Introduction to Particle Accelerator Physics

Final Exam

Numerical results without derivation are not sufficient. Do symbolic derivations first and insert numbers only at the end (unless stated otherwise). You may use books, notes, and tutorials. You can also make use of your tablet or laptop but only if constantly set on plane mode*!*

I. Transverse Beam Dynamics (11 points)

Part I: FODO cell stability

Assume a synchrotron with a circumference of $C = 3\,\text{km}$ composed *only* of symmetric FODO cells. The horizontal tune is $Q_x = 7.28$ and the total number of FODO cells is $N_{\text{FODO}} = 48$.

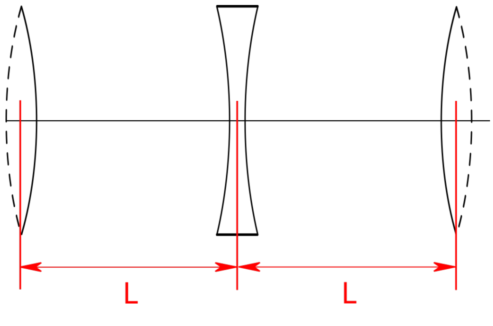

Figure 1: FODO cell, part I.

(a) Calculate the horizontal phase advance per FODO cell μ_{FODO}. (1 point)

The horizontal tune is defined as

$$Q_x = \frac{N_{\text{FODO}} \cdot \mu_{\text{FODO}}}{2\pi} \rightarrow \mu_{\text{FODO}} = \frac{2\pi \cdot Q_x}{N_{\text{FODO}}} = 0.9529\,\text{rad}.$$

(b) Knowing μ_{FODO}, calculate the focal length f of the quadrupoles using the thin-lens approximation. Recall that the FODO cell transport matrix between the centers of two consecutive focusing quadrupoles reads (see Fig. 1) (3 points)

$$M_{\mathrm{FODO,\,I}} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L+\frac{L^2}{f} \\ -\frac{L}{2f^2}\left(1-\frac{L}{2f}\right) & 1-\frac{L^2}{2f^2} \end{pmatrix}.$$

Equate the FODO transport matrix to the common Twiss matrix with $\mu = \mu_{\mathrm{FODO}}$

$$\begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L+\frac{L^2}{f} \\ -\frac{L}{2f^2}\left(1-\frac{L}{2f}\right) & 1-\frac{L^2}{2f^2} \end{pmatrix} = \begin{pmatrix} \cos\mu & \beta_x \sin\mu \\ -\frac{\sin\mu}{\beta_x} & \cos\mu \end{pmatrix}.$$

The *half-length* of the FODO cell is $L = \frac{C}{2\cdot N_{\mathrm{FODO}}} = 31.25\,\mathrm{m}$

$$\rightarrow \cos\mu = 1-\frac{L^2}{2f^2} \rightarrow f = L\sqrt{\frac{1}{2(1-\cos\mu)}} = 34.07\,\mathrm{m}.$$

(c) Is the transverse particle motion stable for the given lattice? Explain why. (1 point)

Yes, since $f > L/2 = 15.625\,\mathrm{m}$.

Part II: FODO cell equivalence

We consider again a FODO cell with focal lengths $\mp f$ and equal drift lengths L between the focusing and defocusing quadrupoles as illustrated in Fig. 2, left. However, other than in part I of the exercise we now define the transport matrix M_{FODO} starting just before the focusing quadrupole instead of at its center

$$M_{\mathrm{FODO,\,II}} = \begin{pmatrix} 1-\frac{L}{f}-\frac{L^2}{f^2} & 2L+\frac{L^2}{f} \\ -\frac{L}{f^2} & 1+\frac{L}{f} \end{pmatrix}.$$

We are going to study the transport matrix M_{OFO} of the drift-lens-drift arrangement shown in Fig. 2, right. It is defined by a single focusing quadrupole of focal length $-\tilde{f}$ between two drift sections of possibly different lengths L_1 and L_2.

(a) Compute the transport matrix M_{OFO} of the drift-lens-drift system analytically. (2 points)

$$M_{\mathrm{OFO}} = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L_2}{\tilde{f}} & L_1+L_2-\frac{L_1 L_2}{\tilde{f}} \\ -\frac{1}{\tilde{f}} & 1-\frac{L_1}{\tilde{f}} \end{pmatrix}.$$

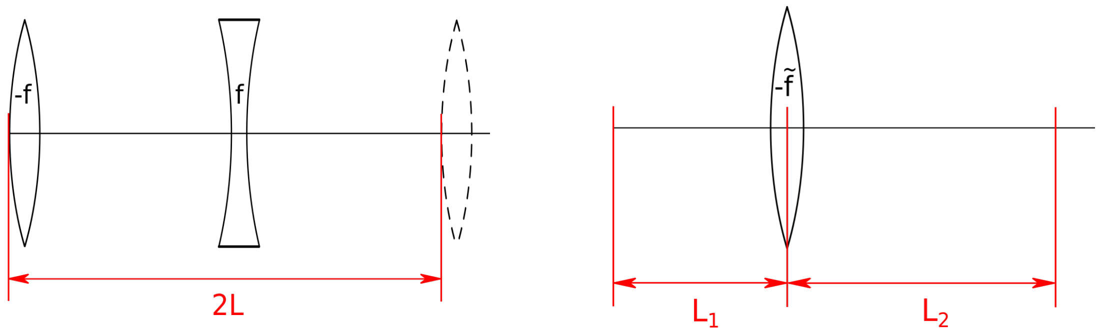

Figure 2: *Left:* FODO cell, part II. *Right:* Drift-lens-drift (OFO) structure.

(b) Show that M_{OFO} is indeed equivalent to M_{FODO} and express $\tilde{f}$, L_1, and L_2 in terms of f and L. (4 points)

We can define 4 equations by setting $M_{\text{FODO}} = M_{\text{OFO}}$. We need 3 equations to express $\tilde{f}$, L_1, and L_2 with the FODO parameters f and L (we use equations from matrix elements M_{21}, M_{22}, and M_{11}, respectively)

$$\tilde{f} = \frac{f^2}{L}, \quad L_1 = -\frac{L\tilde{f}}{f} = -f, \quad L_2 = f + L.$$

The 4$^{\text{th}}$ equation from matrix element M_{12} should be used to demonstrate consistency between found relationships to prove the full equivalence of the two transport matrices.

II. Longitudinal Beam Dynamics (9 points)

Assume a proton storage ring with a circumference of $C = 650\,\text{m}$. The stored protons have a kinetic energy of $E_{\text{kin}} = 4.5\,\text{GeV}$. In this storage ring the bending magnets account for 30 % of the circumference. Furthermore, the average value of the dispersion in the bending magnets, which are all assumed to have identical field strengths, is $\langle D_x \rangle = 0.8\,\text{m}$.

(a) Calculate the total energy E_{tot} of the particles, the relativistic β and γ, the revolution period T_0 of the beam, its momentum p and the magnetic field B in the dipoles. (3 points)

Let $E_0 = m_0 c^2 = 0.938\,\text{GeV}$,

$$E_{\text{tot}} = E_{\text{kin}} + E_0 = 5.438\,\text{GeV},$$

$$\gamma = \frac{E_{\text{tot}}}{E_0} = 5.80,$$

$$\beta = \sqrt{1 - \gamma^{-2}} = 0.985,$$

$$T_0 = \frac{C}{\beta c} = 2.20\,\mu\text{s},$$

$$p = \frac{1}{c}\sqrt{E_{\text{tot}}^2 - E_0^2} = 5.36\,\text{GeV/c},$$

$$\rho = \frac{L_{\text{dip}}}{2\pi} = \frac{0.3\,C}{2\pi} = 31.04\,\text{m} \;\rightarrow\; B = \frac{p[\text{GeV/c}]}{0.29979\,\rho} = 0.58\,\text{T}.$$

(b) Using the smooth approximation calculate the momentum compaction factor α_c of the storage ring as well as the absolute path length change ΔC for a particle with a relative momentum deviation of 1.5 %. (2 points)

$$\alpha_c = \frac{\langle D_x \rangle}{C\,\rho}\,L_{\text{dip}} = \frac{2\pi\,\langle D_x \rangle}{C} = 7.73 \times 10^{-3}.$$

With $\delta = 0.015$,

$$\Delta C = \alpha_c\,C\,\delta = 75.4\,\text{mm}.$$

(c) Explain the concept of transition crossing *(max. 5 sentences)*. (2 points)

(d) Is the beam above or below transition energy? Based on your answer: what is the possible range for the synchronous phase ϕ_s to guarantee phase stability in the longitudinal plane and to ensure particle acceleration? (2 points)

To know whether we are above or below transition, we need to calculate γ_{tr} and compare it to γ calculated earlier.

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha_c}} = 11.37 \;\rightarrow\; \gamma_{tr} > \gamma.$$

We are hence below transition ($\eta < 0$). The possible range for ϕ_s for phase stability is in that case

$$0 < \phi_s < \frac{\pi}{2}.$$

III. Luminosity (11 points)

The Cornell Electron Storage Ring (CESR) is an electron-positron e^-e^+ collider at Cornell University in Ithaca, New York (USA). It has a physical circumference of 768 m. Assume that the machine operates at a beam momentum of 4.5 GeV/c and that the two counter-rotating beams each contain 9 colliding bunches of e^+ and e^-, respectively. The current per bunch is initially 7 mA and their normalized emittances are 100 µm rad and 500 µm rad in the horizontal and the vertical plane, respectively. The β-functions at the interaction point (IP) have values of $\beta^*_{x,y} = 0.3$ m.

(a) Calculate the center-of-mass energy of the e^-e^+ collisions for this machine. (1 point)

One finds that we are in an ultra-relativistic regime and hence $E_{\text{tot}} \approx cp$. The center-of-mass energy is thus $E_{\text{CM}} = 2E_{\text{tot}} = 9\,\text{GeV}$.

(b) What is the desired value of dispersion at the IP? Explain why. (1 point)

The desired value for dispersion at the IP is zero. This is to reduce the beam size at the IP as much as possible ($\sigma_{x,y} = \sqrt{\beta^*_{x,y}\varepsilon^{geo}_{x,y} + D^2_{x,y}\sigma^2_\delta}$) to maximize the luminosity $[\mathcal{L} \propto 1/(\sigma_x\sigma_y)]$.

(c) Assume that the machine is set up to have the desired dispersion at the IP. Calculate the root mean square (rms) horizontal and vertical beam sizes at the collision point. (2 points)

Since we are in the ultra-relativistic regime, $p \approx E_{\text{tot}}$, and $\beta \approx 1$.
From $E_{\text{tot}}, m_e \rightarrow (\beta\gamma)_{\text{rel}} \approx \gamma \approx 8806$.
Given that we have $D_{x,y} = 0$ at the IP, we can compute $\sigma_{x,y}$ using

$$\varepsilon^{\text{geo}}_{x,y} = \frac{\varepsilon^{\text{norm}}_{x,y}}{(\beta\gamma)_{\text{rel}}} \quad \text{and} \quad \sigma_{x,y} = \sqrt{\beta^*_{x,y}\varepsilon^{geo}_{x,y}},$$

with $\varepsilon^{\text{norm}}_x = 0.2\,\text{mm rad}$ and $\varepsilon^{\text{norm}}_y = 5.0\,\text{mm rad}$. The numerical values are hence $\sigma_x = 58.4\,\mu\text{m}$ and $\sigma_y = 130.5\,\mu\text{m}$.

(d) Calculate the luminosity $\mathcal{L}_0$ for the given parameters. (2 points)
If you did not manage to solve c), use $\sigma_x = 40\,\mu m$ and $\sigma_y = 200\,\mu m$ to continue.

As explained above, we can safely use the approximation $\beta \approx 1$.

$$f_{\text{rev}} = \frac{\beta c}{C} \approx \frac{c}{C} = 390.4\,\text{kHz},$$

$$N = N_{1,2} = \frac{I_{1,2}}{e f_{\text{rev}}} = 1.12 \times 10^{11}\ \text{e}^{\pm}/\text{b},$$

$$\mathcal{L}_0 = \frac{n_b N^2 f_{\text{rev}}}{4\pi\sigma_x\sigma_y} = 4.6 \times 10^{31}\,\text{cm}^{-2}\text{s}^{-1}.$$

Or, if student did not manage to solve c)

$$\mathcal{L}_0 = 4.4 \times 10^{31}\,\text{cm}^{-2}\text{s}^{-1}.$$

(e) To avoid multiple collision points near the IP we now introduce a crossing angle in the horizontal plane $\phi_x = 600\,\mu\text{rad}$. Given the rms bunch length of $\sigma_s = 6.5\,\text{cm}$, calculate by what percentage this reduces the luminosity obtained in d). (2 points)

Since $\sigma_s \gg \sigma_x$ and $\phi_x \ll 1$, we can safely use

$$S \approx \frac{1}{\sqrt{1 + \left[\frac{\sigma_s}{\sigma_x}\frac{\phi_x}{2}\right]^2}} = 0.949.$$

The luminosity is hence reduced by 5.1 %.

Or, if student did not manage to solve c)

$$S \approx 0.899.$$

The luminosity is hence reduced by 10.1 %.

(f) Assume that the bunch intensities all decay exponentially with a time constant of $\tau = 3\,\mathrm{h}$ (so-called beam lifetime). Given that all the other beam parameters remain constant – what is the luminosity $\mathcal{L}(t)$ after $t = 1.5\,\mathrm{h}$, including the reduction caused by the crossing angle? (3 points)

$$N(t) = N(0)\; e^{-t/\tau},$$
$$\mathcal{L}(t) = \mathcal{L}_0\, S\, e^{-2t/\tau},$$
$$\mathcal{L}(t = 1.5\,\mathrm{h}) = 1.61 \times 10^{31}\,\mathrm{cm^{-2}s^{-1}}.$$

Or, if student did not manage to solve c)

$$\mathcal{L}(t = 1.5\,\mathrm{h}) = 1.45 \times 10^{31}\,\mathrm{cm^{-2}s^{-1}}.$$

IV. Synchrotron Radiation (11 points)

The Diamond Light Source located near Oxford (UK) stores an electron beam with a total energy of 3 GeV at a current of 300 mA. The Diamond dipole bending radius is 7.1 m.

(a) Compute the energy loss per turn due to synchrotron radiation. (1 point)

$\gamma = \frac{3\,\mathrm{GeV}}{0.511 \times 10^{-3}\,\mathrm{GeV}} = 5870.84 \rightarrow U_0 = \frac{4\pi}{3}\alpha\hbar c\frac{\gamma^4}{\rho} = 1.0092\,\mathrm{MeV}.$

(b) The total power consumption of Diamond is $P_{\mathrm{tot}} = 2\,\mathrm{MW}$. What fraction of P_{tot} is required to compensate for the synchrotron radiation losses? *(Hint: compute the total average power lost by the stored beam.)* (3 points)

The total average power loss due to SR is

$$P_{\gamma,b} = N_b^e\, U_0\, f_{\mathrm{rev}},$$

where N_b^e is the total number of stored electrons.

Using $N_b^e = I_b/(e f_{\mathrm{rev}})$, we rewrite

$$P_{\gamma,b} = U_0\, I_b\, /\, e = 1.89 \times 10^{18}\,\mathrm{MeV\,s^{-1}}.$$

Furthermore, $P_{\mathrm{tot}} = 2\,\mathrm{MW} = 1.248 \times 10^{19}\,\mathrm{MeV\,s^{-1}}$, hence $P_{\gamma,b}/P_{\mathrm{tot}} = 15.1\,\%$.

(c) Let us consider operating the Diamond light source with protons (assume the same bending radius). What would be the total energy of a proton beam with a current of 300 mA that radiates the same amount of synchrotron power? (2 points)

It is clear that the proton beam must also be ultra-relativistic to emit this amount of synchrotron radiation. Hence, its revolution frequency will be the same as that of the electron beam. Since protons and electrons have the same absolute charge, the number of protons and electrons in a 300 mA beam will be the same, so
$U_0^e = U_0^p \rightarrow \gamma_e = \gamma_p = 5870.8922 \rightarrow E_p = m_p \gamma c^2 = 5508.49\,\text{GeV}$.

(d) Imagine that the radio-frequency (RF) system suddenly stops restoring the energy lost due to synchrotron radiation. Explain what happens to the radius of the beam orbit? (1 point)

Since the beam is losing energy due to SR it will go on smaller and smaller orbits (spiraling in) and eventually crash into the *inner* pipe walls.

(e) Knowing that the maximum horizontal dispersion is 25 cm and the horizontal aperture at this location is ±2 cm, compute the number of turns the beam survives in the ring without the RF system before it crashes into the wall. For simplicity, assume a point-like beam and that the energy lost per turn is constant over time. (4 points)

Since we are in an ultra-relativistic regime, $p \approx E_{\text{tot}}$, and hence $\delta = \Delta p/p_0 \approx \Delta E/E_0$.

$$\Delta r = -2\,\text{cm} = D_{\text{max}} \cdot \delta(t) = D_{\text{max}} \cdot \left(-\frac{U_0 t}{E_0}\right),$$

$$t = \frac{E_0}{U_0 D_{\text{max}}} \Delta r = 238.1\,\text{turns}.$$

Useful constants:

- $e = 1.602 \times 10^{-19}\,\text{C}$
- $m_e = 0.511\,\text{MeV}/c^2$
- $m_p = 0.938\,\text{GeV}/c^2$
- $c = 299\,792\,458\,\text{m}\,\text{s}^{-1}$
- $\alpha = 1/137$
- $\hbar c = 197\,\text{MeV fm} \quad (1\,\text{fm} = 1 \times 10^{-15}\,\text{m})$
- $1\,\text{eV} = 1.602 \times 10^{-19}\,\text{J}$