon Creation (3/3): State Evolution

Probabilistic Creation of Two Photons

$$|\psi\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|\psi_2\rangle$$

where p is the probability of pair creation

Entangled State at Cone Intersections

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|H_o, V_e\rangle + e^{i\alpha}|V_o, H_e\rangle)$$

where α is the relative phase determined by crystal properties

Source: Kwiat et al., "New High-Intensity Source of Polarization-Entangled Photon Pairs" (1995)

First Application: 4-photons setup

4-Photon Entanglement (1/4)

Goal

Use polarization-entangled (DC) photons to achieve up to 4-photon entanglement

Motivation

- Exploit 1/N scaling in phase sensitivity
- Signature: Effective de Broglie wavelength

$$\lambda_N = \frac{\lambda_1}{N}$$

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

4-Photon Entanglement (2/4): Double Pass Configuration

Experimental Setup

- UV pump passes through crystal twice using a mirror
- Allows for different pair separation probabilities

Resulting State

$$
\begin{aligned}
|\psi\rangle =& (1-p)|0\rangle_{a1,a2}|0\rangle_{b1,b2}+ \\
& \sqrt{p}(|\Phi^+\rangle_{a1,a2}|0\rangle_{b1,b2} + e^{i\Delta\phi}|0\rangle_{a1,a2}|\Phi^+\rangle_{b1,b2})+ \\
& p|\Phi^+\rangle_{a1,a2}|\Phi^+\rangle_{b1,b2}
\end{aligned}
$$

where p is the probability of pair creation

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

4-Photon Entanglement (3/4): Interference Cases

Simple Separation (Two-Photon Case)

- Pair emitted in mode a or b
- Probability: $P \propto 1 - \cos(2\Delta\phi)$

Four-Photon Cases

1. Double pair emitted once:
   - Four photons in a1-a2 or b1-b2
   - Pure four-photon interference
2. Single pair emitted on each pass:
   - One pair in a1-a2, one in b1-b2
   - Potential ambiguity in interpretation

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

4-Photon Entanglement (4/4): Four-Photon States and Discussion

Explicit Formulas

1. Double pair emission:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|4\rangle_{a1,a2}|0\rangle_{b1,b2} + e^{i4\Delta\varphi}|0\rangle_{a1,a2}|4\rangle_{b1,b2})$$

2. Single pair on each pass:

$$|\psi\rangle = e^{i2\Delta\varphi}(|H\rangle_{a3}|H\rangle_{a4}|H\rangle_{b3}|H\rangle_{b4} + |V\rangle_{a3}|V\rangle_{a4}|V\rangle_{b3}|V\rangle_{b4})$$

Discussion

- Ambiguity in case (2) due to two-photon interference contributions
- Extension to $N > 4$ possible with suppression of lower-order interference

Source: Walther et al., "De Broglie wavelength of a non-local four-photon state" (2004)

Second Application: 3-photons setup (actual paper)

Actual Paper (1/3): Experimental Setup

- Uses two down-converted (DC) photons and one local oscillator (LO) photon
- The 3 photons are coming from the same Ti:sapphire laser.
- A small portion of the Ti:sapphire laser is split off:
  - The main output of the Ti:sapphire laser is frequency-doubled to create a UV pump beam for the DC photons
  - This split beam is heavily attenuated to the single-photon level to act as the LO photon.

Source: Mitchell, M. W., "Super-resolving phase measurements with a multi-photon entangled state" (2005)

Actual Paper (1/2): Experimental Results and Analysis

Three-Photon Interference

- Observed probability: $P \propto 1 + \cos(3\phi)$
- Demonstrates tripling of oscillation frequency

State Analysis

- $|3,0\rangle$ and $|0,3\rangle$ states: Main contributors to 3ϕ dependence
- $|2,1\rangle$ and $|1,2\rangle$ states: Present due to experimental imperfections
- Post-selections techniques are therefore needed.

Comparison with Classical Limit

- Achieved phase sensitivity surpasses classical $1/\sqrt{N}$ limit
- Demonstrates potential for quantum-enhanced metrology

Source: Mitchell, M. W., "Super-resolving phase measurements with a multi-photon entangled state" (2005)

Conclusion (1/3)

How is the NOON state created in experiment?

- Ti:sapphire laser is:
  - frequency-doubled to create the UV pump beam fro the SPDC
  - split off and heavily attenuated to create the LO photon
- Post-selection techniques

How is a single photon state at the input created?

- Attenuation to single photon level
- "mode matched" with SPDC photons - indistiguishable from SPDC

Conclusion (2/3)

What is post-selection and why is it required?

- Keeping only those measurement outcomes that satisfy certain criteria, discarding the rest
- Required to:
  - Filtering out unwanted events: It removes cases where fewer than three photons were detected, or where additional noise photons were present. (proof: detection at "Dark" port, lower-order interference)
  - State preparation: It effectively prepares the desired three-photon entangled state. (eliminates high order terms for example)

What is meant by super-resolution? What does it compare to in a classical measurement?

- Definition: Ability to resolve phase shifts smaller than classical limit
- Comparison:
  - Classical limit: $\Delta\phi \propto 1/\sqrt{N}$ (Shot Noise)
  - Quantum limit: $\Delta\phi \propto 1/N$ (Heisenberg limit)

Conclusion (3/3)

What are the experimental signatures of super-resolution and what signal bears these signatures?

- N-fold increase in fringe frequency
- Probability: $P \propto 1 + \cos(N\phi)$

Is it possible to scale to large N state?

- Different set up:
  - Cascade down-conversion process (double-, triple-pass, ...)
  - Optical loop configuration
- Higher-order terms from DC photons:
  - Probability of creating more than one pair of photons per pump pulse.
  - BUT probability of these higher-order events decreases rapidly with N