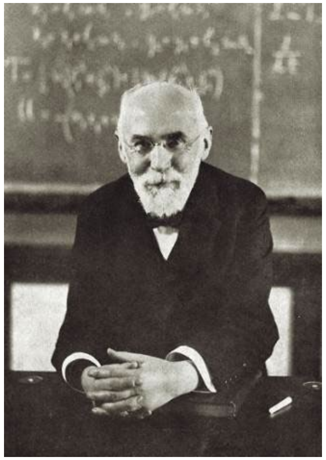
Longitudinal Dynamics

Laboratory for Particle Accelerator Physics, EPFL

Lorentz Force



H.A.Lorentz
1853-1928

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$$

electric field

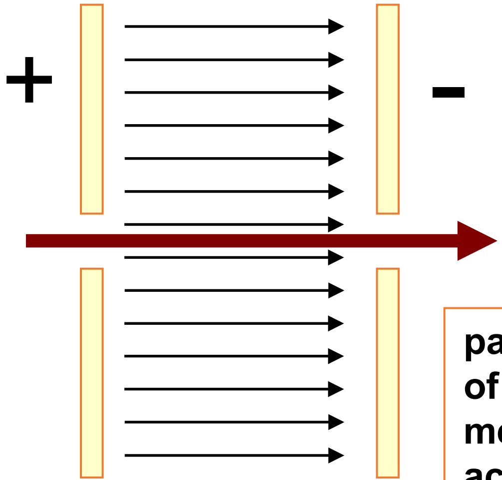
energy gain: $\Delta E_k = eU$

→ this lesson

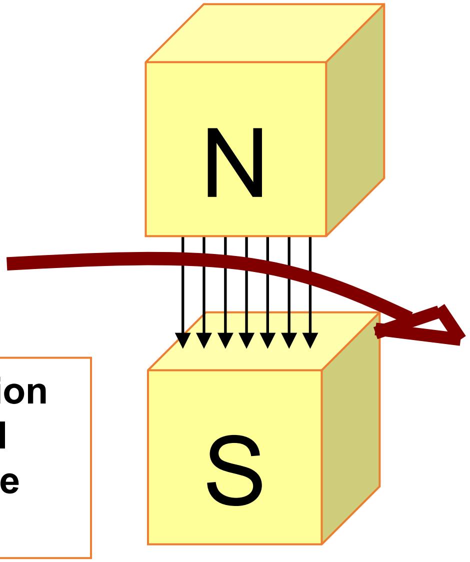
magnetic field

bending: $B\rho = p/e, \Delta E_k = 0$

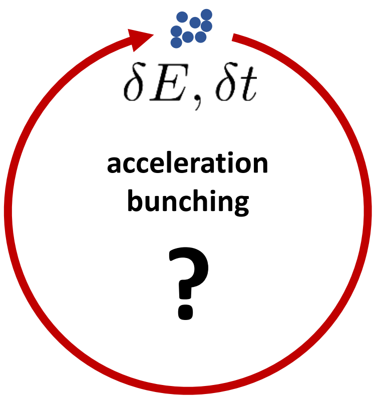
cyclotron frequency: $\omega_c = \dfrac{eB}{\gamma m_0}$



path bending and acceleration of particles are fundamental mechanisms of each particle accelerator



Introduction Longitudinal Dynamics



circular accelerators:

- cyclotron
- **synchrotron**

questions to be answered:

- How can a beam be accelerated continuously?
- For electrons: How can energy loss be compensated?
- What are conditions for longitudinal stability?
- How to calculate properties like bunch length, energy spread, oscillation frequency around synchronous particle?
- How can slow and fast particle beams be compressed longitudinally?

Reminder: particles and relativistic factors

particles with varying rest mass:

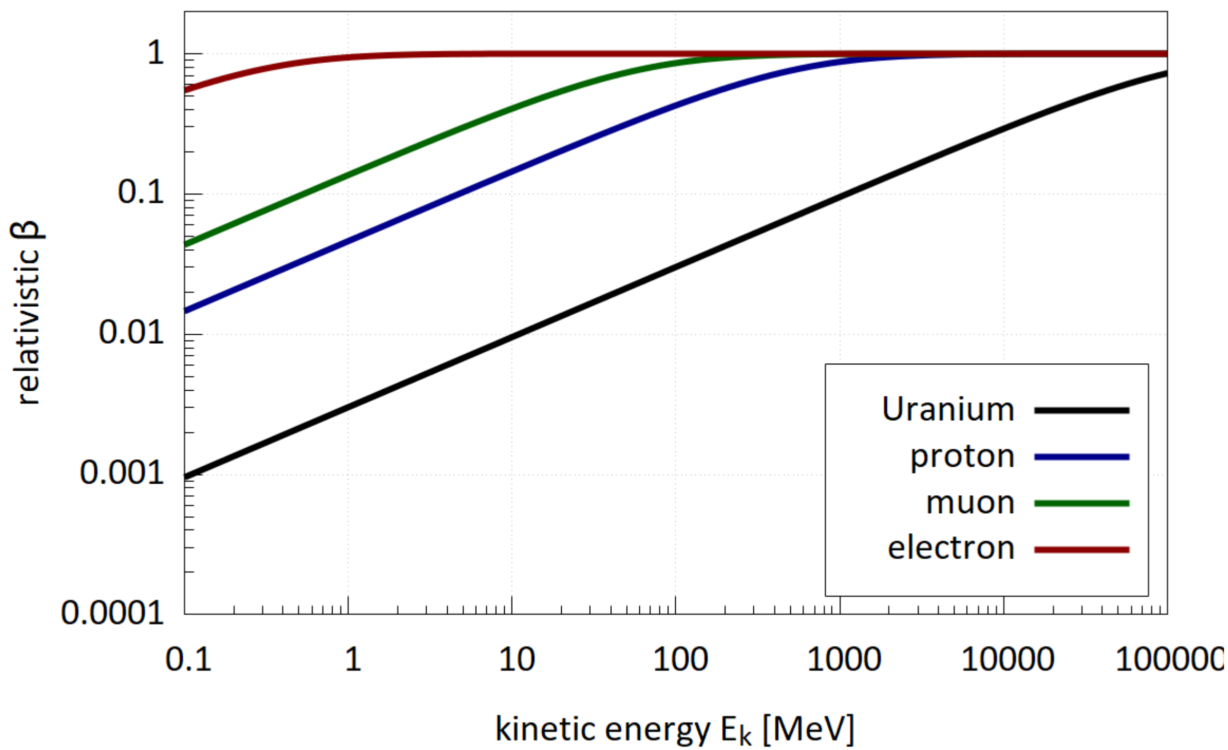
- electron: E_0 = 0.511 MeV
- muon: E_0 = 106 MeV
- proton: E_0 = 938 MeV
- uranium: E_0 = 220 GeV

$$E^2 = c^2p^2 + m_0c^2$$

$$E = \gamma \cdot m_0c^2$$

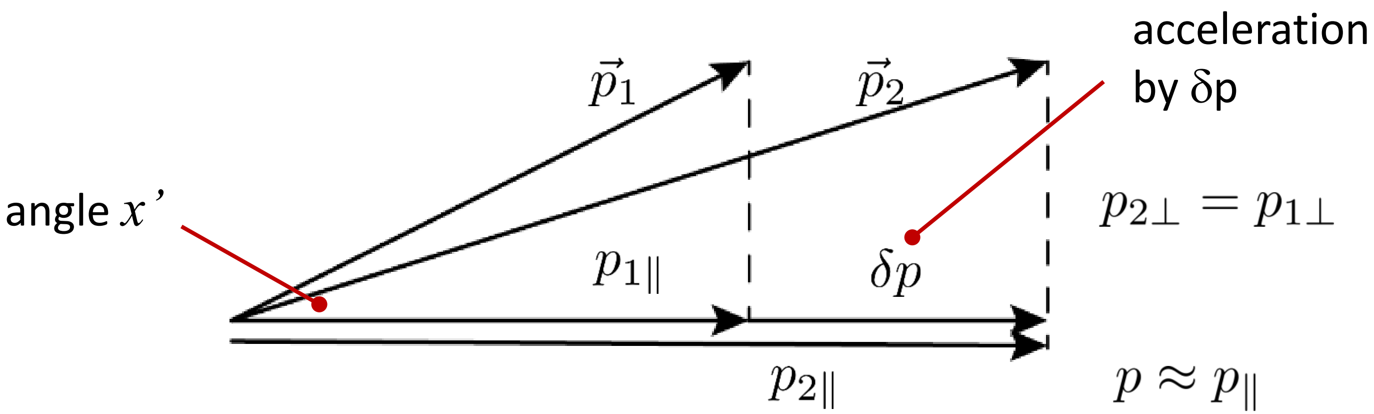
$$\gamma = 1 + \frac{E_k}{m_0c^2}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$



Impact of Acceleration on Transverse Oscillations

acceleration in the direction of the design orbit reduces transverse oscillations



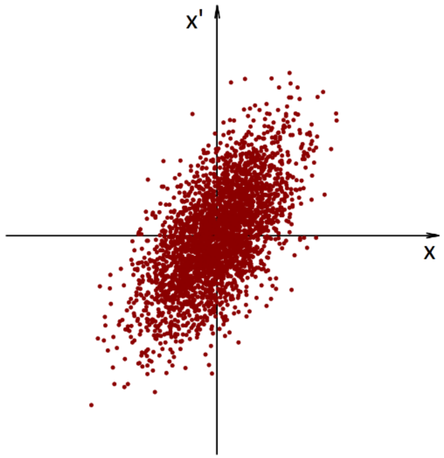
the angular deviation is reduced.

$$\boxed{\begin{gathered}\frac{p_{2\perp}}{p_2} < \frac{p_{1\perp}}{p_1} \\ x_2' < x_1'\end{gathered}}$$

$$\vec{x}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_1}{p_2} \end{pmatrix} \vec{x}_1 = \boldsymbol{R}_x \vec{x}_1$$

$$\mathrm{Det}\,\boldsymbol{R_x} < 1!$$

Adiabatic Damping



$$\Sigma_x = \begin{pmatrix} < x^2 > & < xx' > \\ < xx' > & < x'^2 > \end{pmatrix} = \langle \vec{x} \cdot \vec{x}^{\mathrm{T}} \rangle$$

emittance = statistical property of particle coordinates x, x':

$$\varepsilon_x = \sqrt{\det \Sigma_x}$$

after acceleration:

$$\Sigma_2 = \boldsymbol{R}_x \, \Sigma_1 \, \boldsymbol{R}_x^{\mathrm{T}}, \quad \boldsymbol{R}_x = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_1}{p_2} \end{pmatrix}$$

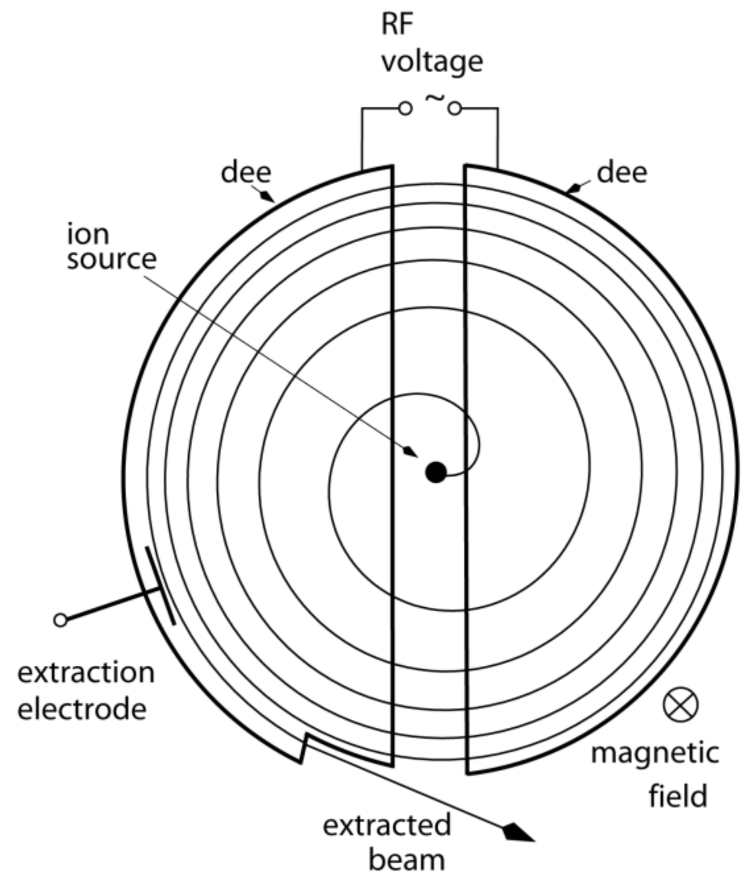
$$\varepsilon_2 = \sqrt{\mathrm{Det}\Sigma_2} = \frac{p_1}{p_2}\varepsilon_1 = \frac{\beta_1\gamma_1}{\beta_2\gamma_2}\varepsilon_1$$

The **normalised emittance** is invariant during acceleration: $\varepsilon_n = \beta\gamma\varepsilon$

see also Wiedemann eq. 10.146

Circular Accelerators: Classical Cyclotron

- two capacitive electrodes „Dees“, two gaps per turn
- internal ion source, homogenous B_y field
- **constant revolution time for low energy**



magnetic rigidity

$$\frac{\beta\gamma m_0 c}{e} = B_y R$$

$$\frac{\beta c}{R} = \omega_c = \frac{eB_y}{\gamma m_0}$$

cyclotron frequency: not exactly constant

Cyclotron: Isochronicity and Scalings

continuous acceleration → revolution time should stay constant, though E_k, R vary

magnetic rigidity: $BR = \frac{1}{e}p = \beta\gamma\frac{m_0 c}{e}$

orbit radius from isochronicity: $R \propto \beta$

deduced scaling of *B*: $\rightarrow B(R) \propto \gamma(R) \rightarrow \frac{dB}{dR} > 0$

needs $\frac{dB}{dR} < 0$

but focusing! → $$\nu_r^2 = 1 + \frac{R}{B}\frac{dB}{dR}, \quad \nu_z^2 = -\frac{R}{B}\frac{dB}{dR}$$

[radial, vertical tunes in cyclotron, without proof]

two solutions to overcome energy limitation:

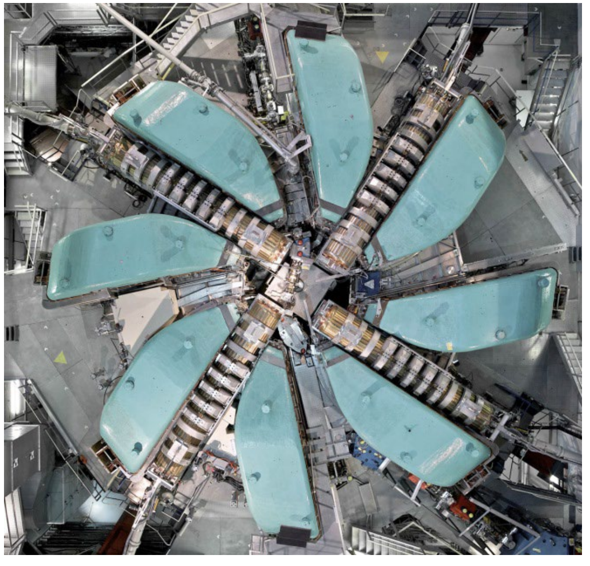
isochronous cyclotron, $B_{avg} \propto \gamma$

synchro-cyclotron, ω_{RF} ramps down

Summary Cyclotrons

longitudinal dynamics:

- isochronous cyclotron ensures constant circulation time throughout acceleration process; RF frequency fixed, but field more complicated
- for synchrocyclotron RF frequency must scale down in certain relation to B field; gain: simple magnet; loss: low intensity
- cyclotron = single pass machine, no longitutinal oscillations, few hundred turns only



PSI isochronous Ring Cyclotron, 590MeV
∅ 15m

vertical focusing through spiral sectors

Next: Synchrotron

- phase stability and strong focusing
- RF cavity, harmonic number
- comparison of circular accelerators

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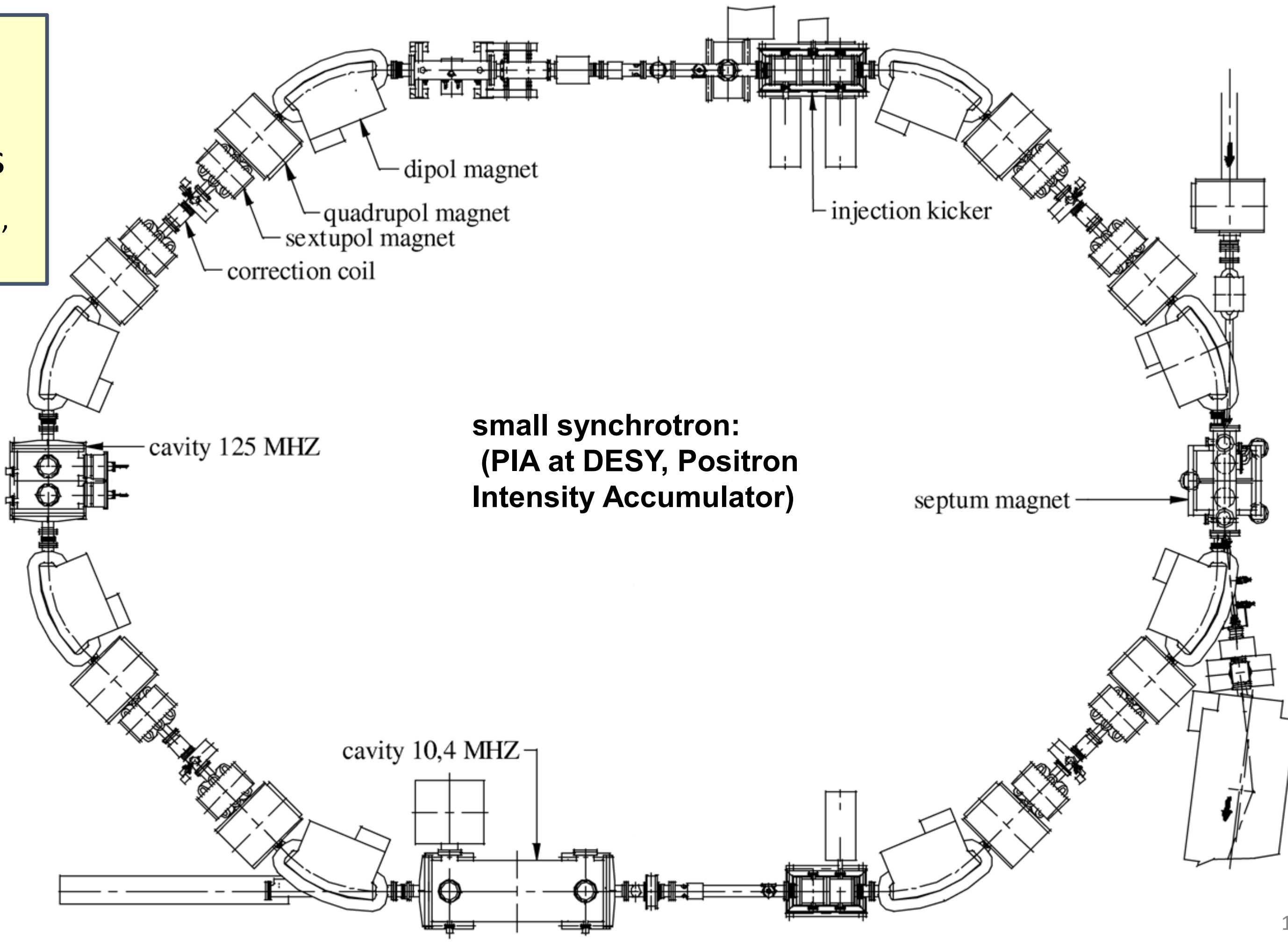
Synchrotron

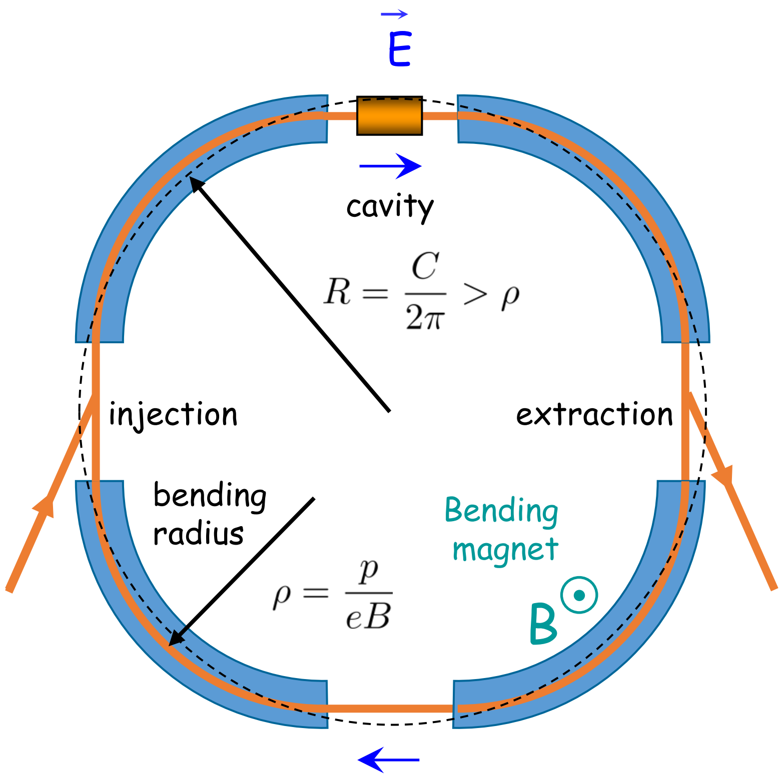
- McMillan (USA) and Veksler (UdSSR) independently in 1945
- concept of phase stability (longitudinal focusing) and alternating gradient focusing (lessons on linear dynamics), magnets are „ramped“ with the particles energy

- RF system
- dipoles
- quadrupoles

(+ sextupoles, injection, extraction …)



Synchrotron

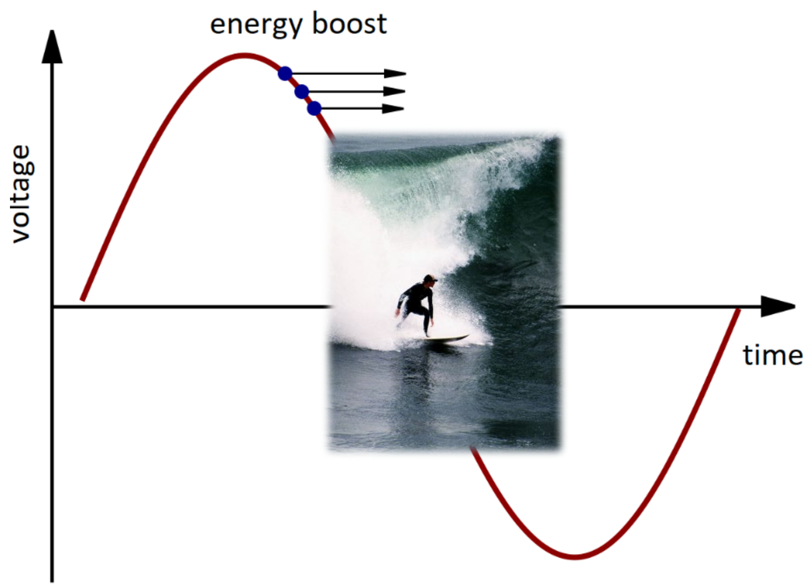


- particles are synchronous with RF wave on every turn
- energy loss per turn (electrons, SR) is compensated in cavity
- during acceleration the required increment of energy per turn is provided by RF
- deviating particles are focused back / are oscillating around the synchronous particle
- for v≈c the RF frequency can be constant

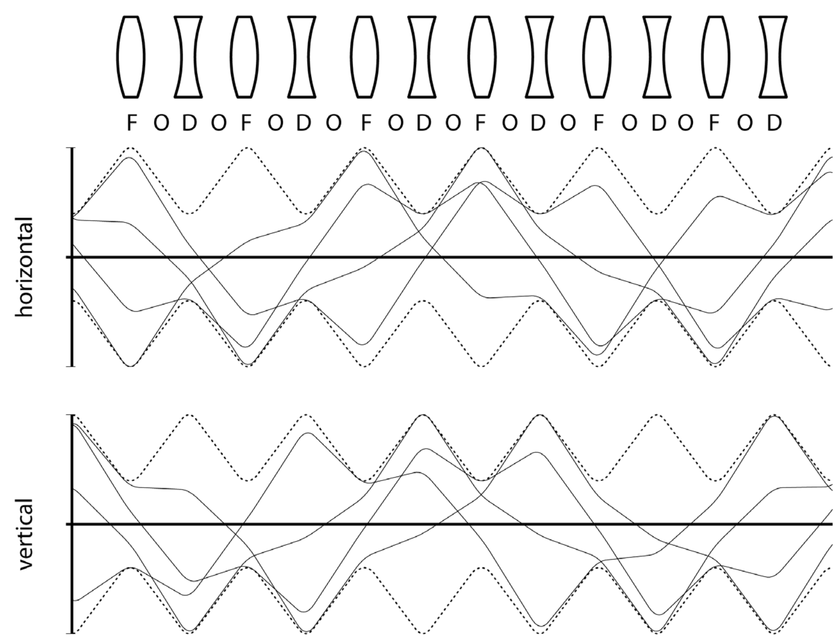
R = average bending radius
R > ρ due to straight sections

Synchrotron: Phase Stability & Strong Focusing

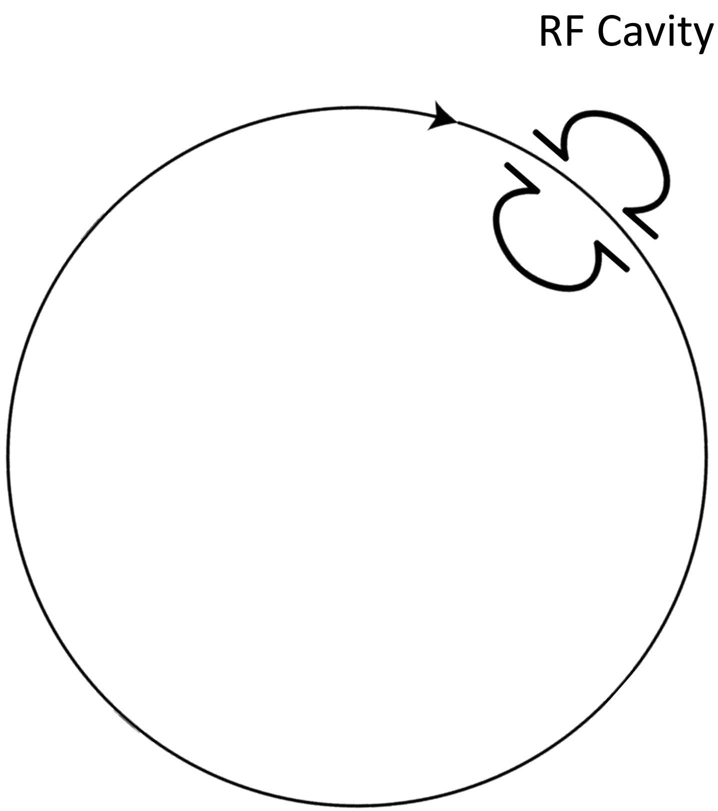
phase stability:
“surfing on the RF-wave” including restoring force; constant bending radius!



strong focusing:
alternating gradient, beam size independent of machine size



Acceleration in a Radio-Frequency (RF) Cavity

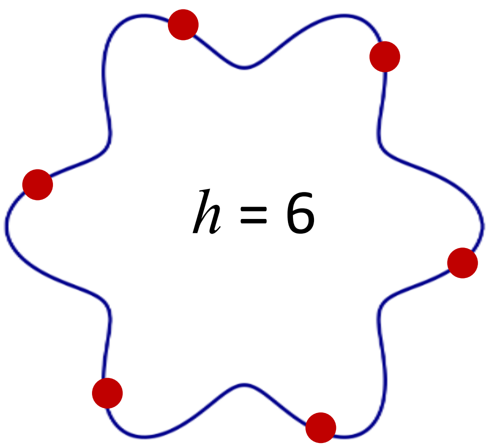


$$V(t) = \hat{V} \sin(\omega_{\mathrm{rf}} t)$$

$$\Delta E = e \cdot V(t)$$

- cavity voltage varies harmonically
- the energy gain of a particle depends on the arrival time
- the RF frequency must be synchronized with the revolution time of the particle
- multiple cavities at same frequency act like one strong cavity

Harmonic Number in a Ring



the RF frequency must be a multiple of the revolution frequency

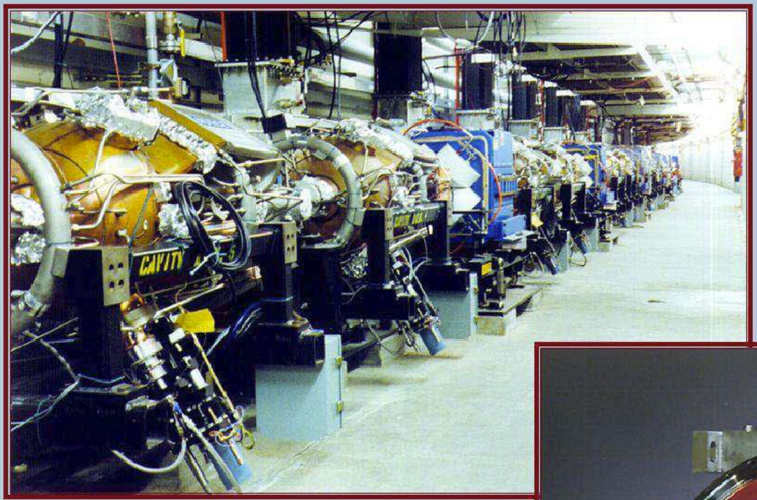
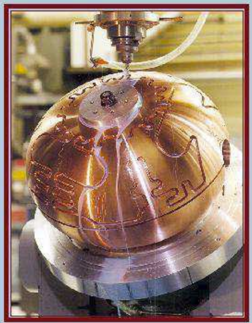
[imagine a toothed wheel with integer number of buckets in which the particles are sitting]

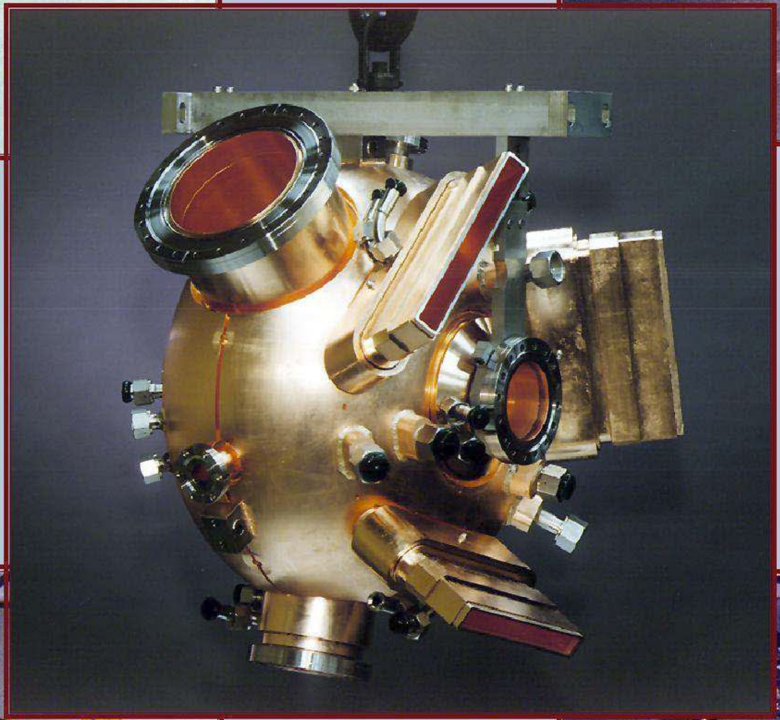
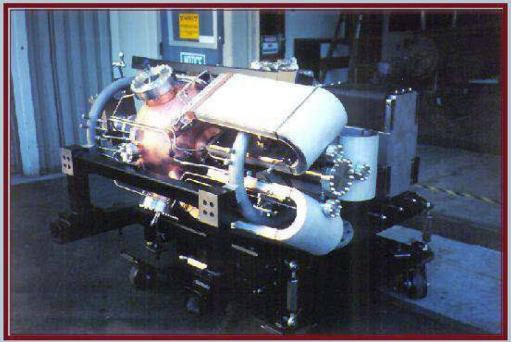
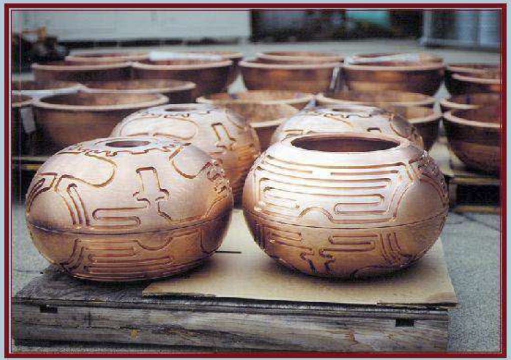
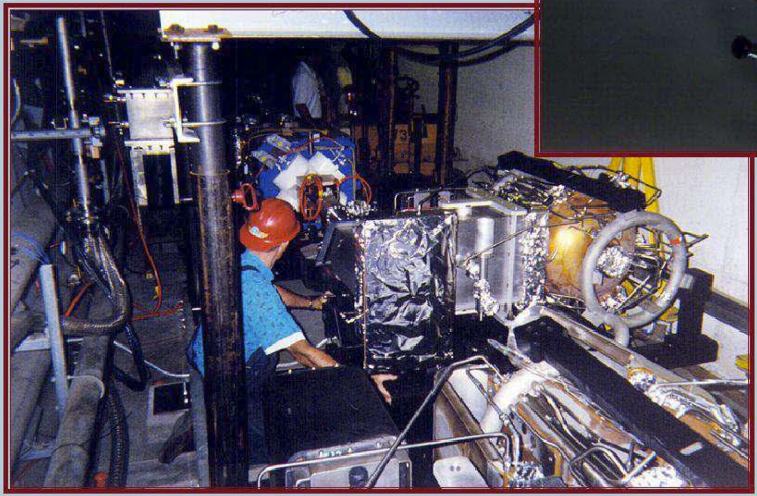
the ***harmonic number h*** denotes the number of RF buckets for ring and is an integer number

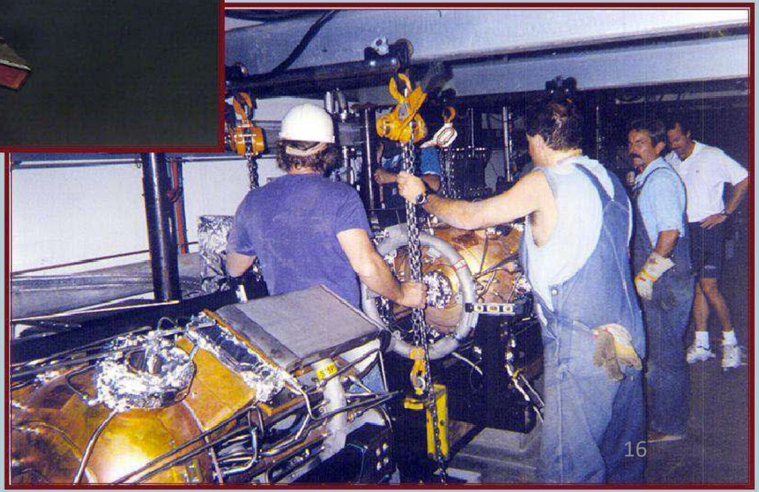
$$\boxed{\omega_{\mathrm{rf}} = h \cdot \omega_{\mathrm{rev}}}$$

see also Wiedemann sec. 3.4.5

PEP-II Ring Cavities [J.Seeman, SLAC et al]

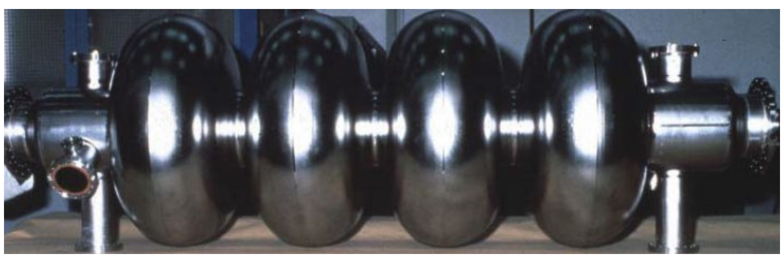





Examples superconducting multi-cell cavities for Rings

DESY/HERA, 500MHz



CERN/LEP, 352MHz

low losses allow continuous operation
$Q_0 \sim 10^{10}$

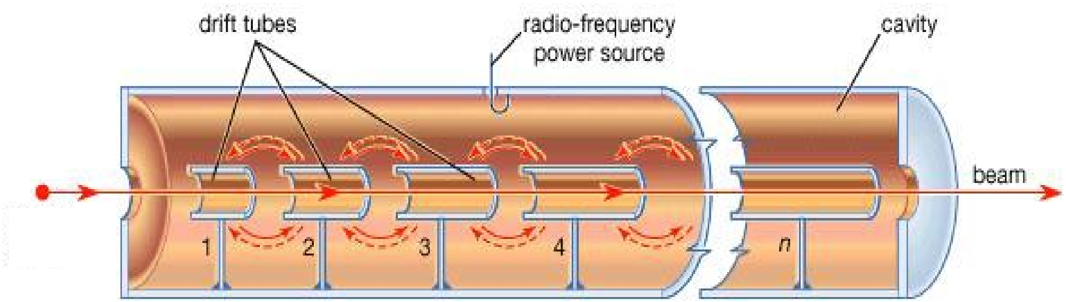
classification of circular accelerators

	bending radius vs. time	bending field vs. time	bending field vs. radius	RF frequency vs. time	operation mode (pulsed/CW)	
isochronous cyclotron	↗	→	↗	→		suited for high power!
synchro-cyclotron	↗	→	→ ↘	↘		higher E_k, but low P
synchrotron	→	↗		→ ↗		high E_k, strong focus

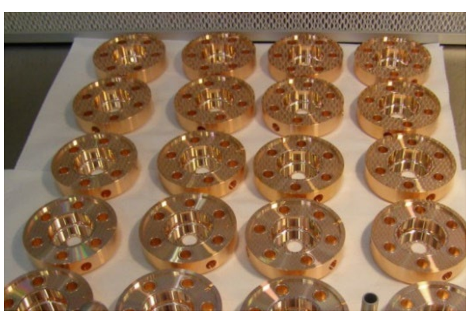
What is different in a linear accelerator?

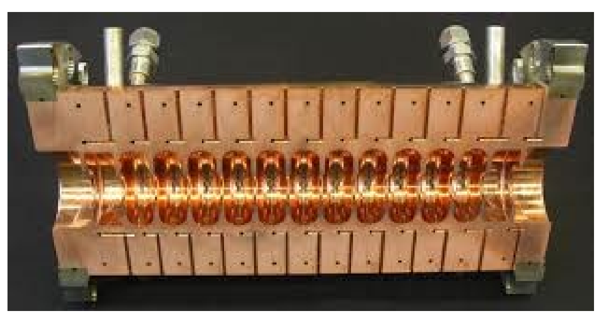
the beam is not recirculated, so there is no harmonic number; no dispersion $\rightarrow$ no path length change

however: the beam must still be in phase with the RF frequency when it travels along the linac and is accelerated



low β structure: drift length increases with speed





high β structure: fixed cell length for speed of light

Next: longitudinal motion in synchrotrons

- circulation time, momentum compaction, slippage factor
- stable and unstable motion
- synchrotron oscillations

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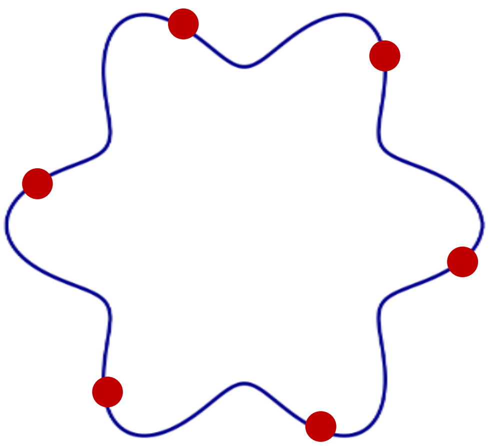
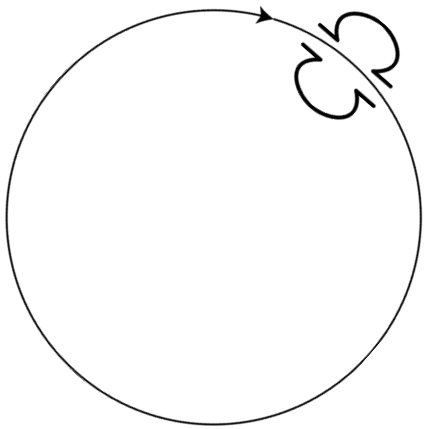
Protons and Acceleration of Particles

- only during acceleration **energy is transferred to the beam,** in storage mode **protons generate no synchrotron radiation**, i.e. no power is transferred from the RF system (term: “stationary bucket”)
- nevertheless the **RF system is needed to keep the beam bunched**
- with the RF system running, protons **are accelerated simply by ramping the magnets** – when moving to a smaller radius particles arrive at a different time, experience more voltage and accelerate to their old radius



when ramping magnets, particles accelerate automatically to stay in their fixed pattern of buckets

Acceleration of Electrons



$$V(t) = \hat{V}\sin(\omega_{\mathrm{rf}} t)$$

$$\Delta E = e \cdot V(t)$$

- electrons in a ring generate synchrotron radiation
- this energy loss must be compensated in the RF cavity by transferring the missing energy $\boldsymbol{U_0}$
- energy loss $\boldsymbol{U_0}$ by SR:

$$U_0 = C_\gamma \frac{E^4}{\rho}$$

$$U_0[\mathrm{GeV}] = 8.86 \cdot 10^{-5} \frac{E^4[\mathrm{GeV}^4]}{\rho[\mathrm{m}]}$$

extreme example:
LEP ring @ 104GeV
C = 27km, ρ = 3.100m, I_{beam} = 6mA
U_0 = 3.3 GeV, P_{RF} = 20MW

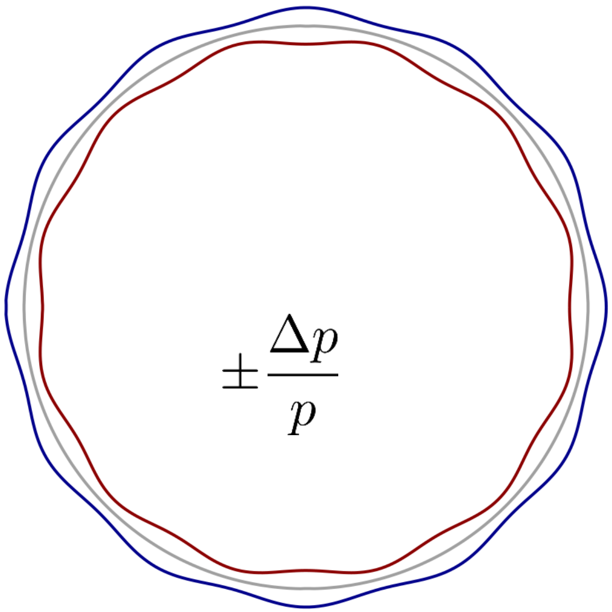
Circulation Time and Synchronous Particle

the **synchronous particle** arrives at the same phase ϕ_s at the accelerating civility during every turn

the circulation time τ of arbitrary particles varies due to:

1. varying velocity v
2. varying path length C

$$\frac{\Delta\tau}{\tau} = \frac{\Delta C}{C} - \frac{\Delta v}{v}$$



we use the momentum compaction factor α_c:

$$\frac{\Delta C}{C} = \alpha_c \frac{\Delta p}{p}, \quad \frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

time delay vs momentum deviation through slip factor η_c:

$$\frac{\Delta\tau}{\tau} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = \eta_c \frac{\Delta p}{p}$$

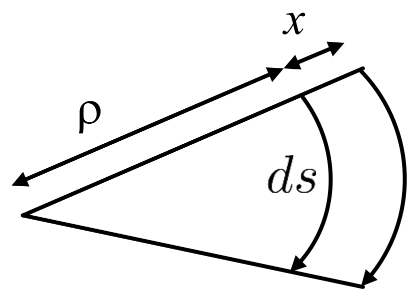
note definition of η_c in some literature with opposite sign!

reminder: Momentum Compaction

for an off-momentum particle the path length changes:

$$\Delta C = \oint \frac{x_D(s)}{\rho(s)} ds$$

$$x_D(s) = D(s)\frac{\Delta p}{p_0}$$



$$ds' = ds\left(1 + \frac{x_D}{\rho}\right)$$

momentum compaction factor α_c:

$$\boxed{\frac{\Delta C}{C} = \alpha_c \frac{\Delta p}{p}, \ \ \alpha_c = \frac{1}{C}\oint \frac{D(s)}{\rho(s)} ds}$$

$$\alpha_c \approx \frac{< D >}{R} = \frac{1}{Q_x^2}$$

Transition Energy

$$\frac{\Delta\tau}{\tau} = \underbrace{\left(\alpha_c - \frac{1}{\gamma^2}\right)}_{\eta_c} \frac{\Delta p}{p}$$

τ = circulation time
p = momentum

the α_c in this formula can be expressed by a transition energy:

$$\boxed{\frac{\Delta\tau}{\tau} = \left(\frac{1}{\gamma_{\mathrm{tr}}^2} - \frac{1}{\gamma^2}\right) \frac{\Delta p}{p}}$$

$$\frac{1}{\gamma_{\mathrm{tr}}^2} = \alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds$$

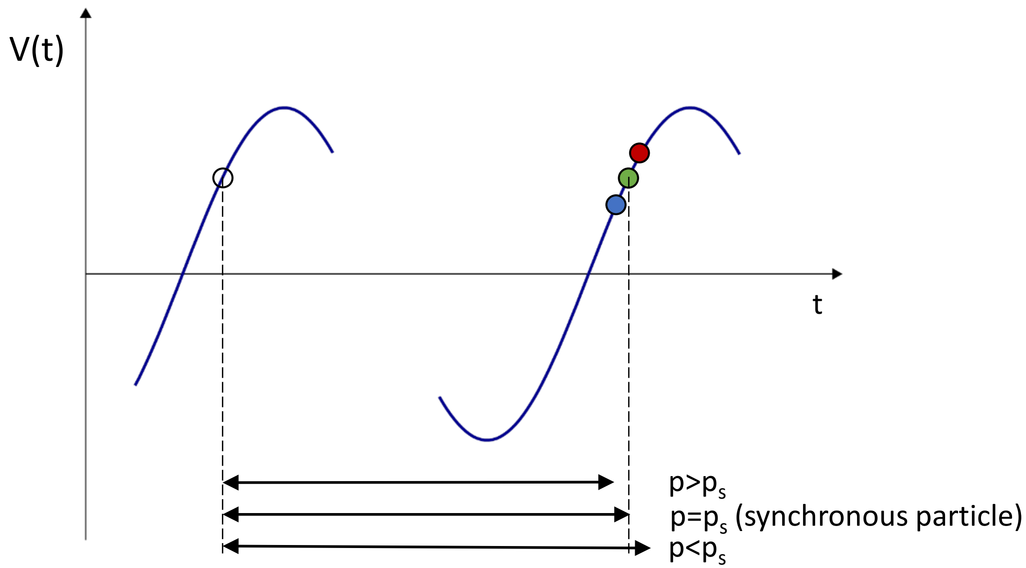
$$E_{\mathrm{tr}} = \gamma_{\mathrm{tr}} m_0 c^2$$

the **transition energy** is a parameter of the magnet lattice

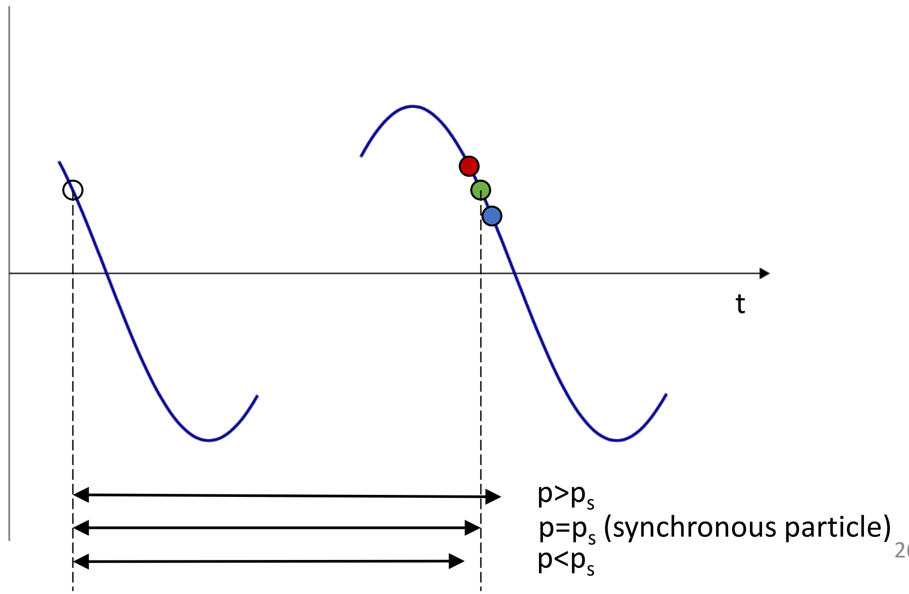
see also Wiedemann
sec. 9.2.2

Above or below transition ?

E<E_{tr}: with increasing energy circulation time is reduced (dominated by velocity)



E>E_{tr}: with increasing energy circulation time is increased (dominated by path length)



Transition Energy: Discussion

- electron rings operate above transition since electrons are quickly relativistic
- for proton and ion rings crossing transition during the acceleration process might be required; the phase has to be shifted to keep the particles stable
- crossing transition is possible since at η_c=0 the circulation time is independent of energy and the synchrotron oscillation stops

estimate of transition energy from smooth approximation:

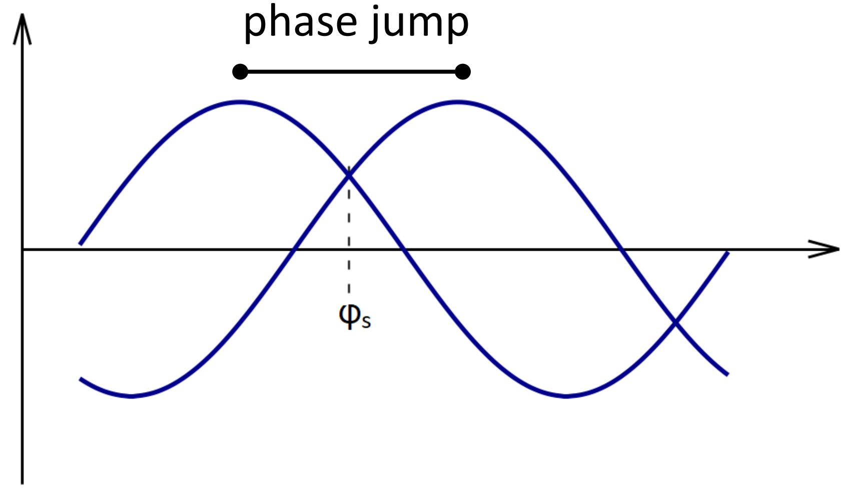
$$\alpha_c \approx \frac{1}{Q_x^2}, \ \gamma_t = \frac{1}{\sqrt{\alpha_c}} \approx Q_x$$

see also Wiedemann eq. 10.104

Crossing Transition

Synchrotron (all CERN)	E_{tr}	E_{inj}	E_{top}
PS	**6 GeV**	**1.4 GeV**	**27.7 GeV**
SPS	22.7 GeV	27.7 GeV	450 GeV
LHC	55 GeV	450 GeV	7 TeV

crossing required



when crossing transition energy the RF signal must be shiftet in phase: **„phase jump“**

Equation of Motion - Energy

energy change per turn for arbitrary particle (k = turn number):

$$E_n - E_{n-1} = \Delta E = e\hat{V}\sin\phi$$

$\hat{V}$ = acc. voltage

$E = \gamma m_0 c^2$ = total energy

we are interested in the deviation of the arb. particle from the synchronous particle δE:

$$\Delta(\delta E) = e\hat{V}(\sin\phi - \sin\phi_s)$$

comment: two “Deltas”:
Δ: change between turns
δ: difference to E_s

$$\frac{d(\delta E)}{dt} \approx \frac{\Delta(\delta E)}{\tau_s} = \frac{e\omega_{\text{rev}}}{2\pi}\,\hat{V}(\sin\phi - \sin\phi_s)$$

this is one equation **relating energy change and phase of particles**
$\rightarrow$ next we look at the phase change

Equation of Motion – Phase Change

phase change per turn for arbitrary particle:

$$\frac{d\phi}{dt} \approx \frac{\phi_n - \phi_{n-1}}{\tau_s} = h\,\omega_{\text{rev}}\,\frac{\Delta\tau}{\tau_s} \qquad \omega_{\text{RF}} = h\omega_{\text{rev}}$$

h = harmonic number
τ_s = circulation time

γ_{tr} = at transition energy

using the slip factor η_c:

$$\frac{d\phi}{dt} = h\,\omega_{\text{rev}}\eta_c\,\frac{\delta p}{p_s} = h\,\omega_{\text{rev}}\eta_c\,\frac{1}{\beta_s^2}\frac{\delta E}{E_s} \qquad \eta_c = \frac{1}{\gamma_{\text{tr}}^2} - \frac{1}{\gamma^2}$$

E_s = tot. energy synchr. p.
β_s = velocity synchr. p.

another derivative w.r.t. t and insertion in energy equation from previous slide yields equation of motion for phase of particle ϕ:

$$\boxed{\frac{E_s\beta_s^2}{h\eta\omega_{\text{rev}}^2}\cdot\frac{d^2\phi}{dt^2} = \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s)}$$

Solution for small oscillations

expand for small deviations from ϕ_s:

$$\Delta\phi = \phi - \phi_s \ll 1$$

$$\sin\phi = \sin(\phi_s + \Delta\phi)$$

$$\approx \sin\phi_s + \Delta\phi\cos\phi_s$$

results in harmonic oscillations with **synchrotron frequency** Ω:

$$\boxed{\begin{gathered}\frac{d^2\Delta\phi}{dt^2} + \Omega_s^2\Delta\phi = 0 \\ \Omega_s^2 = -\eta_c\cos\phi_s\frac{e\hat{V}h\omega_{\mathrm{rev}}^2}{2\pi E_s\beta_s^2}\end{gathered}}$$

→ $\Omega^2{}_s$ must be positive for stable solutions, depending on ϕ_s and sign of e

→ exponential growth for negative values

see also Wiedemann sec. 9.2.1

Discussion Synchrotron Oscillations

synchrotron tune =
synchrotron oscillations per turn

$$Q_s = \frac{\Omega_s}{\omega_{\mathrm{rev}}}$$

- synchrotron oscillations typically slower that transverse oscillations
- in electron rings higher voltages than in proton rings needed (energy loss), so often Q_s higher for electrons

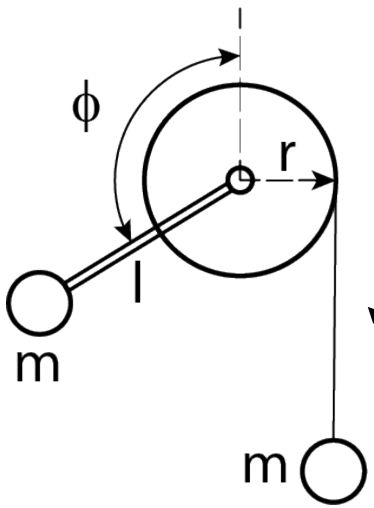
ring examples:

	f_{rev}	Q_x/Q_y	Q_s
SLS (288m)	**1 MHz**	**20.38 / 8.16**	**0.002..0.005**
LEP (27km)	**11.2 kHz**	**60..100**	**0.08..0.13**
LHC (27km)	**11.2 kHz**	**64.28 / 59.31**	**0.006**

Next: Equation of motion discussed using a mechanical example

- biased pendulum with equivalent dynamics
- Hamiltonian and Separatrix

Biased Pendulum = equivalent dynamics



set torques equal:

$$m_1 g l \sin\phi = m_2 g r$$

simple with: $m_1 = m_2, \; l = 2r$

$$\sin\phi_s = \frac{1}{2} \rightarrow \phi_s = \; 150\text{deg}, 30\text{deg}$$

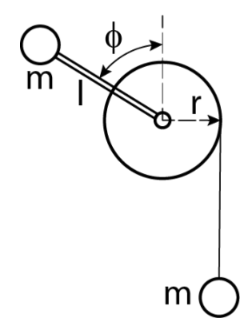
stable fix point: oscillations

unstable fix point: runs away after small distortion

equation of motion (see appendix slide):

$$\boxed{I\frac{d^2\phi}{dt^2} = mgl(\sin\phi - \sin\phi_s)}$$

I = moment of inertia



→ same form as longitudinal particle motion!

Hamiltonian Function

energy method of integration: multiply function by $\dot{\phi}$ and integrate once:

$$I \int dt \ \dot{\phi}\ddot{\phi} \ = \ mgl \int dt \ \dot{\phi}(\sin\phi - \sin\phi_s)$$

results in:

$$\underbrace{\frac{1}{2} I \dot{\phi}^2}_{\text{kinetic energy}} + \underbrace{mgl(\cos\phi + \phi \sin\phi_s)}_{\text{potential energy}} = \underbrace{\text{const}}_{\text{total energy}}$$

I = moment of inertia

kinetic energy **potential energy** **total energy**

$$\boxed{\mathcal{H} = \frac{1}{2I} L^2 + V(\phi)}$$

with ϕ corresponding to q
and angular momentum L corresponding to p

$$L = \dot{\phi} I, \ \ V(\phi) = mgl(\cos\phi + \phi \sin\phi_s)$$

Equations of Motion

$$\boxed{\mathcal{H} = \frac{1}{2I}L^2 + mgl(\cos\phi + \phi\sin\phi_s)}$$

in this case the canonical coordinates are:

- $q \mathrel{\widehat{=}} \phi$, angle
- $p \mathrel{\widehat{=}} L = I\dot{\phi}$, angular momentum

results in equations of motion:

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial L} = \frac{L}{I}$$

$$\dot{L} = -\frac{\partial \mathcal{H}}{\partial \phi} = mgl(\sin\phi - \sin\phi_s)$$ = sum of the torque moments

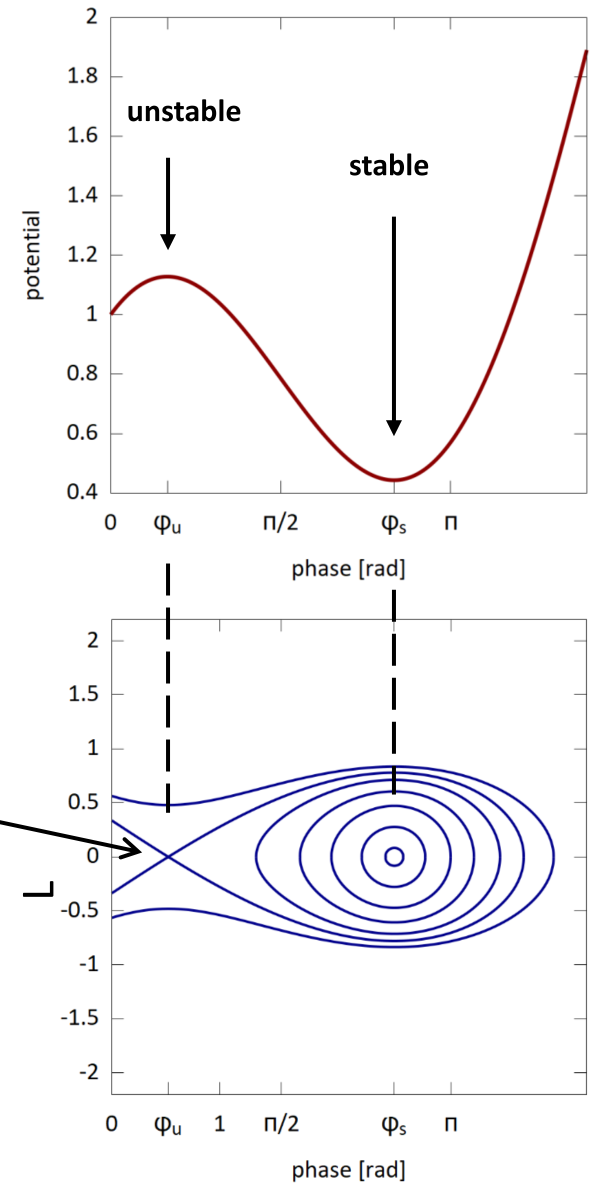
Potential and Separatrix for Pendulum

fix-points: $\dot{\phi} = 0, \quad \dot{L} = 0$

$$\rightarrow L = 0, \quad \sin\phi = \sin\phi_s$$

stable fix-point: $\phi_{\text{stab}} = \phi_s$

unstable fix-point: $\phi_{\text{unstab}} = \phi_u = \pi - \phi_s$



phase space: draw lines for fixed H

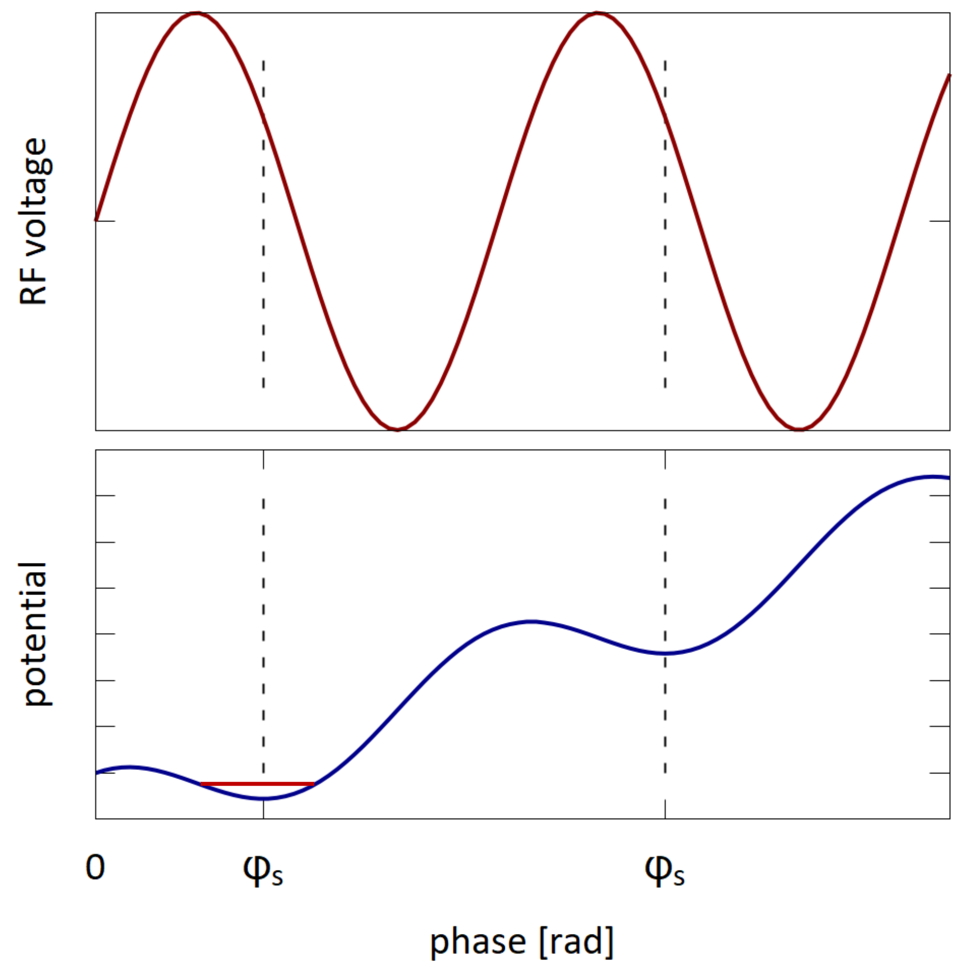
separatrix is boundary between stable and unstable motion, defined by unstable fixpoint

$$L = \pm\sqrt{2I\left(\mathcal{H} - mgl(\cos\phi + \phi\sin\phi_s)\right)}$$

Compare RF Voltage

ϕ_s = 150deg is the situation above transition energy in an accelerator

$$V(\phi) \propto$$
$$\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s$$



Next: Application to Longitudinal Potential

- Hamiltonian and Separatrix
- derived: energy acceptance, bucket size
- longitudinal emittance

back to the accelerator: Hamiltonian

$$\ddot{\phi} = \frac{e\hat{V}h\eta_c\omega_{\mathrm{rev}}^2}{2\pi E_s\beta_s^2}(\sin\phi - \sin\phi_s)$$

$$= -\frac{\Omega_s^2}{\cos\phi_s}(\sin\phi - \sin\phi_s)$$

$$\Omega_s^2 = -\frac{e\hat{V}h\omega_{\mathrm{rev}}^2\eta_c\cos\phi_s}{2\pi E_s\beta_s^2}$$

after integration with energy method:

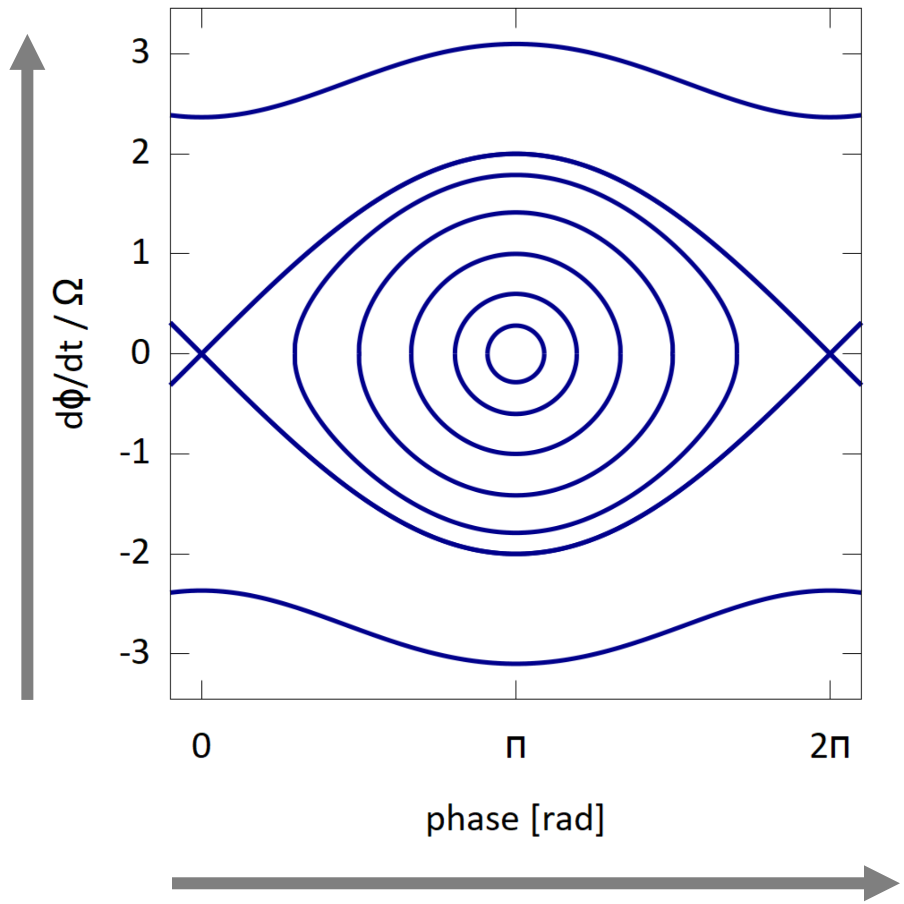
$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 - \frac{\Omega_s^2}{\cos\phi_s}(\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s)$$

small $\Delta\phi = \phi - \phi_s$ - circle in phase space (see Appendix):

$$\mathcal{H} \approx \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\Omega_s^2\Delta\phi^2$$

Longitudinal Coordinates

- rate of phase change $d\phi/dt$
- energy deviation δE



stationary bucket
$\phi_s = \pi$
(no avg. slope)

conversion factors
phase $\rightarrow$ time
d/dt phase $\rightarrow$ energy
see appendix

- phase ϕ
- time τ

Regions of Stability from Hamiltonian

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 - \frac{\Omega^2}{\cos\phi_s}\left(\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right)$$

use relation with δE (deviation from synchronous energy):

$$\dot{\phi} = \frac{h\,\omega_{\rm rev}\eta_c}{\beta_s^2}\frac{\delta E}{E_s}$$

determine H_{sep} from unstable fixpoint, $\dot{\phi} = 0, \phi = 0$, $\phi_s = \pi$:

$$\mathcal{H}_{\rm sep} = 2\Omega^2$$

then evaluate $\dot{\phi}$ on H_{sep} at $\phi = \phi_s = \pi$:

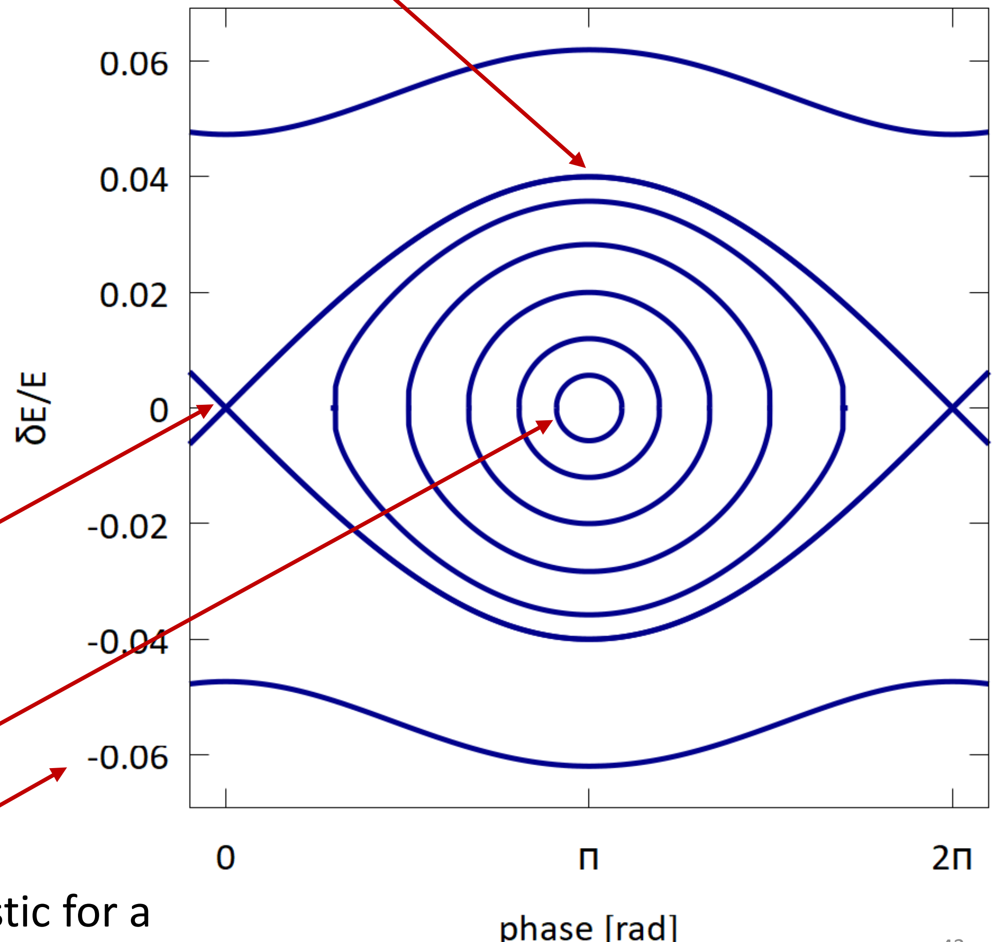
$$\dot{\phi}^2_{\rm max} = 4\Omega^2, \quad \text{or} : \quad \left(\frac{\delta E}{E_s}\right)_{\rm max} = \sqrt{\frac{2e\beta_s^2\hat{V}}{\pi h\eta_c E_s}}$$

Stationary Bucket and Separatrix

maximum energy acceptance on separatrix

energy acceptance of stationary bucket:

$$\left(\frac{\delta E}{E_s}\right)_{\max} = \sqrt{\frac{2e\beta_s^2\hat{V}}{\pi h \eta_c E_s}}$$



unstable fixpoint defines H_{sep}, separatrix

circular trajectories for small ϕ

see also Wiedemann sec. 9.3.2

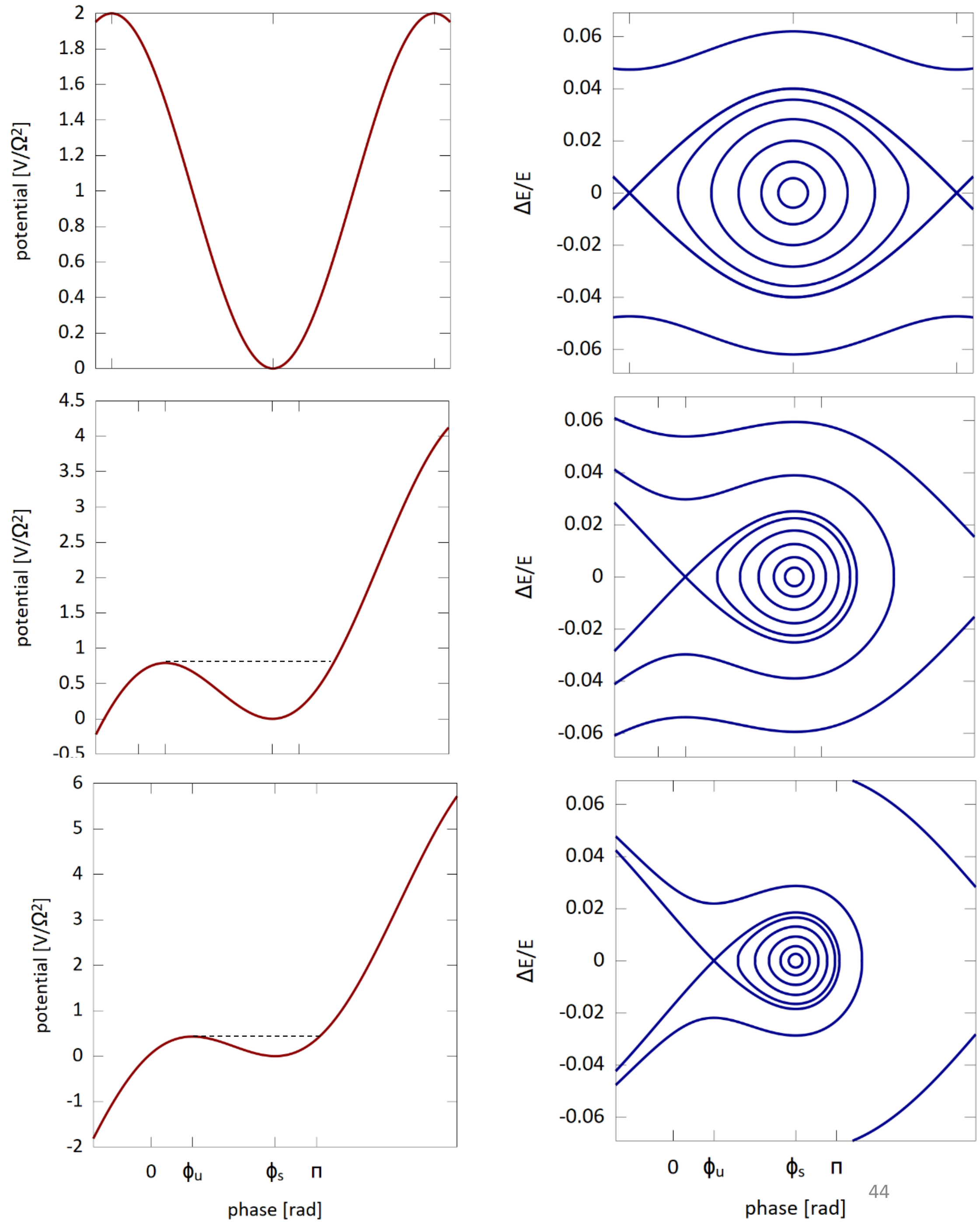
numbers are realistic for a small electron synchrotron

shrinking bucket size with increasing voltage for synchronous particle

ϕ_s = 180deg
$V(\phi_s) = 0$
(stationary)

ϕ_s = 150deg
$V(\phi_s) = 50\%\ \widehat{V}$

ϕ_s = 135deg
$V(\phi_s) = 71\%\ \widehat{V}$



Bucket Area

goal: compute phase space area inside bucket / separatrix

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 - \frac{\Omega_s^2}{\cos\phi_s}(\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s)$$

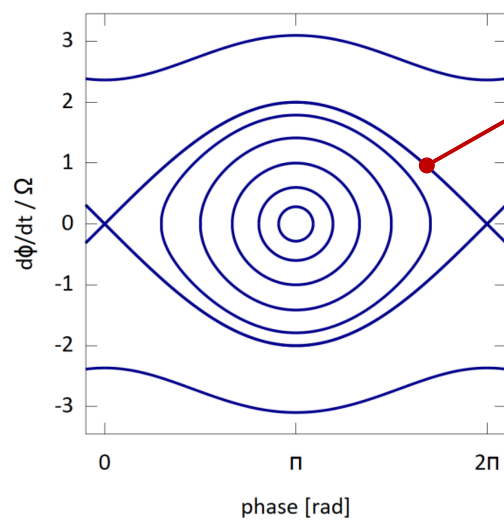
insert values for separatrix:

$$\mathcal{H}_{\text{sep}} = 2\Omega_s^2 = \frac{1}{2}\dot{\phi}^2 + \Omega_s^2(\cos\phi + 1) \quad \text{for} \quad \phi_s = \pi$$

reorder for $\dot{\phi}$:

$$\dot{\phi}^2 = 2\,\Omega_s^2(1 - \cos\phi)$$

$$= 4\,\Omega_s^2\sin^2(\phi/2)$$



$$\boxed{\dot{\phi} = 2\,\Omega_s\sin(\phi/2)}$$

Bucket Area II

compute area by integration:

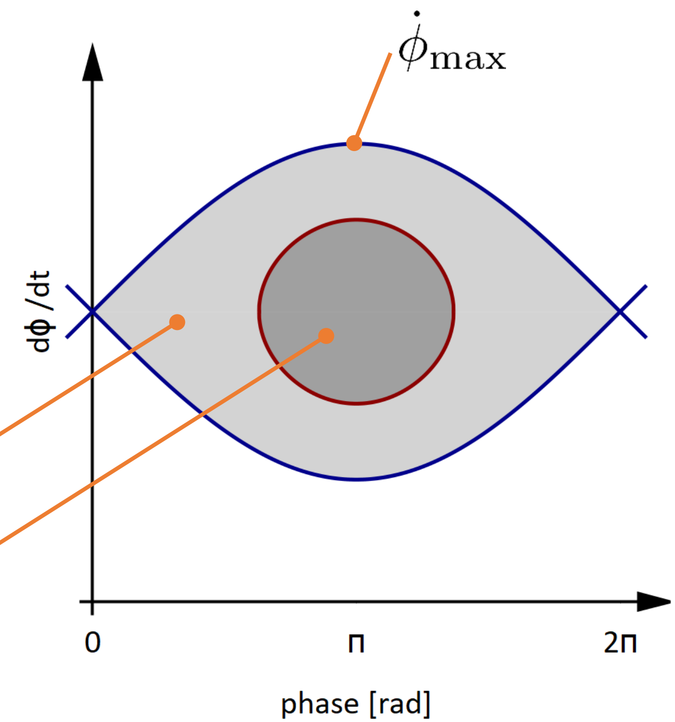
$$2\int_{\phi=0}^{2\pi} d\phi\,\dot{\phi} = 4\,\Omega_s \int_{\phi=0}^{2\pi} d\phi\,\sin(\phi/2)$$

$$= 16\Omega_s$$

$$= 8 \times \dot{\phi}_{\max}$$

Bucket Area = long. Acceptance

long. Emittance $\propto \pi \times \tau_{\mathrm{rms}} \times (\delta E)_{\mathrm{rms}}$



Energy and Time as practical phase space units

practical variables are δE, τ while we are working so far with $\dot{\phi}, \phi$
conversion (see appendix):

$$\delta E = \frac{\beta_s^2 E_s}{h\,\omega_{\mathrm{rev}}\eta_c}\dot{\phi}, \quad \tau = \frac{\phi}{\omega_{\mathrm{rf}}}$$

$$\int (\delta E)_{\mathrm{sep}} d\tau = 16\Omega_s \times \frac{\beta_s^2 E_s}{\omega_{\mathrm{rf}}\eta_c} \times \frac{1}{\omega_{\mathrm{rf}}}$$

size of stationary bucket (acceptance):

$$A_{\delta E,\,\tau} = 8\frac{\beta_s}{\omega_{\mathrm{rf}}}\sqrt{\frac{2e\hat{V}E_s}{\pi h \eta_c}}$$

acceptance is:

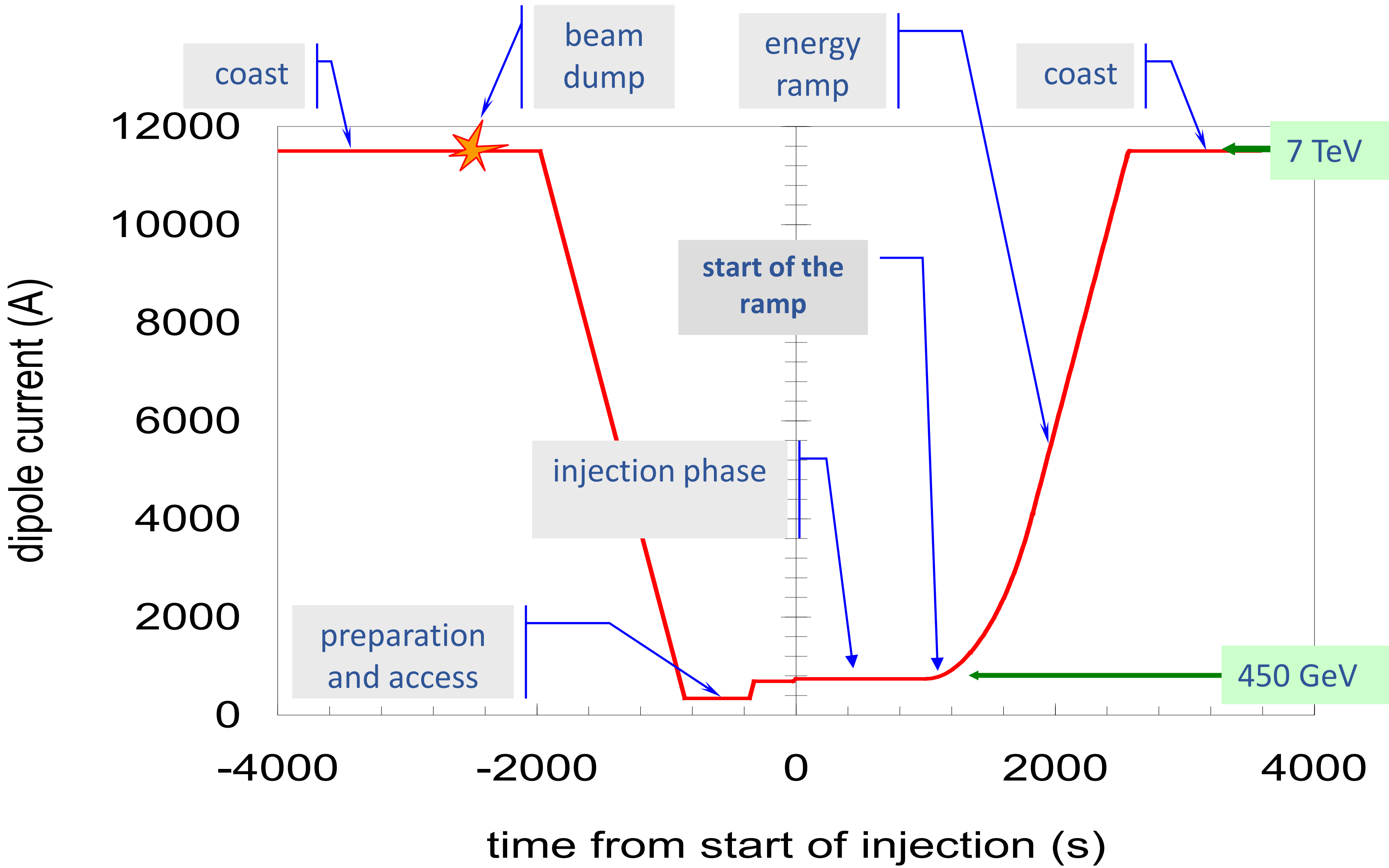
- reduced with RF frequency
- increased with RF voltage
- reduced with slippage factor

Next: acceleration in storage rings

- energy ramping
- frequency change
- storage of electrons (compensate radiation losses)

The Synchrotron – LHC Operation Cycle

The magnetic field (dipole current) is increased during the acceleration.



Energy Ramping in Synchrotrons

in a synchrotron the beam is accelerated by ramping the bending magnets; particles move to the synchronous phase and gain energy

$$B\rho = \frac{p}{e} \rightarrow \frac{\Delta p}{\tau_{\text{rev}}} \approx e\rho\dot{B}$$ (small momentum change per turn)

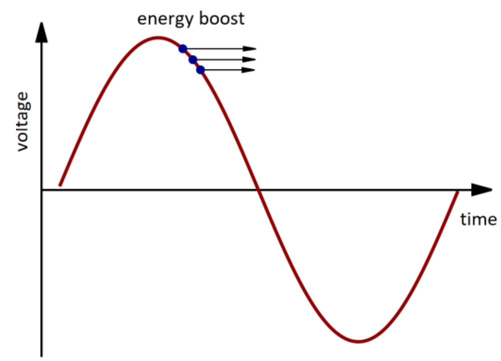
relating energy and momentum change, circulation time and circumference C:

$$E^2 = c^2p^2 + m_0^2c^4 \rightarrow \Delta E = v\Delta p$$

$$e\rho\dot{B} = \frac{\Delta E}{\tau_{\text{rev}}v} = \frac{\Delta E}{C} = \frac{e\hat{V}\sin\phi_s}{C}$$

results in an expression for the synchronous phase as a function of ramp rate $\dot{B}$ and peak voltage $\hat{V}$:

$$\boxed{\phi_s = \arcsin\left(\frac{\rho C}{\hat{V}}\dot{B}\right)}$$



Circulating Electrons

the energy loss of electrons by SR must be continuously compensated
energy loss per turn for reference electron:

$$U_0[\mathrm{keV}] = 88.5 \cdot \frac{E_s^4[\mathrm{GeV}]}{\rho[\mathrm{m}]} \qquad \text{for } E_s \gg m_0 c^2$$

for arbitrary particle:

$$U(\delta E) = U_0 + U'\delta E \quad \delta E = E - E_s,\ U' = \left(\frac{dU}{dE}\right)_{E=E_s}$$

results in energy change per turn:

$$\Delta E = E_k - E_{k-1} = e\hat{V}\sin\phi - U(\delta E)$$

Circulating Electrons II

differentiation, small change per revolution time τ_0 per turn:

δ: difference to E_s

$$\frac{d(\delta E)}{dt} \approx \frac{\delta E_n - \delta E_{n-1}}{\tau_0} = \frac{e\hat{V}\omega_{\mathrm{rev}}}{2\pi}(\sin\phi - \sin\phi_s) - \frac{U'}{\tau_0}\delta E$$

relate δE to $\dot{\phi}$, expand for small ϕ-ϕ_s:

$$\boxed{\ddot{\phi} + \frac{U'}{\tau_0}\dot{\phi} + \Omega_s^2(\phi - \phi_s) = 0}$$

This equation describes a **harmonic oscillator with a damping term**.

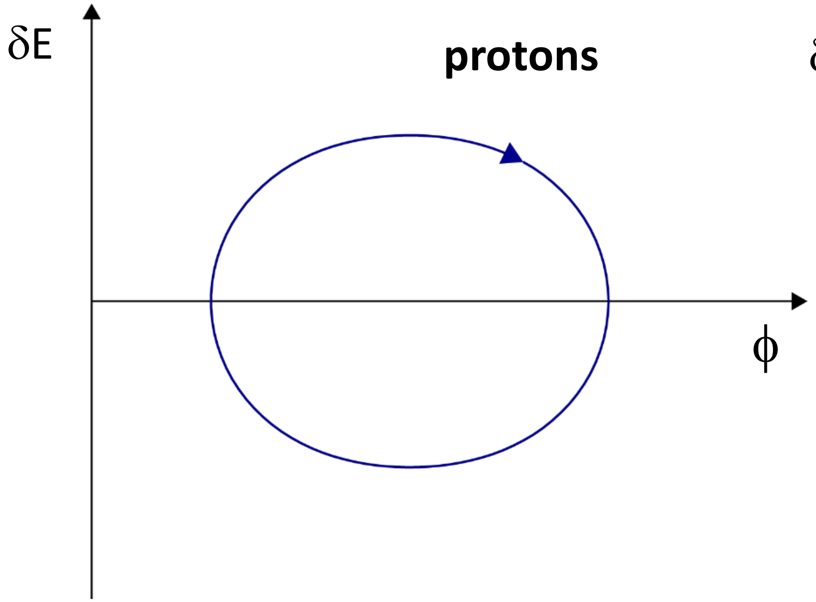
Circulating Electrons III

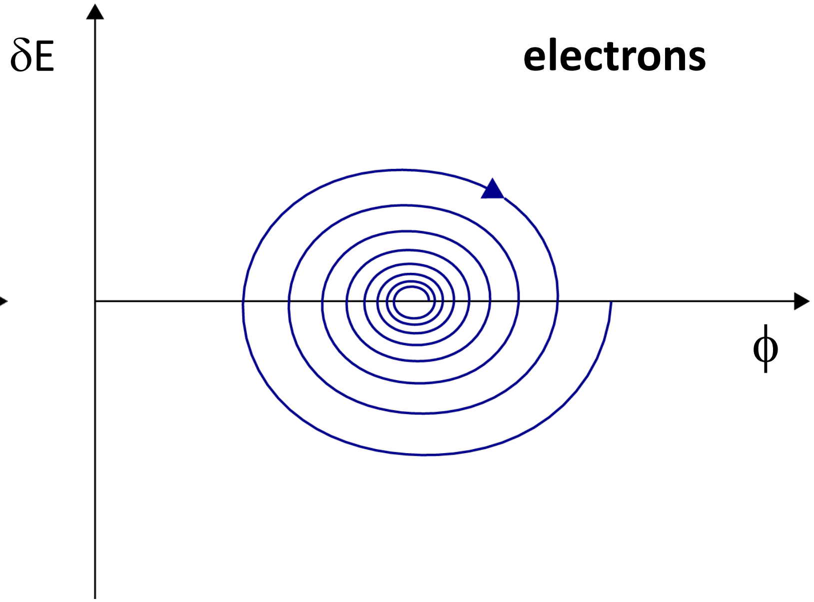
$$\ddot{\phi} + \frac{U'}{\tau_0}\dot{\phi} + \Omega_s^2(\phi - \phi_s) = 0$$

(compute U' in next lecture)

solution is damped oscillator:

$$\phi(t) = \phi_s + a\exp(-t/\tau_E)\cos(\Omega_s t - \phi_0), \quad \tau_E = 2\tau_0/U'$$

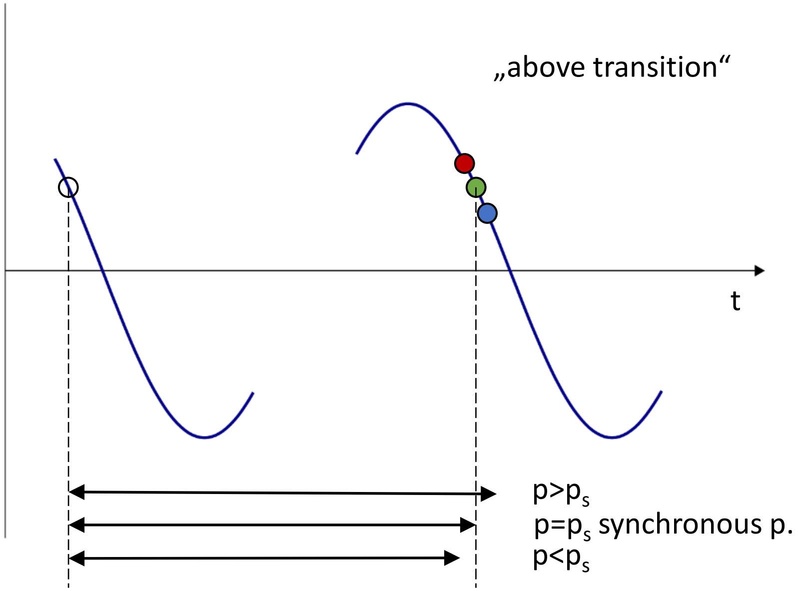




$\rightarrow$ treatment ignores quantum excitation, see lecture on synchrotron radiation

Summary Longitudinal Dynamics in Synchrotrons

- particles are "focused" = oscillate around synchronous particle
- protons/ions during acceleration and electrons (SR!) in general need a net accelerating voltage
- oscillation frequency is much lower than for transverse planes, $Q_s \approx 10^{-1}..10^{-3}$
- scaling: $\Omega_s \propto Q_s \propto \text{sqrt(voltage)}$



- longitudinal acceptance = (time window) × (energy window)

Longitudinal Dynamics: Summary

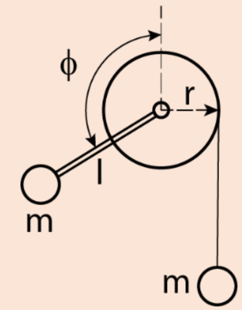
- The **normalized emittance** is conserved during acceleration: **$\beta\gamma\cdot\varepsilon$**
- **RF systems and cavities provide harmonic potentials** in the longitudinal phase space for acceleration and phase stability
- particles oscillate around a synchronous particle with a **synchrotron tune** much lower than in the transverse planes ($Q_s = 10^{-3}$ .. 10^{-1})
- the range of stable motion in phase space is called bucket; phase space is measured in **time-offset × energy-offset**
- bucket area and synchrotron frequency scale with square root of RF voltage

- longitudinal dynamics is also relevant for single pass accelerators, for example **bunch compression** in FEL's and also proton/ion cyclotrons

Appendix: Biased Pendulum, Equation of Motion

I) compute fixpoints:

set torques equal: $m_1 gl \sin\phi = m_2 gr$



simple by choice: $m_1 = m_2,\ l = 2r$

result: $\sin\phi_s = \frac{1}{2} \rightarrow \phi_s = \ 150\text{deg}, 30\text{deg}$

II) angular acceleration: $I\ddot{\phi} = mgl\sin\phi - mgr$

(non-zero for $\phi \neq \phi_s$)

moment of interia:

$$I = ml^2 + mr^2$$

replace: $r = l\sin\phi_s$

result EQM:

$$\boxed{I\frac{d^2\phi}{dt^2} = mgl(\sin\phi - \sin\phi_s)}$$

Appendix: long. motion, Hamiltonian small Δϕ

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 - \frac{\Omega_s^2}{\cos\phi_s}(\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s), \;\; \phi = \phi_s + \Delta\phi$$

$$= \frac{1}{2}\dot{\phi}^2 - \Omega_s^2 \, \frac{\cos\phi_s\cos\Delta\phi - \sin\phi_s\sin\Delta\phi - \cos\phi_s + \Delta\phi\sin\phi_s}{\cos\phi_s}$$

$$= \frac{1}{2}\dot{\phi}^2 - \Omega_s^2\,(\underbrace{\cos\Delta\phi - 1}_{\approx -\frac{1}{2}\Delta\phi^2} + \tan\phi_s\underbrace{(\Delta\phi - \sin\Delta\phi)}_{\mathcal{O}(\Delta\phi^3)\approx 0})$$

$$\boxed{\mathcal{H} \approx \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\Omega_s^2\Delta\phi^2}$$

Appendix: Longitudinal Phase Space conversion

phase change per turn for arbitrary particle:

$$\dot{\phi} \approx \frac{\Delta\phi}{\tau_s} = \omega_{\rm rf}\frac{\Delta\tau}{\tau_s}$$

$$= \omega_{\rm rf}\eta_c\frac{\delta p}{p_s}$$

$$= \frac{\omega_{\rm rf}\eta_c}{\beta_s^2}\frac{\delta E}{E_s}$$

circulation time change vs. phase change:

$$\phi = \omega_{\rm rf}\,\tau$$

$$\frac{\Delta\tau}{\tau_s} = \eta_c\frac{\delta p}{p}$$

$$\eta_c = \frac{1}{\gamma_{\rm tr}^2} - \frac{1}{\gamma^2}$$

$$\frac{\delta E}{E_s} = \beta_s^2\frac{\delta p}{p}$$

$$\omega_{\rm rf} = h\omega_{\rm rev}$$

τ_s = circulation time
η_c = slip factor
γ_{tr} = at transition energy
E_s = energy synchronous p.
β_s = velocity synchronous p.
h = harmonic number
ω_{rev} = revolution frequency
ω_{rf} = RF frequency

conversion factors:

$$\boxed{\delta E = \frac{\beta_s^2 E_s}{\eta_c\omega_{\rm rf}}\cdot\dot{\phi},\ \ \tau = \frac{\phi}{\omega_{\rm rf}}}$$