Exercises in preparation for the exam

A set of short exercises to train typical questions of the form that could appear in the exam.

Exercise 1: consider a Dirac field triplet ψ_i and a scalar triplet ϕ_i.

- Write the most general relativistic Lagrangian invariant under $SO(3)$ up to terms with dimension $d \leq 4$. Does it change if we require invariance under $O(3)$?

Exercise 2: Consider a real neutral scalar field. The ladder operators satisfy:

$$[a(\mathbf{k}), a(\mathbf{p})] = (2\pi)^3 2E_k \delta^3(\mathbf{k} - \mathbf{p}) \, .$$

Given the following state,

$$|\psi\rangle = \int d\phi d\theta \sin\theta \left(e^{2i\phi} \sin^2\theta \, a^\dagger(\mathbf{k}_{\theta,\phi})|0\rangle \right) ,$$

where $\mathbf{k}_{\theta,\phi} = |\mathbf{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$,

- show that

$$J^3|\psi\rangle = 2|\psi\rangle \qquad \text{and} \qquad J^i J^i |\psi\rangle = 6|\psi\rangle \, ,$$

where $J^i = \frac{1}{2}\epsilon^{ijk} J^{jk}$ is the angular momentum operator:

$$J^{ij} = -i \int d\Omega_k \left[a^\dagger(\mathbf{k}) \left(k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) a(\mathbf{k}) \right] .$$

Hint: Notice that $e^{i\phi}\sin\theta = (k^1 + ik^2)/|\mathbf{k}|$.

Exercise 3: Given $T^i = \frac{1}{2} a^\dagger_\alpha \sigma^i_{\alpha\beta} a_\beta$ with the commutation relations

$$[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = 0$$

- Prove that $[T^i, T^j] = i\epsilon^{ijk} T^k$

Given $S^i = \frac{1}{2} b^\dagger_\alpha \sigma^i_{\alpha\beta} b_\beta$ with the anticommutation relations

$$\{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta}, \qquad \{b_\alpha, b_\beta\} = 0$$

- Prove that $[S^i, S^j] = i\epsilon^{ijk} S^k$

Exercise 4: Consider a $SU(2)$ scalar doublet ϕ_α ($\alpha = 1,2$) with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger_\alpha \partial^\mu \phi_\alpha - V\left(\phi^\dagger_\alpha \phi_\alpha\right) .$$

The $SU(2)$ charges are given by

$$Q_k = -i \int d^3x \left[\left(\partial_0 \phi^\dagger_\alpha\right) \frac{\sigma^k_{\alpha\beta}}{2} \phi_\beta - \phi^\dagger_\alpha \frac{\sigma^k_{\alpha\beta}}{2} \partial_0 \phi_\beta \right] , \qquad k = 1,2,3 .$$

- Compute the commutator $[Q_i, \phi_\alpha]$.
- Why is the Lagrangian invariant under this symmetry?
- *Bonus question:* use the Jacobi identities to compute $[Q_i, Q_j]$.

Exercise 5: Consider four different scalars ϕ_1, ϕ_2, ϕ_3 and ϕ_4.

- Prove that $\varepsilon^{\mu\nu\rho\sigma} \partial_\mu \phi_1 \partial_\nu \phi_2 \partial_\rho \phi_3 \partial_\sigma \phi_4$ is a total derivative.
- Write a Lorentz invariant term involving a the Levi-civita tensor $\varepsilon_{\mu\nu\rho\sigma}$ which is not a total derivative.

Exercise 6: Consider the following Lagrangian for a Dirac fermion ψ:

$$\mathcal{L} = iZ\bar{\psi}\gamma^\mu \partial_\mu \psi - M\bar{\psi}\psi - i\tilde{M}\bar{\psi}\gamma_5 \psi.$$

- Find the equations of motion. What is the mass of the Dirac particle?
- Prove that the above Lagrangian can be recast in the standard Dirac form via a field redefintion of the kind $\psi \to e^{\alpha + i\beta\gamma_5}\psi$ with $\alpha, \beta \in \mathbb{R}$.

Exercise 7: consider the following Lagrangian for a scalar field ϕ:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \lambda(\partial\phi)^2 \Box \phi$$

- What is the dimensionality of λ?
- Find the equations of motion. Can you identify the symmetries of the equations of motion?

Exercise 8: consider two scalars ψ_I and ϕ_I, $I = 1,2,3$ transforming as triplets under an internal $SU(2)$, called Isospin.

- Decompose the product $\psi_I \phi_J$ into irreducible representations of $SU(2)$.
- Given a third scalar χ_{IJ}, what constraint should it satisfy for it to correspond to Isospin 2?
- Write the most general Lagrangian for ϕ_I, ψ_I and χ_{IJ} (satisfying the above constraints) involving only terms with dimension $d \leq 4$.

Exercise 9: Consider N left-handed fermions ψ_L^a and N right-handed fermions ψ_R^a, $a = 1, \dots, N$ with Majorana and Dirac mass matrix

$$\begin{aligned}\mathcal{L} &= i(\psi_L^a)^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L^a + i(\psi_R^a)^\dagger \sigma^\mu \partial_\mu \psi_R^a \\ &- \left[(\psi_L^a)^\dagger\, m_{ab} \psi_R^b + h.c.\right] - \frac{1}{2}\left[(\psi_R^a)^T i\sigma_2 M_{ab} \psi_R^b + h.c.\right]\end{aligned}$$

- Show that the Lagrangian is Lorentz invariant
- For the last term to be non-vanishing, should M_{ab} be symmetric or anti-symmetric?
- Find the equations of motion. What happens if $m_{ab} = 0$?

Remark: when deriving the the equations of motion you should consider the fields and their variation as anticommuting variables. For instance given two anti commuting variables χ_1 and χ_2 we have $\delta(\chi_1\chi_2) = \delta\chi_1\chi_2 + \chi_1\delta\chi_2 = \delta\chi_1\chi_2 - \delta\chi_2\chi_1$.

Exercise 10: Given two scalar fields H_α and ϕ_α both transforming as doublet under an internal $SU(2)$, prove that

- - $H_\alpha^* H_\alpha \equiv H^\dagger H$
  - $\phi_\alpha^* \phi_\alpha \equiv \phi^\dagger \phi$
  - $H_\alpha^* \phi_\alpha \equiv H^\dagger \phi$
  - $H_\alpha \epsilon_{\alpha\beta} \phi_\beta \equiv H^T \epsilon \phi \qquad\qquad \epsilon = i\sigma_2$

  are $SU(2)$ singlets,

- - $H^\dagger \sigma^i H$
  - $H^T \epsilon \sigma^i H$

  are $SU(2)$ triplets,

- finally decompose explicitly into irreducible representations the following $SU(2)$ tensors
  - $(H^\dagger \sigma^i H)(H^\dagger \sigma^j H) = \mathbf{2} \oplus \mathbf{0}$
  - $(H^\dagger \sigma^i H)(\phi^\dagger \sigma^j \phi) = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}$

Exercise 11: Given a Dirac spinor field ψ,

- study the transformation properties of the following bilinears

  $$\bar{\psi}\psi\,, \qquad \bar{\psi}\gamma^\mu\psi\,, \qquad \bar{\psi}\gamma^\mu\gamma^\nu\psi$$

  under the field tranformation $\psi \to e^{i\alpha + i\beta\gamma_5}\psi$, with $\alpha, \beta\ \in \mathbb{R}$.

Exercise 12: Consider the action

$$S = \int d^4x (i\bar{\psi}_j \gamma^\mu \partial_\mu \psi_j - m_{ij} \bar{\psi}_i \psi_j), \quad i = 1, 2$$

- Under which condition is it invariant under the transformation

$$\psi_j \to U_{jk} \psi_k$$

  where U is an unitary 2×2 matrix?

- What are the conserved currents associated to this symmetry?