Transverse Dynamics :: Equation of Motion and Solutions

Laboratory for Particle Accelerator Physics, EPFL

EPFL

Introduction Linear Dynamics

- accelerator = series of elements for **beam guiding** (bending, focusing) and **acceleration**; often arranged in a **closed loop** (ring)
- guiding fields must ensure long term (h) stability of circulating particles

questions to be answered:

- How to ensure bound motion of a particle beam?
- What are conditions for stability?
- Amplitude and frequency of particle oscillations?
- Statistical beam properties like beam width and angular spread?
- How to design magnet lattices (arrangements of dipoles and quads in a line)?
- What is the impact of field errors in bending and focusing magnets?

References and Accessible Reading Material

available on the internet:

P. Schmüser & J. Rossbach, Basic course on accelerator optics:

https://cds.cern.ch/record/247501/files/p17.pdf

F.Tecker, Longitudinal Dynamics:

https://arxiv.org/pdf/1601.04901.pdf

Book, H.Wiedemann, Particle Accelerators, download pdf

https://link.springer.com/book/10.1007%2F978-3-319-18317-6

M. Sands, Physics of Electron Storage Rings: An Introduction.

https://digital.library.unt.edu/ark:/67531/metadc865991/

CERN Accelerator School (CAS) proceedings homepage (huge!)
http://cas.web.cern.ch/cas/CAS_Proceedings.html

CERN Accelerator School on Medical Applications:

https://cds.cern.ch/record/2271793/files/33-8-PB.pdf

books, papers:

M.Conte and W.McKay, An Introduction to the Physics of Particle Accelerators, World Scientific, 2008

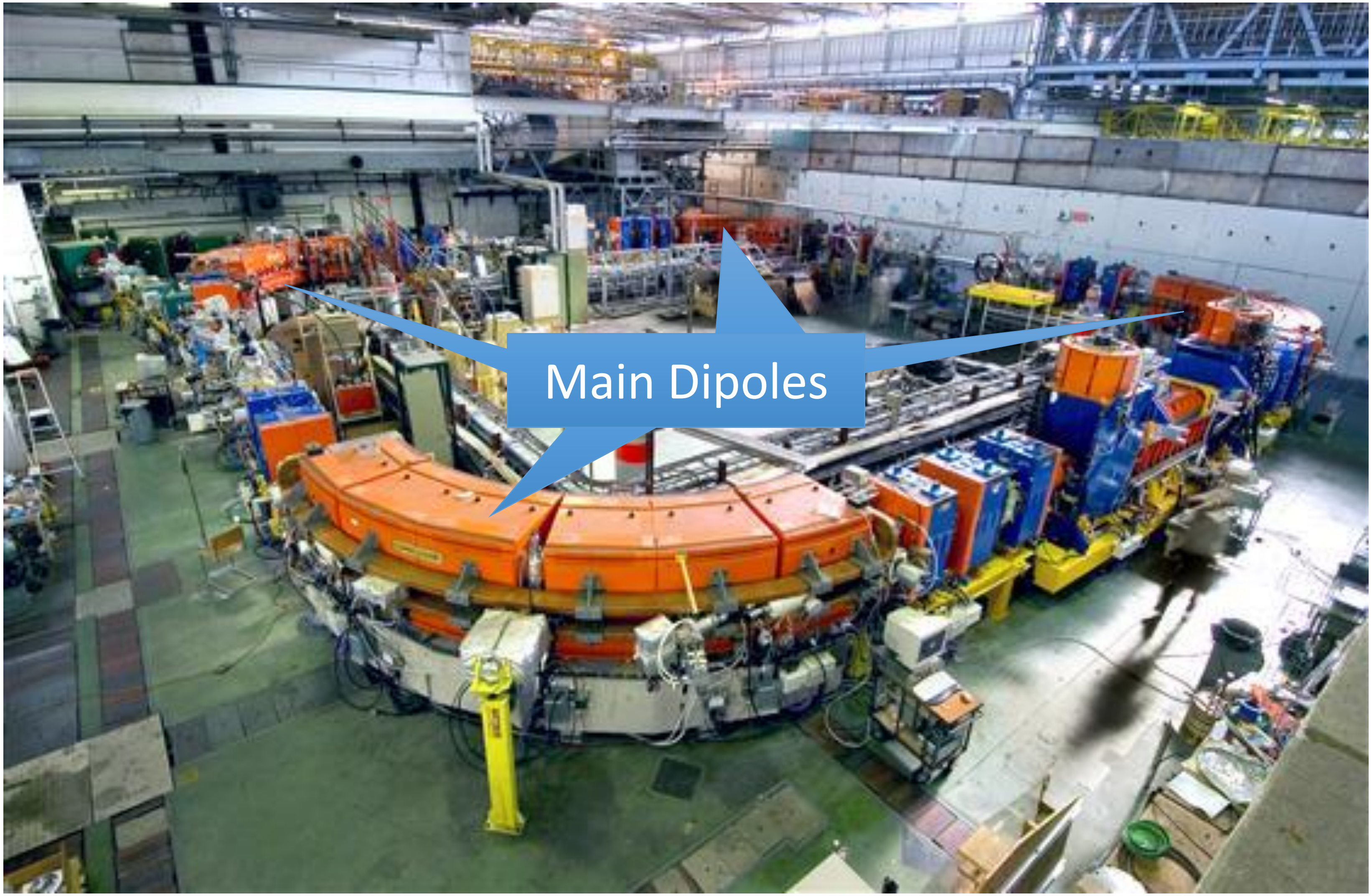
S.Peggs, T.Satogata, *Introduction to Accelerator Dynamics*, Cambridge University Press, 2017

A. Wolski, *Beam Dynamics in high energy particle accelerators*, Imperial College Press, 2014

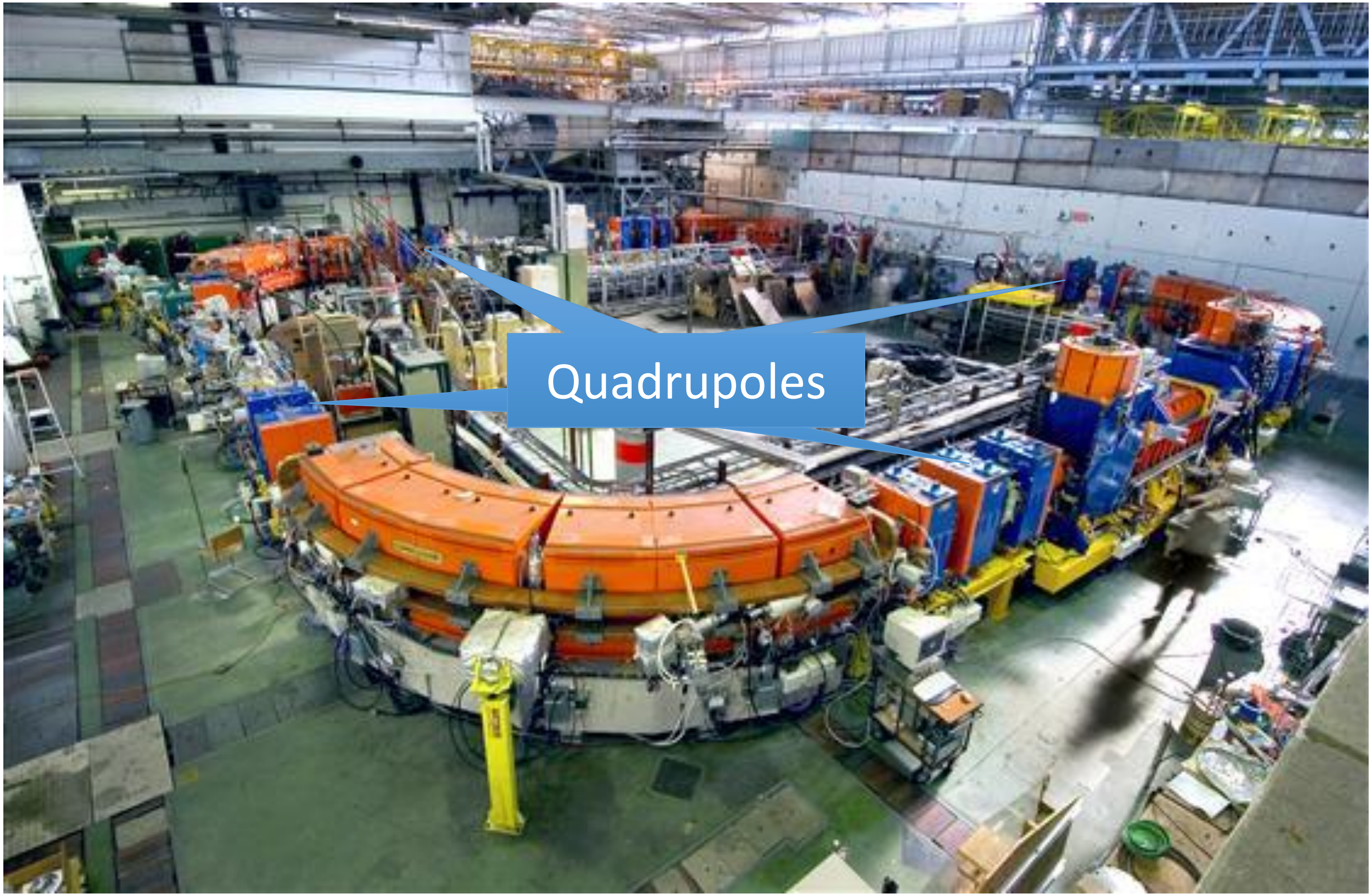
A. W. Chao, M. Tigner, *Handbook of Accelerator Physics and Engineering*, World Scientific 1999

E. D. Courant and H. S. Snyder, Annals of Physics: **3**, 1-48 (1958)

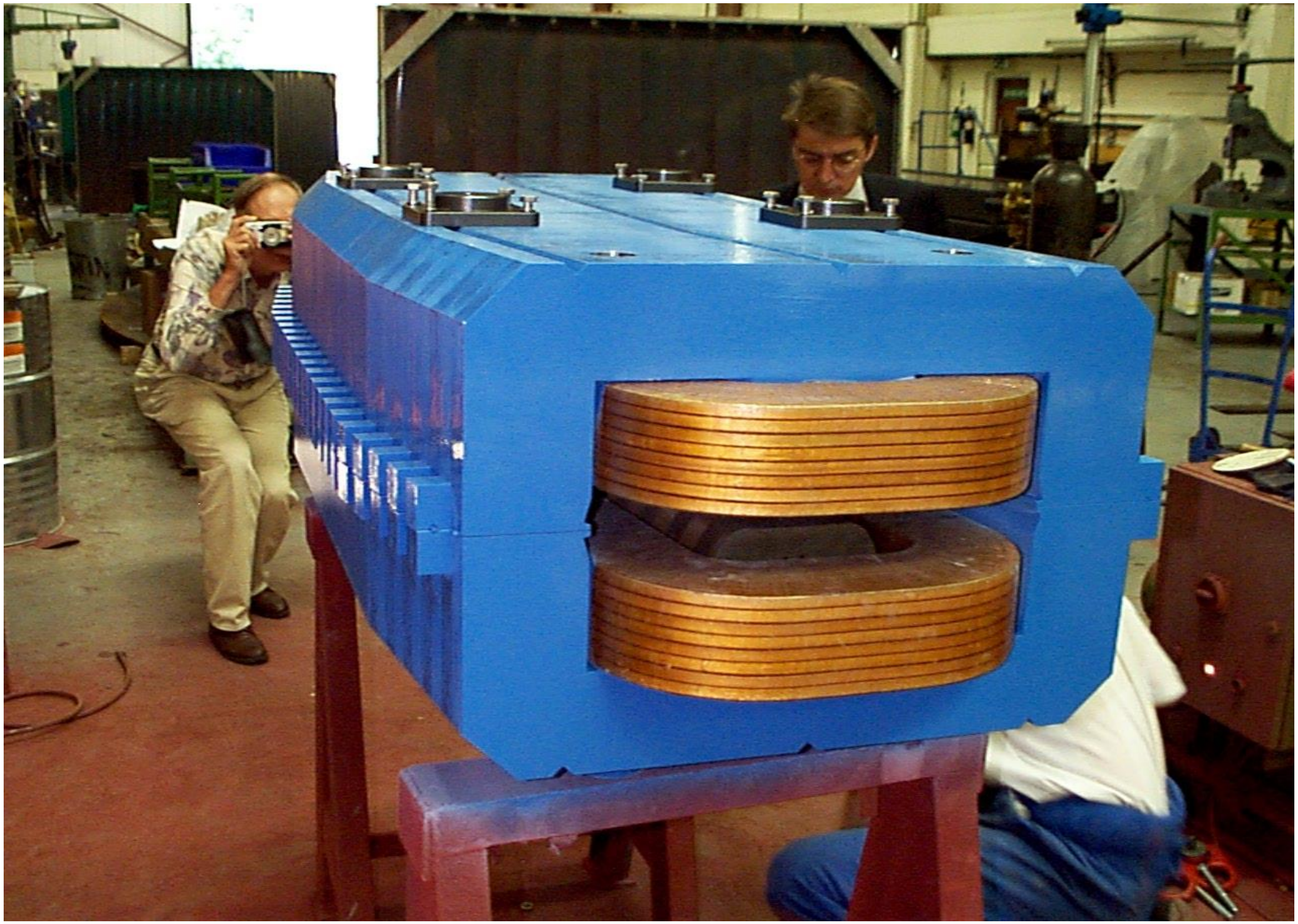
Make Particles Circulate



Focusing the Particles



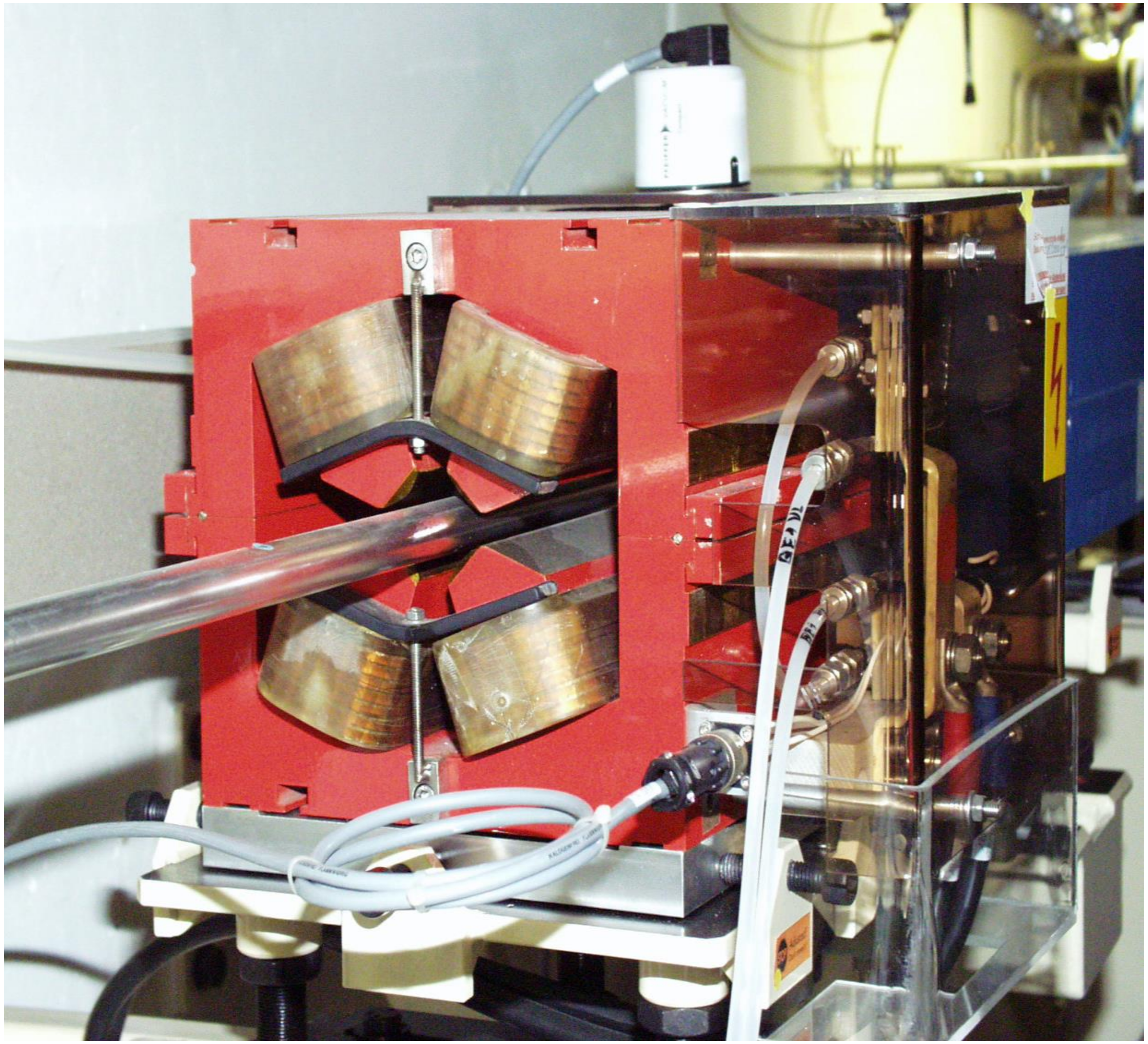
Bending Magnet - SLS dipole



magnetic rigidity:

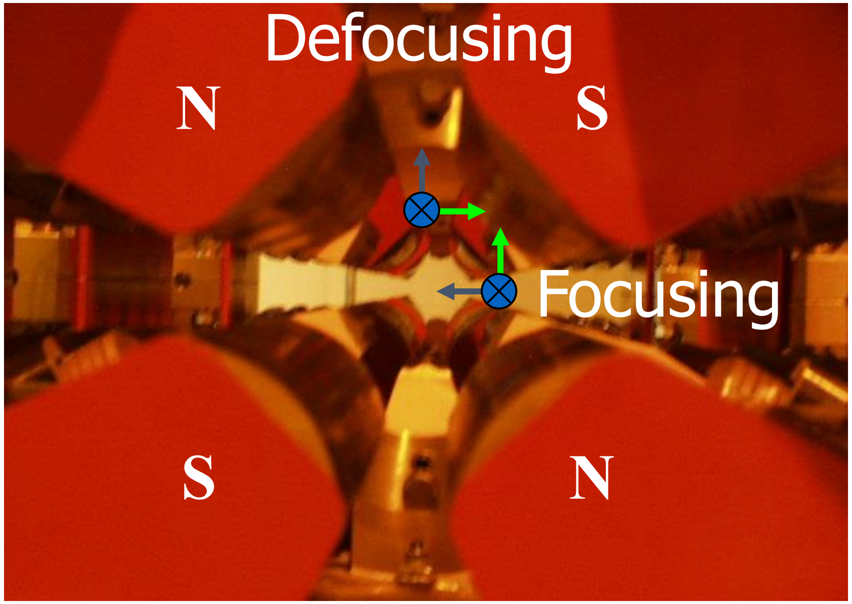
$$B\rho = \frac{p}{e}$$

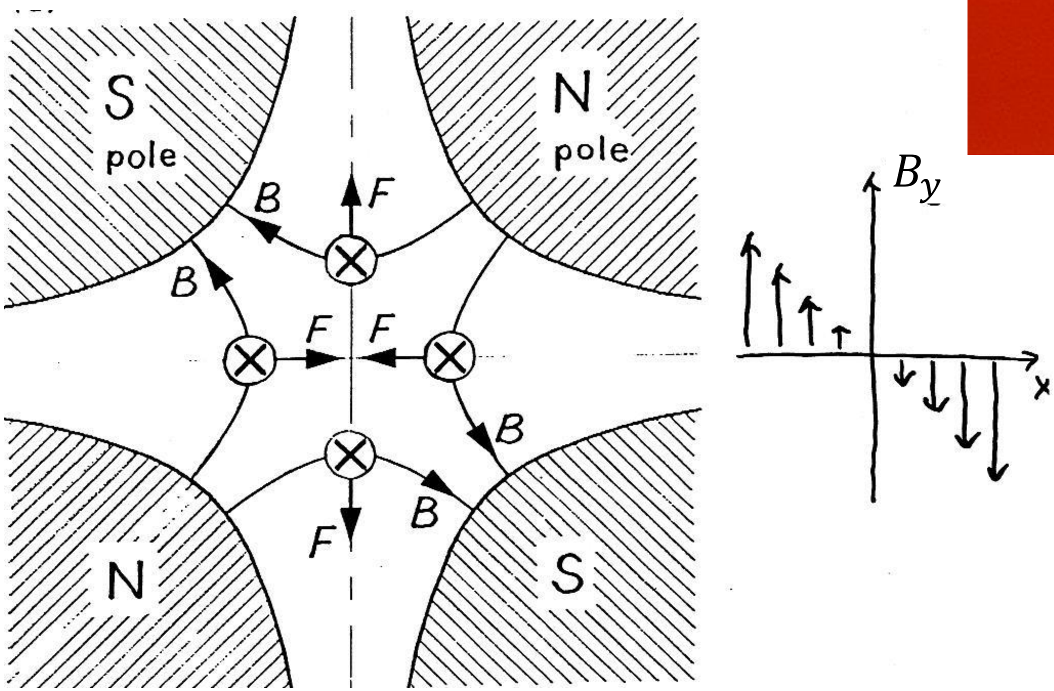
Quadrupole Magnet - Focusing Element



Quadrupole Lens

- Focusing in one plane
- Defocusing in the other plane





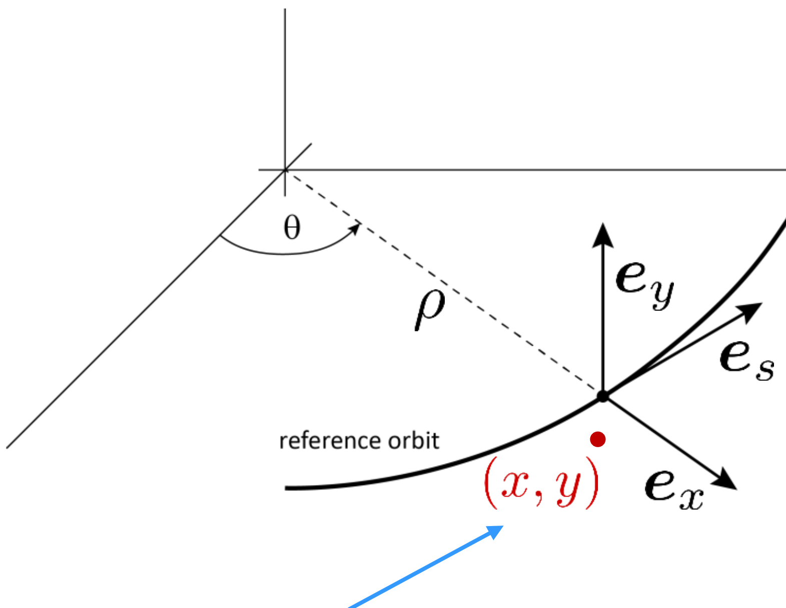
$$\nabla \times \boldsymbol{B} = 0 \rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

Next: Equation of Motion

- suited coordinate system
- linearizing forces and deriving differential equation

Curvilinear Coordinate System

aim: derive a set of equations that describe the motion of a single particle wrt. a curved coordinate system around the reference orbit of a beam, (x, y)



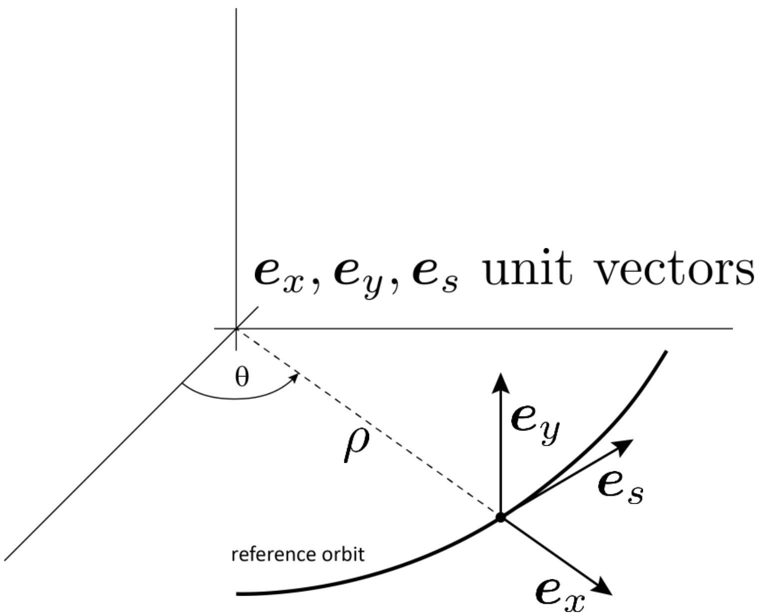
$\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_s$ unit vectors

particle coordinate: $\vec{R} = r\boldsymbol{e}_x + y\boldsymbol{e}_y, \;\; r \equiv \rho + x$

$$x, y \ll \rho$$

see also: Frenet-Serret coordinates, e.g. Wiedemann chap 4.3

Deriving the Equation of Motion (see Appendix)



the effect of the curved coordinate system, i.e. the moving unit vectors e_x, e_s must be *included in the calculation*

$$x' \equiv \frac{\partial x}{\partial s}, \; x'' \equiv \frac{\partial^2 x}{\partial s^2}$$

starting with general equation of motion: $\dfrac{d\vec{p}}{dt} = \gamma m_0 \ddot{\vec{R}} = e\vec{v} \times \vec{B}$

$$B_y = B_0 + gx, \; B_x = gy$$ dipole and quadrupole field

$$g \equiv \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$ field gradient varying sign convention !

$$\frac{1}{\rho} = \frac{eB_0}{\gamma m_0 v}$$ orbit curvature

$$k = \frac{eg}{\gamma m_0 v}$$ k – value [m^{-2}]

$$x'' + \left(\frac{1}{\rho^2} + k\right) x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

k - value

curved coordinates

off momentum term

derivative w.r.t. path-length *s*, not time *t*

Equation of Motion

$$\begin{aligned} x'' + \left(\frac{1}{\rho^2} + k\right) x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\ y'' - ky &= 0 \end{aligned}$$

generalised form:

$$x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

DE is valid for drift spaces, Quadrupoles ($k \neq 0$), combined function magnets ($k \neq 0$, $1/\rho \neq 0$) and for off-momentum particles ($\Delta p \neq 0$, first order)

we discuss solutions of different cases of this equations in single accelerator magnets, depending on K(s), ρ(s), Δp

see also Wiedemann sec. 1.5.8

geometric meaning of coefficients

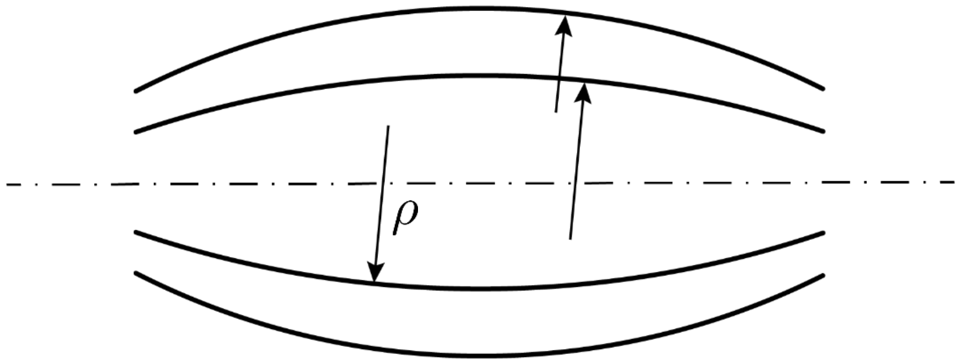
$$\boxed{x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}}$$

$$\frac{d\theta}{ds} = \frac{1}{\rho}(s)$$

= curvature $1/\rho$ [m^{-1}]

$$K(s) \cdot x = \frac{1}{\rho}(x,s)$$

K = amplitude dependent curvature K[m^{-2}]



inside quadrupole field

Summary on Approximations used

- small displacements $x \ll \rho, y \ll \rho, \ddot{s} \approx 0$ **(paraxial optics)**
- only dipole and quadrupole magnets **(linear field changes)**
- design orbit lies in a plane **(flat accelerator)**
- no coupling between motion in hor. and vert. plane **(upright magnets)**
- small momentum deviations **(quasi monochromatic beam)**
- **in general: no quadratic or higher order terms (linear beam optics)**

linear parametrization of magnetic field:

$$\frac{e}{p}\vec{B} = \left(kx + \frac{1}{\rho}\right)\boldsymbol{e}_y + ky\,\boldsymbol{e}_x$$

Next: Solving the Equation of Motion using Matrices

- matrix approach using principal trajectories
- drift space, focusing and defocusing quadrupole

Piecewise solution of trajectory equation in terms of principal trajectories

$$\boxed{x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}}$$

see also Wiedemann sec. 5.5

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\frac{\Delta p}{p}$$

Dispersion Trajectory
(= particular solution of eq.) $D(0) = D'(0) = 0$

Sinelike Trajectory $S(0) = 0,\ S'(0) = 1$

Cosinelike Trajectory $C(0) = 1,\ C'(0) = 0$

C(s), S(s) are independent solutions of the homogeneous equation:

$$C'' + K(s)C = 0,\ \ S'' + K(s)S = 0.$$

Formulation as Transport Matrix

$$\boxed{\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} + \frac{\Delta p}{p_0} \begin{pmatrix} D \\ D' \end{pmatrix}}$$

$$\det \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = 1$$

<u>proof:</u>

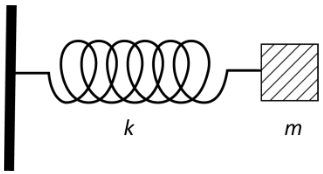
$$\det \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}_{s=s_0} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(s)(CS - SC) = 0$$

$\rightarrow$ det = 1 initially, and it does not change as motion proceeds

Piecewise Solution of Equation

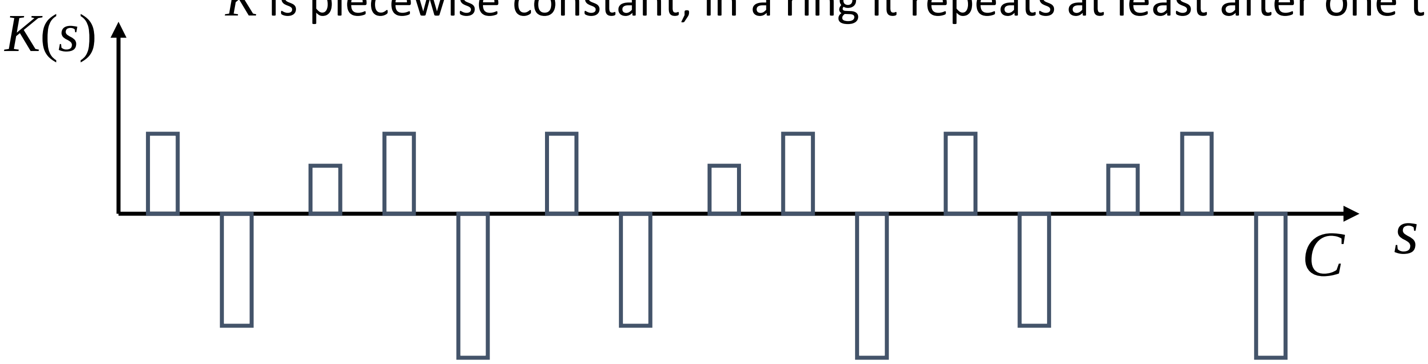
$$\boxed{x'' + K(s)x \quad = \quad 0}$$



general form of equation similar to harmonic oscillator with three cases: K=0, K<0, K>0

$$m\ddot{x} + kx = 0, \;\; \omega = \sqrt{\frac{k}{m}}$$

K is piecewise constant, in a ring it repeats at least after one turn (s=C)

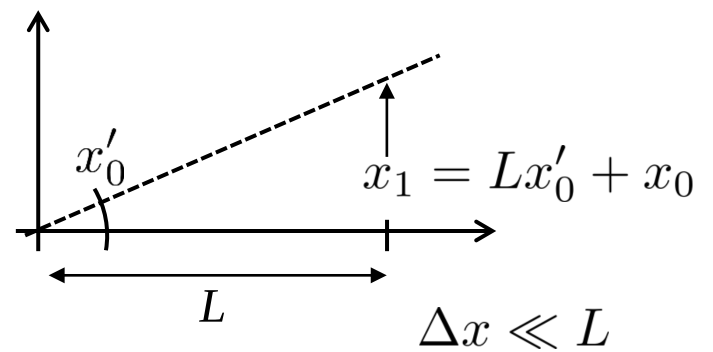


Drift Space

particle moves straight

1.) K=0: Drift Space

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$



Focusing Quadrupole

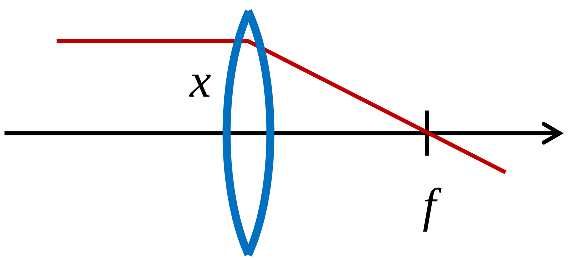
2.) K > 0:   Focusing Quadrupole

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cos(\sqrt{K}L) & \sin(\sqrt{K}L)/\sqrt{K} \\ -\sin(\sqrt{K}L)\sqrt{K} & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$

thin lens approximation:

$$K = \frac{1}{Lf}, \quad \lim_{L\to 0}\left(\sin\left(\sqrt{L/f}\right)\frac{1}{\sqrt{Lf}}\right) = \frac{1}{f}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$



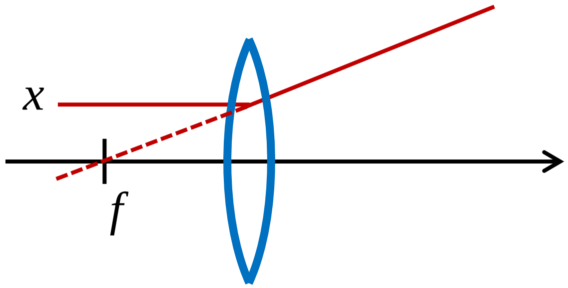
Defocusing Quadrupole

3.) K < 0:   Defocusing Quadrupole

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \sinh(\sqrt{|K|}L)/\sqrt{|K|} \\ \sinh(\sqrt{|K|}L)\sqrt{|K|} & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$

thin lens approximation for defocusing quad:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$

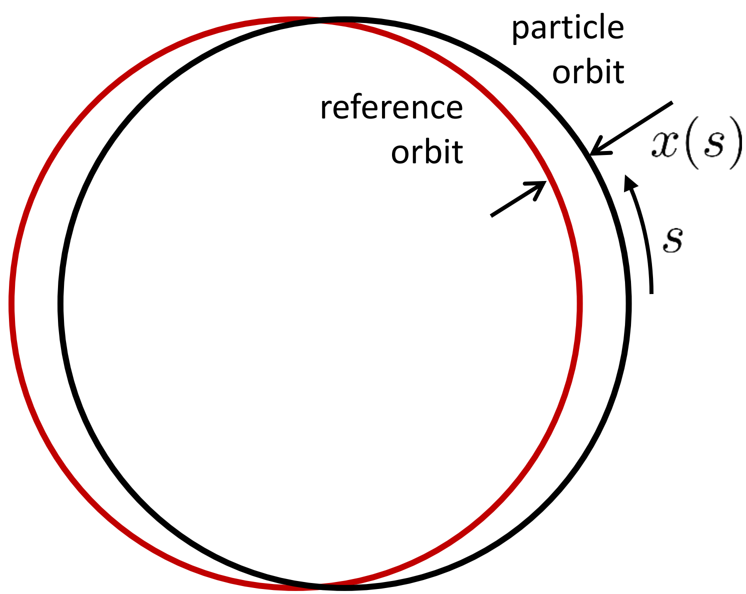


Special Case: Weak Focusing

$$\text{for } g = \frac{\partial B_y}{\partial x} = 0, \frac{1}{\rho} \neq 0 :$$

$$K = \frac{1}{\rho^2}, \; x(s) = A\cos(s/\rho + \varphi_0)$$

→ one oscillation per turn as expected for „weak focusing“



Dispersion Trajectory

is solution of equation:

$$D'' + K(s)D = \frac{1}{\rho(s)}$$

is constructed from homogeneous solutions C(s), S(s):

$$D(s) = S(s)\int_{s_0}^{s} \frac{C(t)}{\rho(t)}dt - C(s)\int_{s_0}^{s} \frac{S(t)}{\rho(t)}dt$$

<u>proof:</u> by insertion in above DE and by using $CS' - C'S = 1$.

compare components of
particle trajectory:

$$x(s) = C(s)x_0 + S(s)x_0' + D(s)\frac{\Delta p}{p}$$

Dispersion Trajectory in combined function magnet (Dipole & Quadrupole)

$$\begin{pmatrix} D \\ D' \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho K}(1-\cos(\sqrt{K}L)) \\ \frac{1}{\rho\sqrt{K}}\sin(\sqrt{K}L) \end{pmatrix}$$

for K > 0

$$\begin{pmatrix} D \\ D' \end{pmatrix} = \begin{pmatrix} -\frac{1}{\rho|K|}(1-\cosh(\sqrt{|K|}L)) \\ \frac{1}{\rho\sqrt{|K|}}\sinh(\sqrt{|K|}L) \end{pmatrix}$$

for K < 0

$$\begin{pmatrix} D \\ D' \end{pmatrix} = \begin{pmatrix} \frac{1}{2\rho}L^2 \\ \frac{1}{\rho}L \end{pmatrix}$$

for K = 0

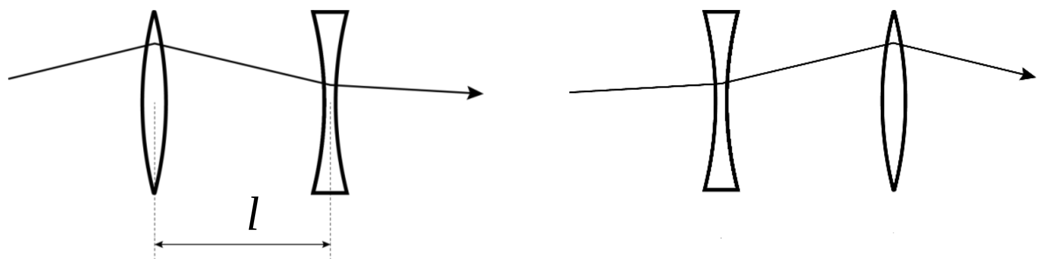
$\rightarrow$ Dispersion is generated by bending fields

$\rightarrow$ Dispersion oscillates in Quadrupoles like particle trajectories

Quadrupole Doublet

concatenation of particle transport through a series of elements:

$$\boldsymbol{M} = \boldsymbol{M}_n \ldots \boldsymbol{M}_2 \cdot \boldsymbol{M}_1$$ (***M*** = transport matrix 2x2)



$$\boldsymbol{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{l}{f} & l \\ -\frac{1}{f^*} & 1 - \frac{l}{f} \end{pmatrix}$$

[thin lens approximation]

$$f^* = \frac{f^2}{l} > 0$$ $\rightarrow \mathrm{M}_{\text{doublet}}$ is always focusing

3x3 transport matrix for off-momentum particles

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} + \frac{\Delta p}{p_0} \begin{pmatrix} D \\ D' \end{pmatrix}$$

can be re-formulated as a single matrix multiplication:

$$\begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_{s_0}$$

this formulation is convenient when computing the dispersion function

Matrix Notation

… is one approach to describe particle trajectories in accelerators – for each element a matrix for linear computation can be defined.

The matrix can be extended to all six dimensions of phase space:
x, x', y, y', s, δ

energy deviation $\delta=\Delta E/E_0$

path length

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ s \\ \delta \end{pmatrix}_{\text{out}} = \boldsymbol{R} \cdot \begin{pmatrix} x \\ x' \\ y \\ y' \\ s \\ \delta \end{pmatrix}_{\text{in}}$$

In this case R contains 36 parameters, some of which have practical importance for example in coupling horizontal and vertical motion.

The matrix notation can be generalised to higher order dependencies. For example a tensor of rank 3 holds coefficients $\mathrm{T}_{\mathrm{ijk}}$ for a quadratic map of particle coordinates.

Next: Stability and Twiss Matrix

- Eigenvalues and Stability
- Twiss Matrix Decomposition

Stability Criterion

for stable oscillation in a ring these coordinates must stay finite for n → ∞:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \boldsymbol{M}^n \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$

$\boldsymbol{M}$ = transport matrix of entire ring

to assess stability use decomposition of $\boldsymbol{M}$ in Eigenvectors:

$$\boldsymbol{M}\vec{v}_1 = \lambda_1 \vec{v}_1, \;\; \boldsymbol{M}\vec{v}_2 = \lambda_2 \vec{v}_2$$

then:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}} = A\vec{v}_1 + B\vec{v}_2, \;\; \boldsymbol{M}^n \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}} = A\lambda_1^n \vec{v}_1 + B\lambda_2^n \vec{v}_2$$

thus it is sufficient analysing the Eigenvalues to assess stability of linear motion
λ^n must be bounded for $n \to \infty$ (λ = complex number in general)

Stability Criterion :: Matrix & EV Properties

transfer matrix M has properties that can be used:

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

from that follows:

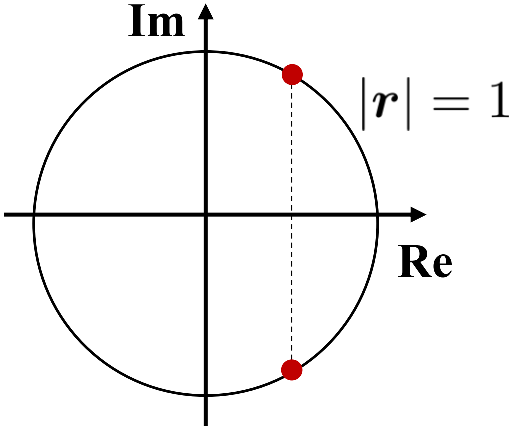
$$\det\boldsymbol{M} = ad - bc = 1 \longrightarrow \lambda_1 \cdot \lambda_2 = 1$$

$$a, b, c, d = \text{real numbers} \longrightarrow \lambda_1 = \lambda_2^* \text{ (complex conjugate)}$$

this can be extended to 4dim matrices
(coupled horizontal + vertical)

$\lambda_{1,2}^n$ bounded for $\lambda_1 = e^{+i\mu}, \lambda_2 = e^{-i\mu}$

with μ = real number



stable = EVs on unit circle

Stability Criterion – the Trace of M

Computation of eigenvalues via characteristic polynomial:

$$\det(\boldsymbol{M} - \lambda \boldsymbol{I}) = 0,\ \ \boldsymbol{M} - \lambda \boldsymbol{I} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$\lambda^2 - (a+d)\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{1}{2}(a+d) \pm \sqrt{\ldots}$$

2x2 matrix: characteristic polynomial
quadratic equation for eigenvalues λ
solution: values λ_1, λ_2

$$\lambda_1 + \lambda_2 = a + d = \mathrm{Tr}\,\boldsymbol{M}$$

$$\lambda_1 = e^{-i\mu}, \lambda_2 = e^{i\mu}$$

$$\rightarrow \mathrm{Tr}\,\boldsymbol{M} = \lambda_1 + \lambda_2 = 2\cos(\mu)$$

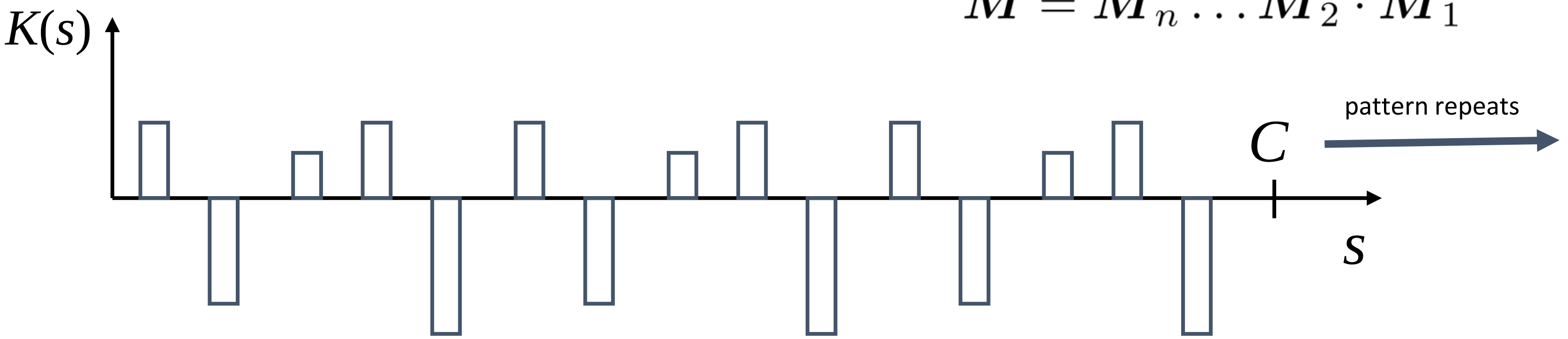
motion is stable if $|\mathrm{Tr}\,\mathbf{M}|<2$, which also means that μ is real

see also Schmüser/Rossbach, and sect. 4.1, Wiedemann chap. 10.1.2

Summary Matrix Treatment

- equation of motion is piecewise solved for constant K(s)
- x, x' are propagated by concatenated multiplication with 2x2 matrices
- defocusing and focusing quadrupoles are combined in overall focusing doublets
- linear motion in a ring is stable if $|\mathrm{Tr}\,\mathbf{M}|<2$
- $\det \mathbf{M}$ = 1 (also: **M** symplectic)

$$\boldsymbol{M} = \boldsymbol{M}_n \ldots \boldsymbol{M}_2 \cdot \boldsymbol{M}_1$$



Next: Analytical Solution

- Hills equation
- Beta function
- phase space ellipse
- include momentum offset
- tune Q_x, Q_y

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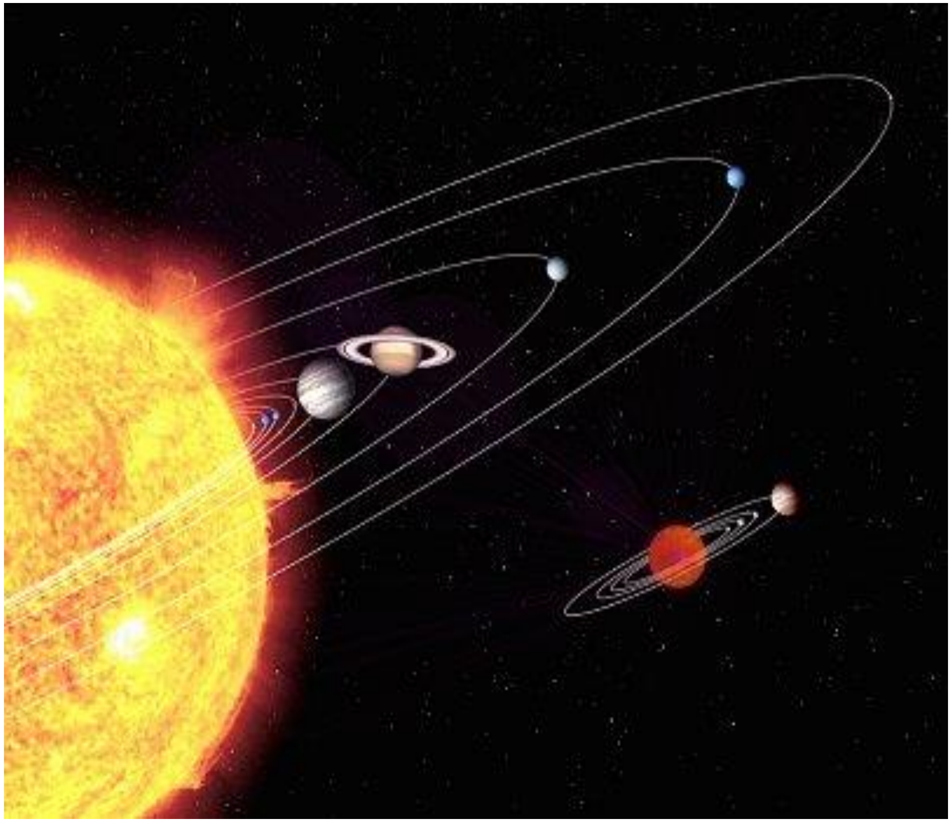
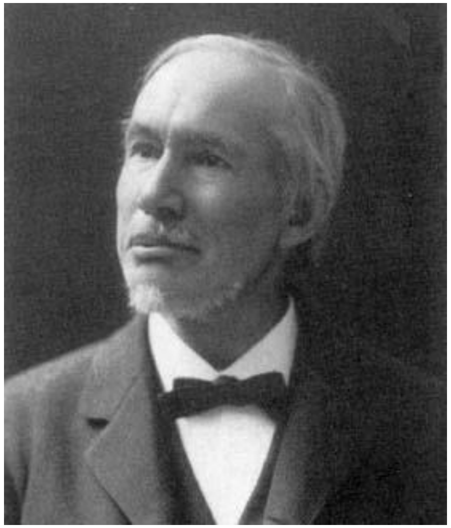
Solution of Hill´s equation

First used by an astronomer George Hill in his studies of the motion of celestial bodies, a motion under the influence of periodically changing forces





1838 -- 1914

Hill: Solution for periodic *K* $K(s+C) = K(s)$

$$x(s) = A\sqrt{\beta(s)}\cos(\varphi(s) - \varphi_0),\ \ \varphi(s) = \int_{t=s_0}^{s} \frac{dt}{\beta(t)}$$

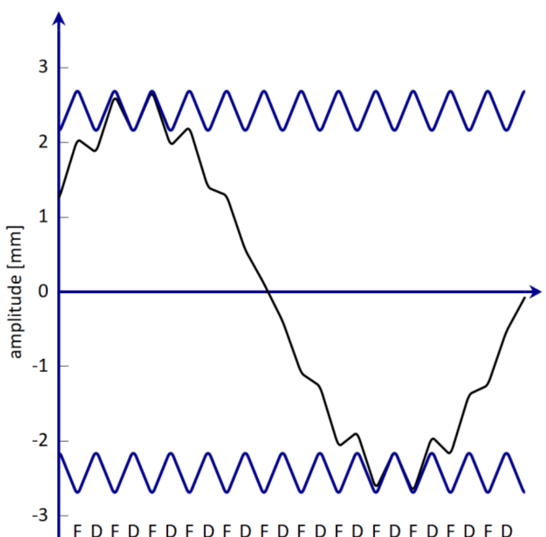
$\rightarrow$ the **beta function [m] is a scaling factor** for the amplitude of orbit oscillations, and their **local wavelength**

A, φ_0 are constants of motion

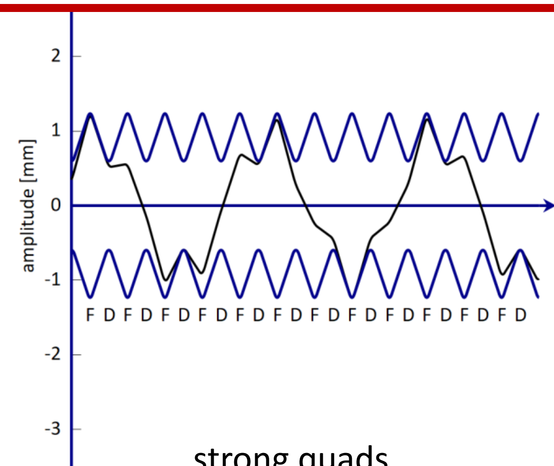
β - well known **Betafunction** of accelerators

variables **are not related to relativistic factors**

(sorry for the historic nomenclature)



weak quads



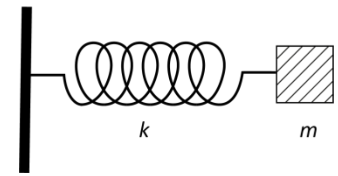
$A\sqrt{\beta(s)}$

strong quads

Comparison to Classical Harmonic Oscillator

$$\ddot{u} + \omega^2 u = 0$$

$$u(t) = A\cos\omega t, \;\; \omega = \sqrt{\frac{k}{m}}$$



amplitude is fixed: $A = \text{const}$

phase grows linear with time: $\sqrt{\frac{k}{m}}\, t$

conserved (energy): $\frac{k}{2}u^2 + \frac{m}{2}\dot{u}^2 = k\, A^2$

Hill Equation (pseudo harmonic equation)

$$\boxed{\begin{aligned} x(s) &= \sqrt{2J\beta}\cos(\varphi) \\ x'(s) &= -\sqrt{\frac{2J}{\beta}}\left(\alpha\cos(\varphi) + \sin(\varphi)\right) \end{aligned}}$$

Twiss Parameters:

$\beta(s)$ [meter]

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha^2}{\beta}$$

amplitude varies: $x(s) \propto \sqrt{\beta(s)}$

phase increases monotonically
but growth rate varies as 1/β : $d\varphi = \dfrac{ds}{\beta(s)}$

conserved (action J): $\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J = \text{const}$

Computing the Beta Function in three ways

1.) from the transport matrix of a closed ring

$$\boldsymbol{M}_{\text{ring}}(s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\cos(\mu) = \frac{(a+d)}{2},\ \beta = \frac{b}{\sin(\mu)},\ \alpha = \frac{(a-d)}{2\sin(\mu)}$$

2.) by solving a DE for β (DE without proof)

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1$$

3.) by using a transport matrix for Twiss parameters (C,S principal trajectories)

$$\begin{pmatrix} \beta_s \\ \alpha_s \\ \gamma_s \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

Computing the transport matrix from Twiss Parameters

transport matrix $s=s_0 \rightarrow s_1$ (arbitrary section):

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\Delta\varphi + \alpha_0 \sin\Delta\varphi) & \sqrt{\beta\beta_0}\sin\Delta\varphi) \\ -\frac{1}{\sqrt{\beta\beta_0}}((\alpha-\alpha_0)\cos\Delta\varphi + (1+\alpha\alpha_0)\sin\Delta\varphi) & \sqrt{\frac{\beta_0}{\beta}}(\cos\Delta\varphi - \alpha\sin\Delta\varphi) \end{pmatrix}$$

β_0, α_0 at s_0
β, α at s_1
$\Delta\varphi$ = phase advance between s_0, s_1

The one turn matrix $\boldsymbol{M}_{\text{rev}}$

transport matrix for one revolution, "One Turn Matrix"

- same location: $\beta = \beta_0$
- $\Delta\varphi = 2\pi Q$ phase advance for complete turn, Q = "Tune" of accelerator

$$\boldsymbol{M}_{\text{rev}} = \begin{pmatrix} \cos 2\pi Q + \alpha \sin 2\pi Q & \beta \sin 2\pi Q \\ -\frac{1+\alpha^2}{\beta} \sin 2\pi Q & \cos 2\pi Q - \alpha \sin 2\pi Q \end{pmatrix}$$

- special case: choose symmetry point $\alpha = 0$

$$\boldsymbol{M}_{\text{rev}} = \begin{pmatrix} \cos 2\pi Q & \beta \sin 2\pi Q \\ -\frac{1}{\beta} \sin 2\pi Q & \cos 2\pi Q \end{pmatrix}$$

Twiss Matrix

$$\boldsymbol{M} = \boldsymbol{I}\cos(\mu) + \boldsymbol{J}\sin(\mu)$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \boldsymbol{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

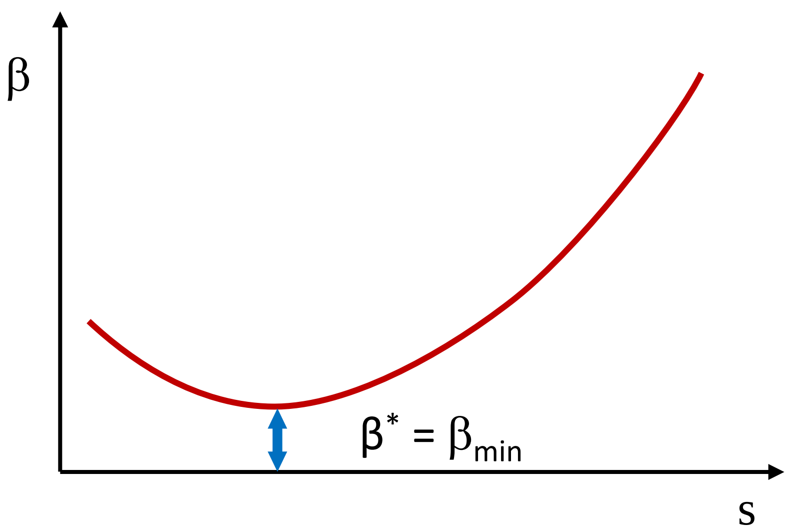
$$\boldsymbol{J}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

this expression has similar properties as $\cos\mu + i\sin\mu$

it holds: $\boldsymbol{M}^n = \boldsymbol{I}\cos(n\mu) + \boldsymbol{J}\sin(n\mu)$

Beta Function in a Drift

$$\beta(s) = \beta_0 + \beta_0' \cdot s + \frac{1}{2}\beta_0'' \cdot s^2$$
$$= \beta_0 - 2\alpha_0 \cdot s + \gamma_0 \cdot s^2$$



- in a field free region β follows a quadratic function
- the coefficients are related to the TWISS parameters
- at β_{min} we have a waist

Adding Dispersion

$$x(s) = A\sqrt{\beta(s)}\cos(\varphi(s) - \varphi_0) + D(s)\frac{\Delta p}{p}$$

where D(s) is a solution of: $$D'' + K(s)D = \frac{1}{\rho}(s)$$

$\uparrow$

outside bending magnets D behaves like particle trajectories

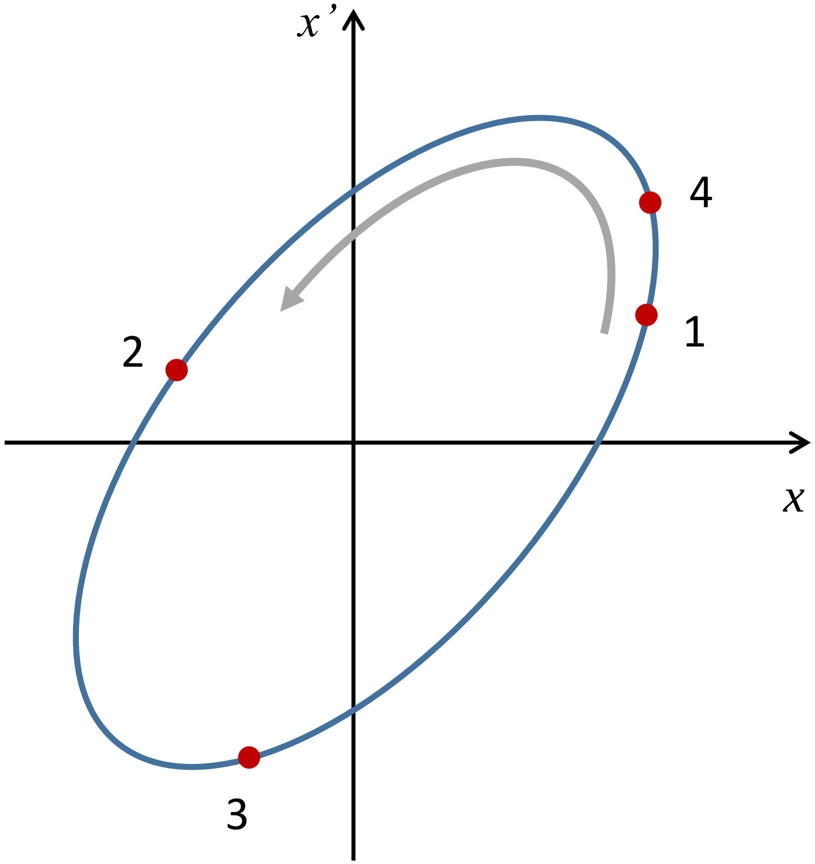
$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \oint dt\ \frac{\sqrt{\beta(t)}}{\rho(t)} \cos\left(\varphi(t) - \varphi(s) - \pi Q\right), \quad D(s+C) = D(s)$$

note:

D(s) = dispersion function as lattice solution – not exactly the same as the previously introduced dispersion trajectory for single elements

η(s) = used in some literature instead dispersion function D(s)

Phase Space Ellipse



$$x(s) = \sqrt{2J\beta}\cos(\varphi)$$

$$x'(s) = -\sqrt{\frac{2J}{\beta}}\left(\alpha\cos(\varphi) + \sin(\varphi)\right)$$

J = particle action (oscillation amplitude)

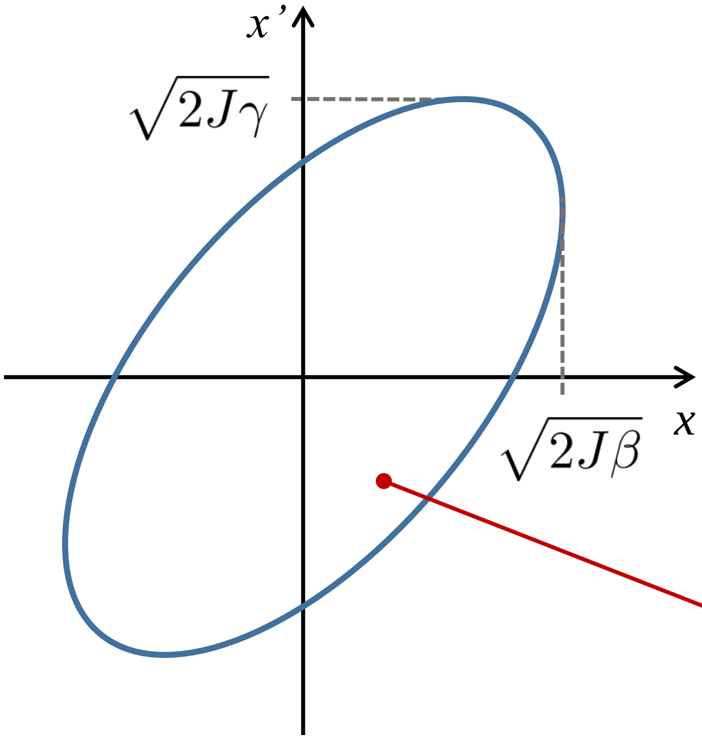
observing the coordinates of a particle at one location in a ring for consecutive turns

x, x' describe an ellipse in phase space when φ is varied

example $\varphi = Q \times 2\pi \approx 0.37 \times 2\pi$ phase advance per turn

see also Schmüser/Rossbach sect. 4.3/4.4, Wiedemann chap. 8.1.2

Courant-Snyder Invariant



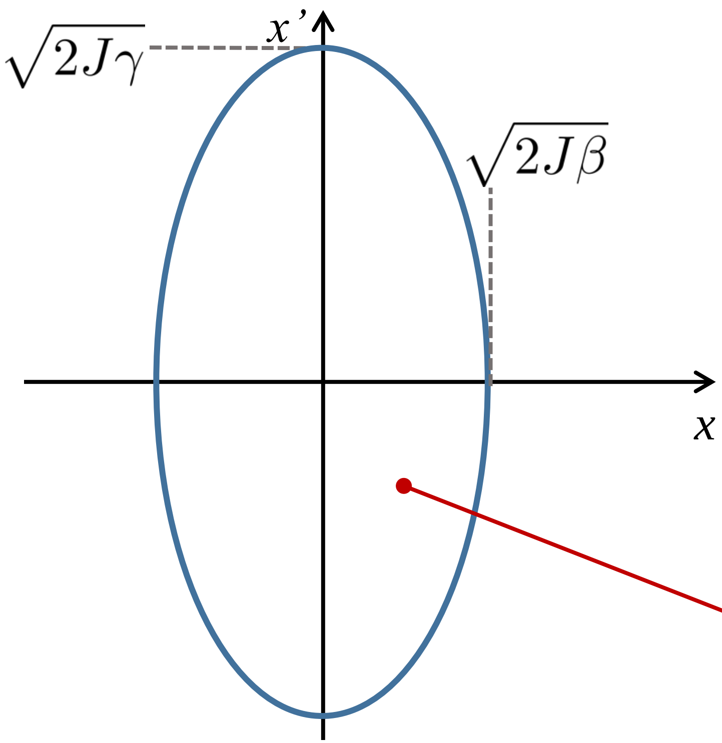
- at one location this phase space ellipse is sampled by a particle with action J
- when moving along the magnet lattice, the ellipse is changing size, but it stays an ellipse
- the area of the ellipse is invariant (Courant-Snyder Invariant) and equals 2π × action.
- the unit is [m·rad], or often [mm·mrad]
- this constant of motion refers to a **single particle**, **emittance** is a statistical property

$$\text{area} = 2\pi J = \pi(\gamma x^2 + 2\alpha x x' + \beta x'^2)$$

$$\text{with} : \beta\gamma - \alpha^2 = 1$$

E.D. Courant and H.S. Snyder, Theory of the Alternating Synchrotron, Annals of Physics 3 (1958) 1

Courant-Snyder Invariant



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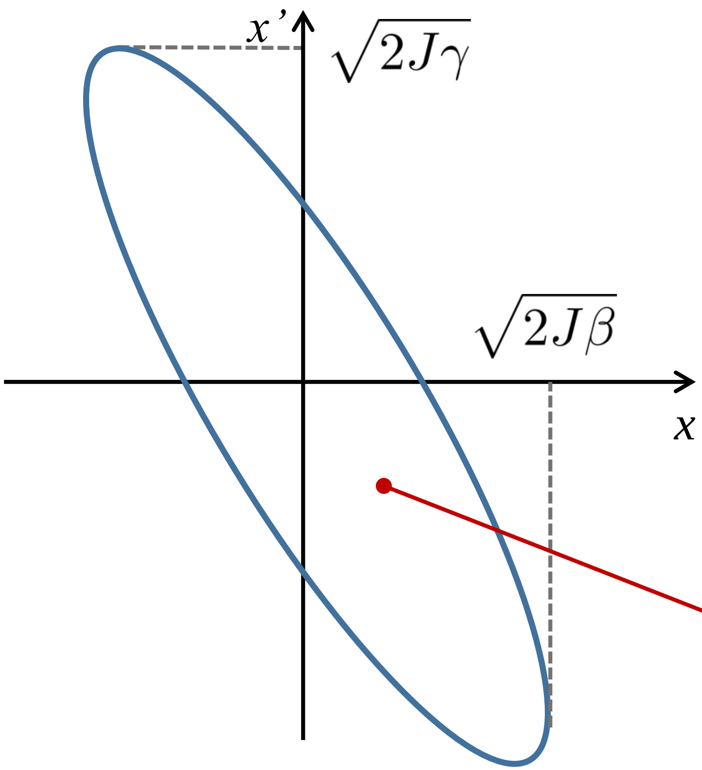

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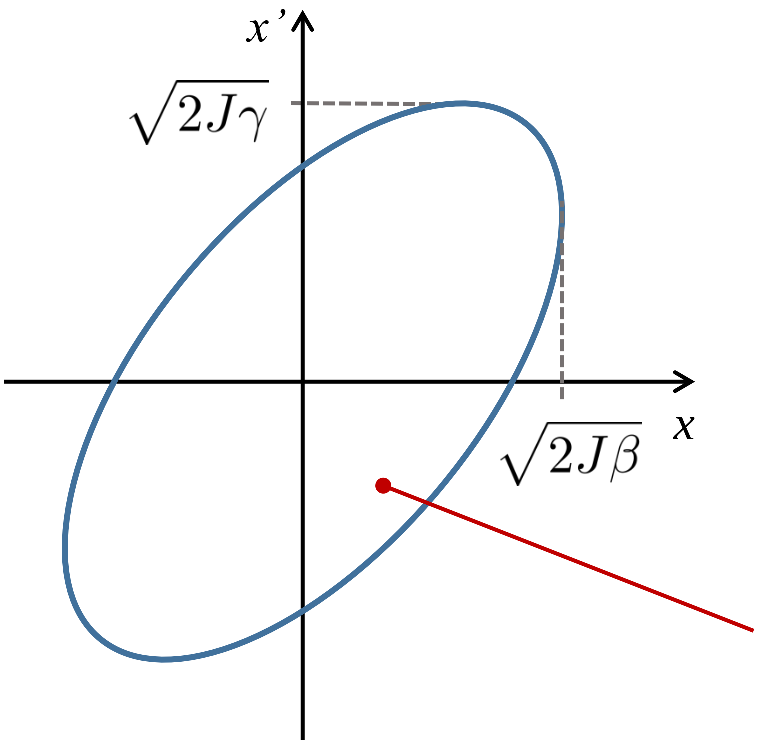

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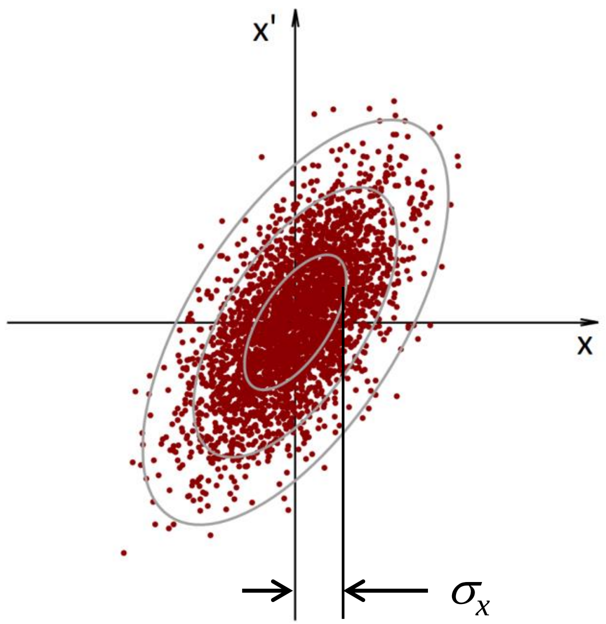
Action vs. Emittance



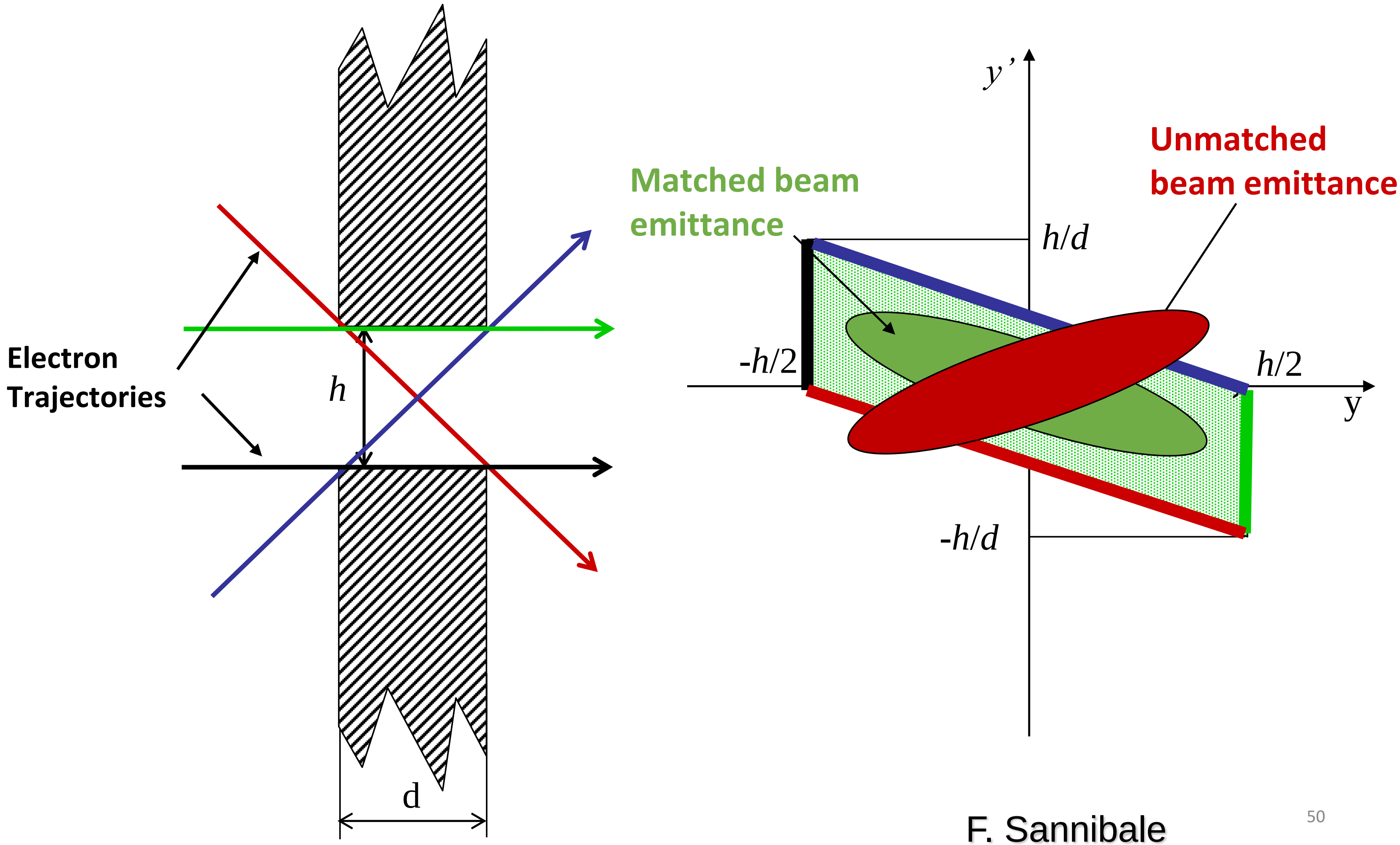
- single particles are associated with a particular ellipse
- when observing a beam with a wire scanner a (projected) rms width σ_x is measured; one ellipse corresponds to σ_x
- emittance ε is the average value of action *J*

$$\boxed{\varepsilon = \langle J \rangle}$$

$\rightarrow$ more on emittance next lesson

What beam will fit into the machine?

Example: Acceptance of a lengthy aperture restriction (limits amplitude and angle)



F. Sannibale

Admittance of an accelerator

admittance = the largest ellipse that can be accepted

- at any point along the accelerator
- units = phase space area [mm·mrad]
- if the available half-aperture is *d(s)*
  there is at some *s* a minimum of $\frac{d(s)}{\sqrt{\beta(s)}}$
- so in general: $$\boxed{\mathrm{Admittance} = \left(\frac{d^2}{\beta}\right)_{\min}}$$
- Special case: uniform aperture $d(s) = d$
  the minimum occurs at β_{max}

$$\mathrm{Admittance} = \frac{d^2}{\beta_{\max}}$$

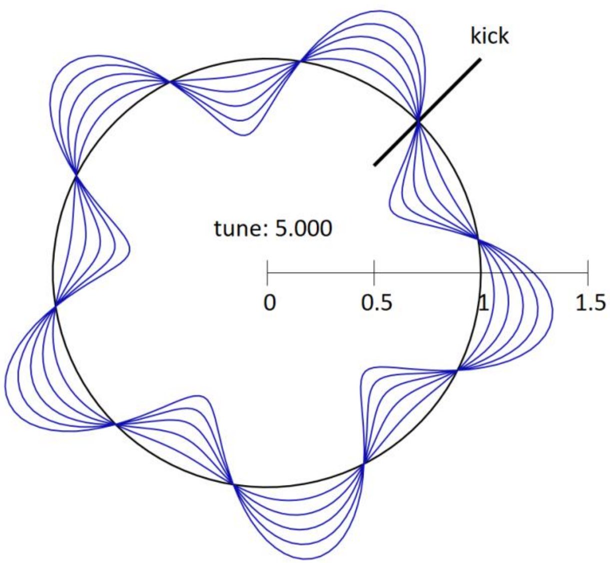
The Betatron Frequency Q (tune of accelerator)

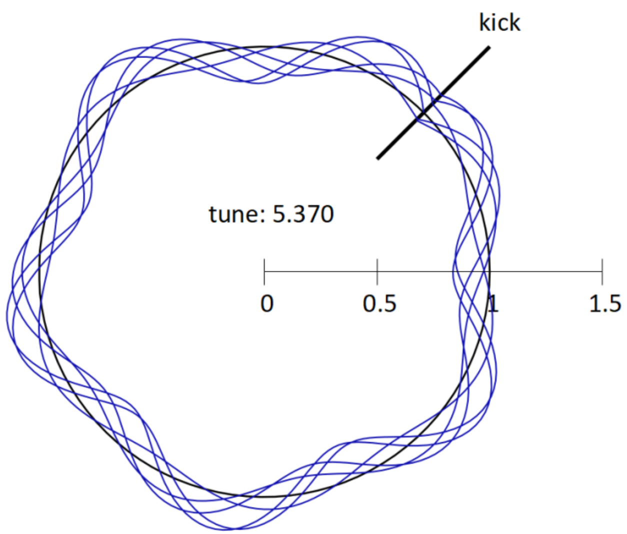
$$Q_x = \frac{1}{2\pi} \oint \frac{ds}{\beta_x(s)}$$

around ring

Tune = Number of Betatron Oscillations per Turn (remember Q≈1 for purely weak focusing) the choice of tune is important to avoid resonances

Both integer and fractional part are important for machine design

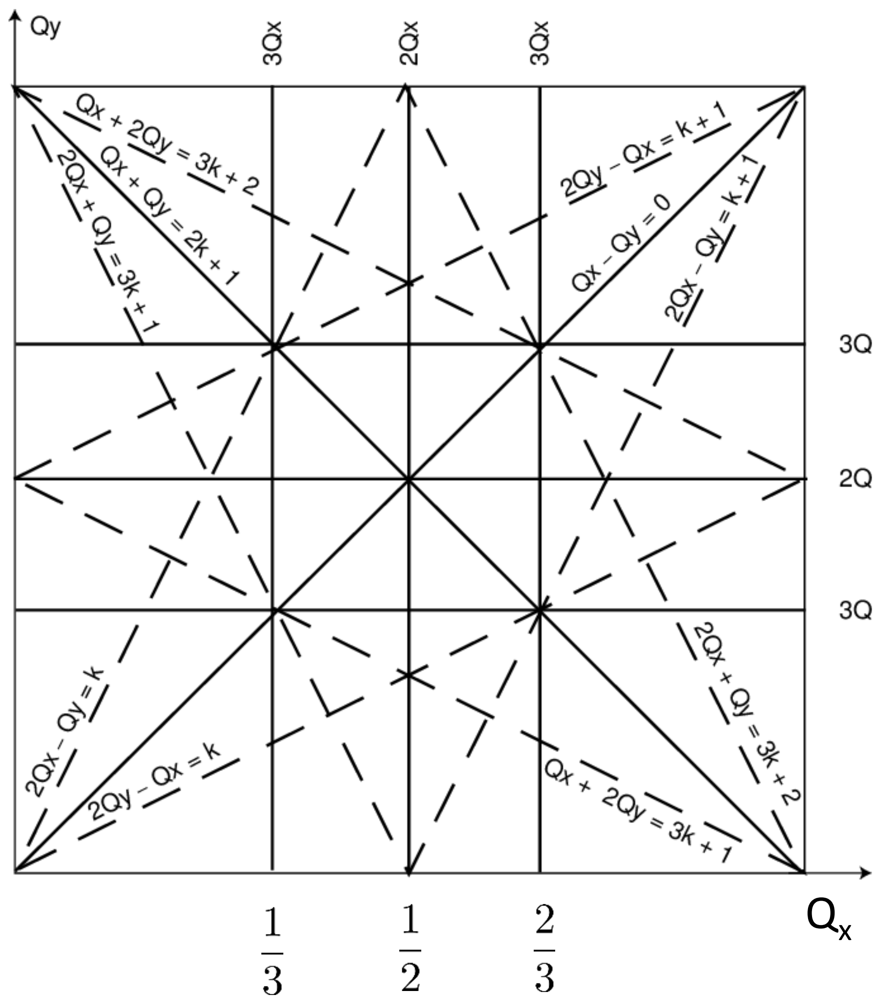




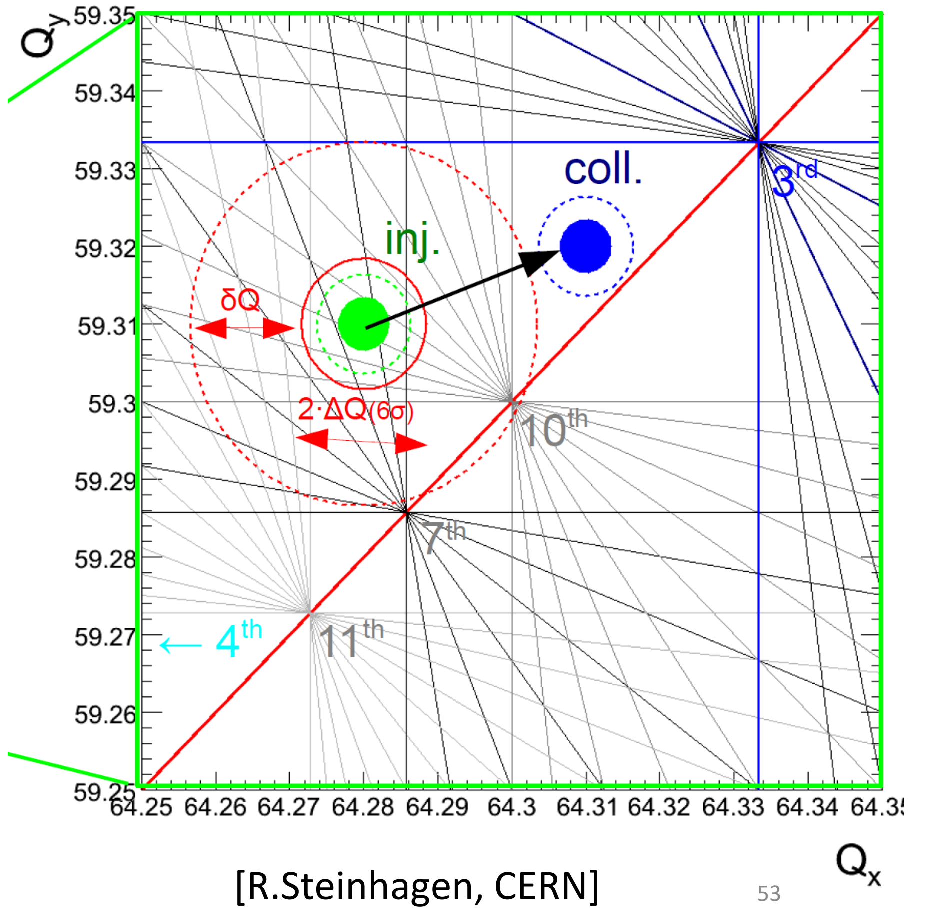
Resonance Plot

condition for resonance: $j \cdot Q_x + k \cdot Q_y = n$

order of resonance: $|j| + |k|$



LHC working point



[R.Steinhagen, CERN]

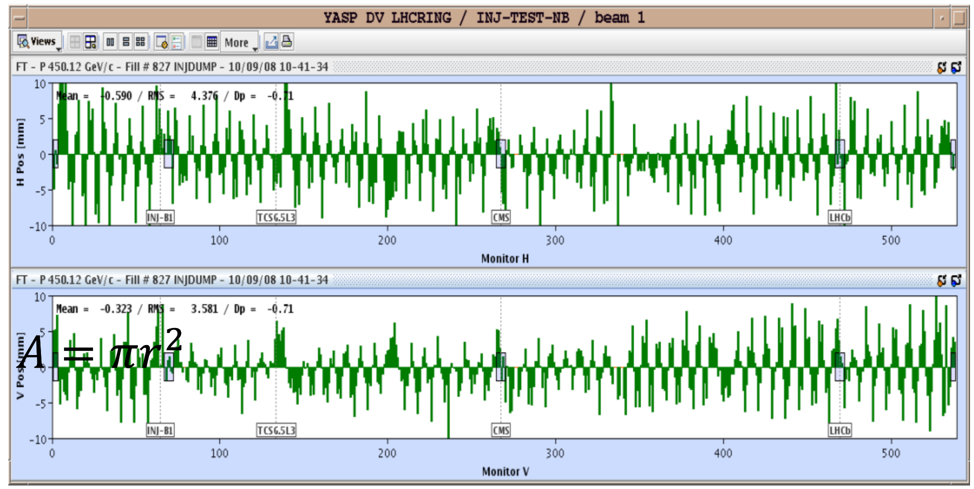
Tunes and Orbit in LHC

Example Orbit Oscillations:

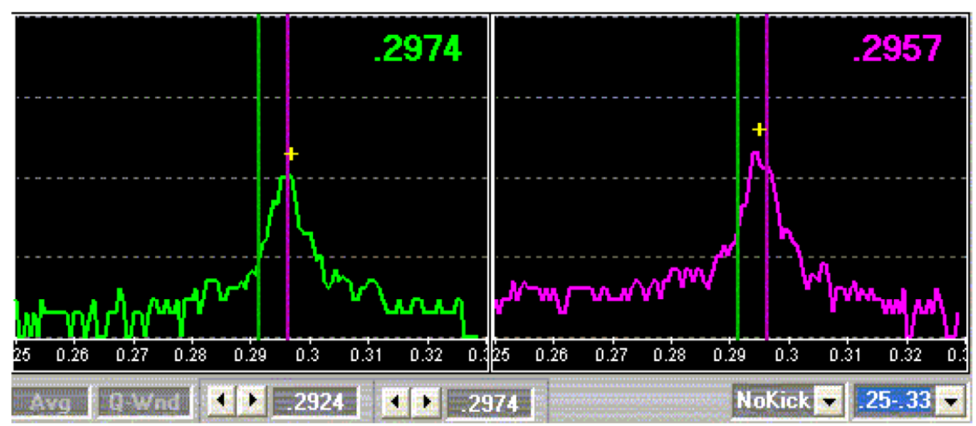
LHC Tunes:

Q_x = 62.31
Q_y = 60.32

relevant for stability: non-integer part



Example Measured Beam Spectrum:



LHC Revolution Frequency: 11.3kHz
peak position: 3.5kHz = 0.31×11.3kHz

$$\frac{oscillations}{second} = \frac{oscillations}{revolution} \times \frac{revolution}{second}$$

Smooth Approximation

$$x(s) = A\sqrt{\beta(s)}\cos(\varphi(s) - \varphi_0), \;\; \varphi(s) = \int_{t=s_0}^{s} \frac{dt}{\beta(t)}$$

simplify: $\beta_{\mathrm{avg}} = \langle \beta(s) \rangle = \mathrm{const}$

$$x(s) \approx A\sqrt{\beta_{\mathrm{avg}}}\cos\left(\frac{s}{\beta_{\mathrm{avg}}} - \varphi_0\right), \;\; x'' + K_{\mathrm{eff}} x = 0$$

can be used to estimate important parameters:

$$\boxed{K_{\mathrm{eff}} = \frac{1}{\beta_{\mathrm{avg}}^2}}$$

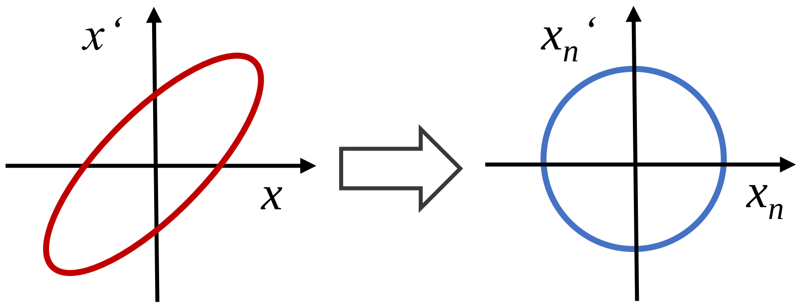
$$\boxed{Q = \frac{1}{2\pi} \oint \frac{ds}{\beta_{\mathrm{avg}}} = \frac{R}{\beta_{\mathrm{avg}}}}$$

note: $Q \propto R$, i.e. proportional to size
compare cyclotron: $Q \propto \gamma$, independent of size!

Normalized Coordinates

in normalized coordinates the particle describes a circle in phase space

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = \boldsymbol{T} \begin{pmatrix} x \\ x' \end{pmatrix}$$



$$x_n(s) = \frac{1}{\sqrt{\beta(s)}} x(s)$$

unit of normalized coordinates: $\sqrt{\text{length}}$

$$x'_n(s) = \frac{\alpha(s)}{\sqrt{\beta(s)}} x(s) + \sqrt{\beta(s)} x'(s)$$

$\rightarrow$ this re-scaling of coordinates is related to the Floquet theorem (see appendix)

Next: Summary of this Lesson

Summary Linear Beam Dynamics I

1) Matrix Treatment

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} + \frac{\Delta p}{p_0} \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

2) Analytical Treatment

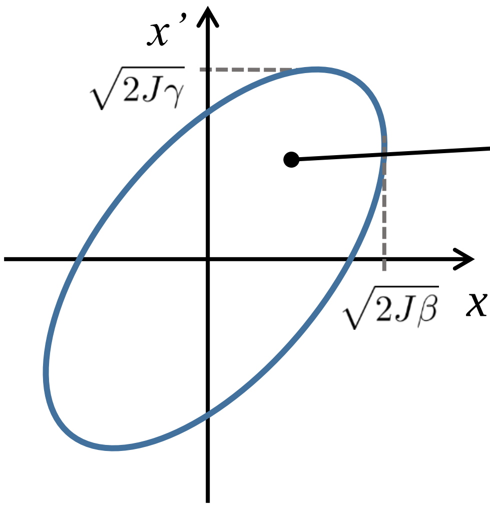
varies with *s*

off-momentum part

$$x(s) = A\sqrt{\beta(s)}\cos(\varphi(s)) + D(s)\frac{\Delta p}{p_0}$$

$$x'(s) = -\frac{A}{\sqrt{\beta(s)}}\left(\alpha\cos(\varphi(s)) + \sin(\varphi(s))\right) + D'(s)\frac{\Delta p}{p_0}$$

Summary Beam Dynamics - I



$$\mathrm{area} = 2\pi J = \pi(\gamma x^2 + 2\alpha x x' + \beta x'^2)$$

phase space area is preserved

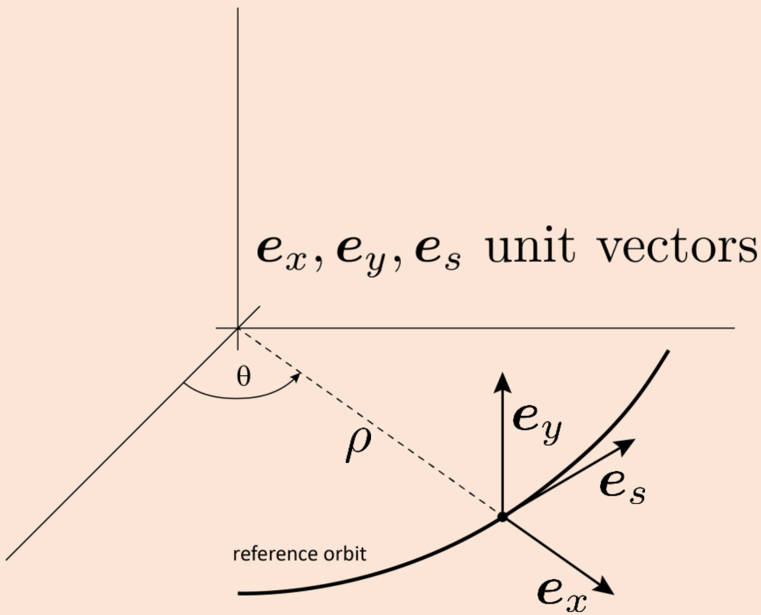
single particle vs. emittance

tune = number of oscillations per turn:

$$Q_x = \frac{1}{2\pi} \oint \frac{ds}{\beta_x(s)}$$

Appendix, Derivation: Equation of Motion I

starting with general equation of motion: $$\frac{d\vec{p}}{dt} = \gamma m_0 \ddot{\vec{R}} = \vec{F}$$



$$\begin{aligned}
\vec{R} &= r\boldsymbol{e}_x + y\boldsymbol{e}_y, \;\; r \equiv \rho + x \\
\dot{\vec{R}} &= \dot{r}\boldsymbol{e}_x + r\dot{\boldsymbol{e}}_x + \dot{y}\boldsymbol{e}_y \\
\dot{\vec{R}} &= \dot{r}\boldsymbol{e}_x + r\dot{\theta}\boldsymbol{e}_s + \dot{y}\boldsymbol{e}_y \\
\ddot{\vec{R}} &= \ddot{r}\boldsymbol{e}_x + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_s + r\dot{\theta}\dot{\boldsymbol{e}}_s + \ddot{y}\boldsymbol{e}_y \\
\ddot{\vec{R}} &= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_x + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_s + \ddot{y}\boldsymbol{e}_y
\end{aligned}$$

used here: $\dot{\boldsymbol{e}}_x = \dot{\theta}\boldsymbol{e}_s, \;\; \dot{\boldsymbol{e}}_s = -\dot{\theta}\boldsymbol{e}_x$

comment: the main purpose here is to correctly treat the effect of the curved coordinate system, i.e. the moving unit vectors e_x, e_s

Derivation: Equation of Motion II

right side of equation, the force:

$$\vec{F} = e\vec{v} \times \vec{B}$$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_s \\ v_x & v_y & v_s \\ B_x & B_y & 0 \end{vmatrix}$$

$$= -v_s B_y \boldsymbol{e}_x + v_s B_x \boldsymbol{e}_y + (v_x B_y - v_y B_x)\boldsymbol{e}_s$$

assumptions:
- no B_s
- $B_x(y=0) = 0$

use:

$$B_y = B_0 + gx$$
$$B_x = gy$$

$$g \equiv \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

in literature g has varying sign conventions
Wiedemann, Table 6.2: $g=+dB_y/dx$
Schmüser/Hillert: $g= -dB_y/dx$

result: two equations hor/vert from *x*,*y* components:

$$\gamma m_0(\ddot{r} - r\dot{\theta}^2) = -ev_s(B_0 + gx)$$
$$\gamma m_0 \ddot{y} = ev_s gy$$

Derivation: Equation of Motion III

introduce path length *s* as independent variable:

$$\begin{aligned}\gamma m_0(\ddot{r} - r\dot{\theta}^2) &= -ev_s(B_0 + gx)\\ \gamma m_0 \ddot{y} &= ev_s g y\end{aligned}$$

$$\begin{aligned}x'' &= \frac{1}{r} - \frac{e}{\gamma m_0 v}(B_0 + gx)\\ y'' &= \frac{e}{\gamma m_0 v} g y\end{aligned}$$

use:

$$v_s = r\dot{\theta} \approx v$$

$$\ddot{r} = \ddot{x}$$

$$\ddot{x} = v^2 x'', \;\; x'' \equiv \frac{\partial^2 x}{\partial s^2}$$

$$\ddot{y} = v^2 y'', \;\; y'' \equiv \frac{\partial^2 y}{\partial s^2}$$

Derivation: Equation of Motion IV

$$\begin{aligned} x'' &= \frac{1}{r} - \frac{e}{\gamma m_0 v}(B_0 + gx) \\ y'' &= \frac{e}{\gamma m_0 v} gy \end{aligned}$$

$$\begin{aligned} x'' &= \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) - kx - \frac{1}{\rho\left(1 + \frac{\Delta p}{p_0}\right)} \\ &= -\left(\frac{1}{\rho^2} + k\right)x + \frac{1}{\rho}\frac{\Delta p}{p_0} \\ y'' &= ky \end{aligned}$$

use:

$$\frac{1}{r} = \frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

$$\frac{eB_0}{\gamma m_0 v} = \frac{eB_0}{p} = \frac{1}{\rho}$$

$$p = p_0\left(1 + \frac{\Delta p}{p_0}\right)$$

$$k = \frac{eg}{\gamma m_0 v}$$

Floquet Theorem :: related to normalized coordinates

$$\boxed{x'' + K(s)x = 0, \quad \text{with } K(s+L) = K(s)}$$

has two solutions in the form:

$$x_1(s) = \exp(+i\mu s/L)\, p_1(s), \;\; x_2(s) = \exp(-i\mu s/L)\, p_2(s)$$

with:

$$p_i(s+L) = p_i(s)$$ [envelope functions]

for accelerators we note:

$$\cos\mu = \frac{1}{2}\mathrm{trace}\boldsymbol{M}$$

see also Schmüser/Rossbach sect. 4.2, Wiedemann chap. 10.2.1

Applying Floquet to transport matrix

transport matrix M can be factorized in an „oscillatory“ and an „envelope“ part; oscillatory part describes a circle in normalised coordinates

$$\boldsymbol{M}(s) = \boldsymbol{T}^{-1} \cdot \boldsymbol{R} \cdot \boldsymbol{T}, \quad \boldsymbol{R}(\mu) = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

$$\boldsymbol{T} = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix}$$

then for *k* turns in accelerator (M = one turn matrix):

$$\begin{aligned} \vec{x}(s)_k &= \boldsymbol{M}^k \vec{x}_0 \\ &= (\boldsymbol{T}^{-1}\boldsymbol{R}\boldsymbol{T}) \cdot (\boldsymbol{T}^{-1}\boldsymbol{R}\boldsymbol{T}) \cdot (\dots) \cdot (\boldsymbol{T}^{-1}\boldsymbol{R}\boldsymbol{T}) \cdot \vec{x}_0 \\ &= (\boldsymbol{T}^{-1}\boldsymbol{R}^k\boldsymbol{T}) \cdot \vec{x}_0 \end{aligned}$$

with:

$$\boldsymbol{R}^k(\mu) = \boldsymbol{R}(k\mu)$$