Quantum Field Theory

Set 11: solutions

Exercise 1

Let us consider the expansion of a scalar field in terms of the ladder operators:

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2k_0}\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right] e^{i\vec{k}\cdot\vec{x}}.$$

- We want to show that this satisfies the Klein-Gordon equation:

  $$(\Box + m^2)\phi(\vec{x},t) = 0.$$

  Indeed:

  $$\begin{aligned}(\Box + m^2)\phi(\vec{x},t) &= (\partial_t^2 - \partial_i^2 + m^2)\int \frac{d^3k}{(2\pi)^3 2k_0}\left[a(\vec{k})e^{-ik_0t+i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k})e^{ik_0t-i\vec{k}\cdot\vec{x}}\right]\\ &= \int \frac{d^3k}{(2\pi)^3 2k_0}\left[(m^2 - k_0^2 + |\vec{k}|^2)\ a(k,t)e^{-ik_0t+i\vec{k}\cdot\vec{x}} + (m^2 - k_0^2 + |\vec{k}|^2)\ a^\dagger(k,t)e^{ik_0-i\vec{k}\cdot\vec{x}}\right] = 0,\end{aligned}$$

  where we used the mass shell condition $k_0^2 = |\vec{k}|^2 + m^2$.

- $$\begin{aligned}[\phi(\vec{x},t),\phi(\vec{y},t)] &= \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0}\frac{d^3\vec{q}}{(2\pi)^3 2q_0}\left([a(\vec{k}),a^\dagger(\vec{q})]e^{i(q_0-k_0)t}e^{i\vec{k}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} + [a^\dagger(\vec{k}),a(\vec{q})]e^{-i(q_0-k_0)t}e^{-i\vec{k}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}\right)\\ &= \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0} d^3\vec{q}\left(\delta^3(\vec{k}-\vec{q})e^{i(q_0-k_0)t}e^{i\vec{k}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} - \delta^3(\vec{k}-\vec{q})e^{-i(q_0-k_0)t}e^{-i\vec{k}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}\right)\\ &= \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0}\left(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\right) = 0\end{aligned}$$

  where the last line is 0 because integral over the whole space of momenta $\vec{k}$ of an odd function of $\vec{k}$.

  $$\begin{aligned}[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)] &= \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0}\frac{d^3\vec{q}}{(2\pi)^3 2q_0}\left(iq_0[a(\vec{k}),a^\dagger(\vec{q})]e^{i(q_0-k_0)t}e^{i\vec{k}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} - iq_0[a^\dagger(\vec{k}),a(\vec{q})]e^{-i(q_0-k_0)t}e^{-i\vec{k}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}\right)\\ &= i\int \frac{d^3\vec{k}}{2(2\pi)^3} d^3\vec{q}\left(\delta^3(\vec{k}-\vec{q})e^{i(q_0-k_0)t}e^{i\vec{k}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} + \delta^3(\vec{k}-\vec{q})e^{-i(q_0-k_0)t}e^{-i\vec{k}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}\right)\\ &= i\int \frac{d^3\vec{k}}{2(2\pi)^3}\left(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\right) = i\int \frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = i\delta^3(\vec{x}-\vec{y})\end{aligned}$$

- With a Lorentz boost we can simply bring the points x, y to be at the same time, where the commutator vanishes according to the previous result. This *microcausality* condition is very important and means that a measure performed at one point x cannot affect another measure performed at one point y such that x and y are space-like separated, i.e. there is no superluminal propagation of information.

Exercise 2

Given a real free massive scalar field ϕ one can obtain the energy momentum tensor using Noether's prescription as usual:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}.$$

In order to compute the Noether's charge one needs

$$\begin{aligned} T_{00} &= \dot{\phi}^2 - \mathcal{L} = \mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right), \\ T_{0i} &= \pi\partial_i\phi. \end{aligned} \tag{1}$$

The decomposition of the fields ϕ, π in terms of the operator $a(\vec{k})$ and $a^\dagger(\vec{k})$ reads:

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3 2k_0}\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right] e^{i\vec{k}\cdot\vec{x}}, \\ \pi(x) &= \int \frac{d^3k}{(2\pi)^3 2k_0}(-ik_0)\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right] e^{i\vec{k}\cdot\vec{x}}, \end{aligned}$$

where we have used the notation $x \equiv (t, \vec{x})$. Therefore:

$$\begin{aligned} P_0 = \int d^3x\; T_{00} \;&=\; \frac{1}{2}\int d^3x \int \frac{d^3k}{(2\pi)^3 2k_0}\frac{d^3q}{(2\pi)^3 2q_0}\Big\{-k_0q_0\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &+ \left(-\vec{k}\cdot\vec{q} + m^2\right)\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Big\} e^{i(\vec{k}+\vec{q})\cdot\vec{x}}. \end{aligned}$$

Using the relation

$$\int d^3x\, e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k}+\vec{q}),$$

one can integrate over d^3k and set $\vec{k} = -\vec{q}$. In addition, $k_0 = \sqrt{m^2 + |\vec{k}|^2} = \sqrt{m^2 + |\vec{q}|^2} = q_0$. Thus:

$$\begin{aligned} P_0 = \frac{1}{4}\int \frac{d^3q}{(2\pi)^3 2q_0}&\Big\{-q_0\left[a(-\vec{q},t) - a^\dagger(\vec{q},t)\right]\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &+ \left(\frac{|\vec{q}|^2 + m^2}{q_0}\right)\left[a(-\vec{q},t) + a^\dagger(\vec{q},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Big\} \\ = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}&q_0\left\{a(\vec{q},t)a^\dagger(\vec{q},t) + a^\dagger(\vec{q},t)a(\vec{q},t)\right\}, \end{aligned}$$

where in the last step we have used the fact that the measure and the extremes are invariant under $\vec{q} \longrightarrow -\vec{q}$, so $a(-\vec{q},t)a^\dagger(-\vec{q},t)$ can be replaced by $a(\vec{q},t)a^\dagger(\vec{q},t)$. Finally one can commute the operators to achieve the normal ordered expression plus an irrelevant infinite constant:

$$P_0 = \int \frac{d^3q}{(2\pi)^3 2q_0}\; q_0\; a^\dagger(\vec{q},t)\; a(\vec{q},t)\; + \text{const}.$$

Similarly:

$$P_i = \int d^3x\; T_{0i} = \int d^3x \int \frac{d^3k}{(2\pi)^3 2k_0}\frac{d^3q}{(2\pi)^3 2q_0}\left\{-k_0q_i\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\right\} e^{i(\vec{k}+\vec{q})\cdot\vec{x}}.$$

Note that the minus sign in front of k_0q_i is due to the fact that $\partial_i e^{i\vec{q}\cdot\vec{x}} = \partial_i e^{iq^ix^i} = iq^ie^{iq^ix^i} = -iq_ie^{i\vec{q}\cdot\vec{x}}$. Again, integrating first over d^3x to generates the delta function on the momentum space and then integrating over one of the momenta one gets:

$$\begin{aligned} P_i &= \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}\left\{-q_i\left[a(-\vec{q},t) - a^\dagger(\vec{q},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\right\} \\ &= \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}q_i\left\{a^\dagger(\vec{q},t)a(\vec{q},t) - a(-\vec{q},t)a^\dagger(-\vec{q},t)\right\} + \frac{1}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}q_i\left\{a^\dagger(\vec{q},t)a^\dagger(-\vec{q},t) - a(-\vec{q},t)a(\vec{q},t)\right\} \\ &= \int \frac{d^3q}{(2\pi)^3 2q_0}\; q_i\; a^\dagger(\vec{q},t)\; a(\vec{q},t) + \text{const}. \end{aligned}$$

The second term in second line vanishes since the integrand is odd under $q \to -q$: indeed $\vec{q}\; a(-\vec{q},t)\; a(\vec{q},t) \to -\vec{q}\, a(\vec{q},t)\; a(-\vec{q},t) = -\vec{q}\; a(-\vec{q},t)\; a(\vec{q},t)$, because $a(\vec{q},t)$ commutes with itself for any q. Finally in the last equality of last equation we have used again $aa^\dagger = a^\dagger a + \text{const}$.

Let us consider the Noether's current associated to rotations and boosts; recalling the transformation properties $\phi'(x) \simeq \phi(x) + \frac{1}{2}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\phi(x)\omega^{\rho\sigma}$, one can define $\Delta_\phi = (x_\rho\partial_\sigma - x_\sigma\partial_\rho)\phi(x)\omega^{\rho\sigma}$ and therefore

$$M_{\mu\rho\sigma} = \partial_\mu\phi(x_\rho\partial_\sigma\phi - x_\sigma\partial_\rho\phi) - (x_\rho\eta_{\mu\sigma} - x_\sigma\eta_{\mu\rho})\mathcal{L} = x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho}.$$

Notice that the Noether's current is defined up to constant rescaling of the transformation parameter: in this case we have considered $\omega^{\rho\sigma}/2$ as parameters. However, if the theory contains objects transforming according some other representation of the Lorentz group, the definition of what the parameters are has to be consistent. In the present case the Noether's charge reads:

$$\int d^3x\ M_{0\rho\sigma} = \int d^3x \left\{x_\rho T_{0\sigma} - x_\sigma T_{0\rho}\right\}.$$

In particular the generator of boosts can be extracted taking the timelike component of previous expression:

$$K_i = \int d^3x\ M_{00i} = \int d^3x \left\{x_0 T_{0i} - x_i T_{00}\right\} = t\, P_i - \int d^3x\, \mathcal{H}\, x_i.$$

The first term is t times the generator of translation and has been already computed, while the second one involves the Hamiltonian density. Let us compute this quantity:

$$\begin{aligned}\int d^3x\, \mathcal{H}\, x_i \quad &= \quad \frac{1}{2}\int d^3x \int \frac{d^3q}{(2\pi)^3 2q_0}\frac{d^3k}{(2\pi)^3 2k_0}\Big\{-k_0q_0\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &\quad + \left(-\vec{k}\cdot\vec{q} + m^2\right)\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Big\}\, x_i e^{i(\vec{k}+\vec{q})\cdot\vec{x}}.\end{aligned}$$

Let us use the following relation:

$$\int d^3x\, x_i\ e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = \int d^3x\, i\frac{\partial}{\partial k^i}e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = i(2\pi)^3\frac{\partial}{\partial k^i}\delta^3(\vec{k}+\vec{q}).$$

We can then integrate by parts the derivative with respect to k^i:

$$\begin{aligned}\int d^3x\, \mathcal{H}\ x_i \quad &= \quad \frac{i}{4}\int \frac{d^3q}{(2\pi)^3 2q_0}d^3k\frac{\partial}{\partial k^i}\Bigg\{q_0\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &\quad - \left(\frac{-\vec{k}\cdot\vec{q} + m^2}{k_0}\right)\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right]\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Bigg\}\, \delta^3(\vec{k}+\vec{q}).\end{aligned}$$

The only subtle point arises in the derivation of the fraction in parentheses:

$$\frac{\partial}{\partial k^i}\left(\frac{-\vec{k}\cdot\vec{q} + m^2}{k^0}\right) = -\frac{q^i}{k_0} + (-\vec{k}\cdot\vec{q} + m^2)\left(-\frac{k^i}{k_0^3}\right),$$

and since the integral contains $\delta^3(\vec{k}+\vec{q})$, after the integration on d^3k it will be $\vec{k} = \vec{q}$, $q_0 = k_0$, and this term will vanish. Therefore we neglect it from now on. Hence

$$\begin{aligned}\int d^3x\, \mathcal{H}\ x_i \quad &= \quad \frac{i}{4}\int \frac{d^3q}{(2\pi)^3 2q_0}d^3k\Bigg\{q_0\left(\frac{\partial}{\partial k^i}\left[a(\vec{k},t) - a^\dagger(-\vec{k},t)\right]\right)\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &\quad - \left(\frac{-\vec{k}\cdot\vec{q} + m^2}{k^0}\right)\left(\frac{\partial}{\partial k^i}\left[a(\vec{k},t) + a^\dagger(-\vec{k},t)\right]\right)\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Bigg\}\, \delta^3(\vec{k}+\vec{q}),\end{aligned}$$

and finally, integrating over d^3k (and then setting $\vec{k} = \vec{q}$):

$$\begin{aligned}\int d^3x\, \mathcal{H}\ x_i \quad &= \quad -\frac{i}{4}\int \frac{d^3q}{(2\pi)^3 2q_0}\, q_0\Bigg\{\left(\frac{\partial}{\partial q^i}\left[a(-\vec{q},t) - a^\dagger(\vec{q},t)\right]\right)\left[a(\vec{q},t) - a^\dagger(-\vec{q},t)\right] \\ &\qquad - \left(\frac{\partial}{\partial q^i}\left[a(-\vec{q},t) + a^\dagger(\vec{q},t)\right]\right)\left[a(\vec{q},t) + a^\dagger(-\vec{q},t)\right]\Bigg\} \\ &= \quad \frac{i}{2}\int \frac{d^3q}{(2\pi)^3 2q_0}\, q_0\left\{\frac{\partial}{\partial q^i}a(-\vec{q},t)\, a^\dagger(-\vec{q},t) + \frac{\partial}{\partial q^i}a^\dagger(\vec{q},t)\, a(\vec{q},t)\right\} \\ &= \quad -i\int \frac{d^3q}{(2\pi)^3 2q_0}\, q_0\left(a^\dagger(\vec{q},t)\, \frac{\partial}{\partial q^i}a(\vec{q},t)\right),\end{aligned}$$

where in the last step we have integrated the second term by parts and commuted the operators $a^\dagger$ and ∂a in the first. This is possible for the following reason: defining $C_i(q) \equiv [a^\dagger(\vec{q},t), \frac{\partial}{\partial q^i} a(\vec{q},t)]$, it is easy to show that $[C_i(q), a(\vec{p},t)] = [C_i(q), a^\dagger(\vec{p},t)] = 0$, so $C_i(q)$ is a $\mathbb{C}$-number, that can be neglected for the purposes of this exercise, as it has been done previously as well. Finally one can compute the generator of boosts:

$$K^i = t\,P^i - \int d^3x\, \mathcal{H}\, x^i = \int \frac{d^3q}{(2\pi)^3 2q_0}\, a^\dagger(\vec{q},t) \left(t q^i - i q_0 \frac{\partial}{\partial q^i} \right) a(\vec{q},t).$$

In the same way one can compute the generators of rotations:

$$J^{ij} = \int d^3x\; M_0{}^{ij} = \int d^3x \left\{ x^i T_0{}^j - x^j T_0{}^i \right\}.$$

Proceeding as before one has

$$\int d^3x\, x^i T_0{}^j = \int d^3x \int \frac{d^3q}{(2\pi)^3 2q_0} \frac{d^3k}{(2\pi)^3 2k_0} \left\{ -k_0 q^j \left[a(\vec{k},t) - a^\dagger(-\vec{k},t) \right] \left[a(\vec{q},t) + a^\dagger(-\vec{q},t) \right] \right\} x^i e^{i(\vec{k}+\vec{q})\cdot\vec{x}},$$

and integrating by parts the derivative with respect to k^i (this time it's straightforward since k_0 simplifies):

$$\int d^3x\, x^i T_0{}^j = -\frac{i}{2} \int \frac{d^3q}{(2\pi)^3 2q_0} d^3k \left\{ q^j \frac{\partial}{\partial k^i} \left[a(\vec{k},t) - a^\dagger(-\vec{k},t) \right] \left[a(\vec{q},t) + a^\dagger(-\vec{q},t) \right] \right\} \delta^3(\vec{k}+\vec{q}).$$

Finally integrating over d^3k gives:

$$\int d^3x\, x^i T_0{}^j = \frac{i}{2} \int \frac{d^3q}{(2\pi)^3 2q_0} q^j \left\{ \frac{\partial}{\partial q^i} \left[a(-\vec{q},t) - a^\dagger(\vec{q},t) \right] \right\} \left[a(\vec{q},t) + a^\dagger(-\vec{q},t) \right].$$

In the end we have:

$$\begin{aligned}
J^{ij} &= \int d^3x\; M_0{}^{ij} = \int d^3x \left\{ x^i T_0{}^j - x^j T_0{}^i \right\} \\
&= \frac{i}{2} \int \frac{d^3q}{(2\pi)^3 2q_0} \left\{ \left[q^j \frac{\partial}{\partial q^i} a(-\vec{q},t) \right] a(\vec{q},t) - \left[q^j \frac{\partial}{\partial q^i} a^\dagger(\vec{q},t) \right] a^\dagger(-\vec{q},t) \right. \\
&\quad \left. + \left[q^j \frac{\partial}{\partial q^i} a(-\vec{q},t) \right] a^\dagger(-\vec{q},t) - \left[q^j \frac{\partial}{\partial q^i} a^\dagger(\vec{q},t) \right] a(\vec{q},t) \right\} - (i \leftrightarrow j).
\end{aligned}$$

The antisymmetrization causes the first line to vanish, while the two terms in the second are identical (up to some infinite constant). Indeed integrating by parts one can show that the former terms are symmetric in i, j:

$$\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3 2q_0} \left(q^j \frac{\partial}{\partial q^i} a(-\vec{q},t) \right) a(\vec{q},t) - (i \leftrightarrow j) = - \left(\int \frac{d^3q}{(2\pi)^3}\, a(-\vec{q},t) \frac{\partial}{\partial q^i} \left[\frac{q^j}{2q_0} a(\vec{q},t) \right] - (i \leftrightarrow j) \right) \\
&= - \left(\int \frac{d^3q}{(2\pi)^3 2q_0}\, a(-\vec{q},t) q^j \frac{\partial}{\partial q^i} a(\vec{q},t) + \int \frac{d^3q}{(2\pi)^3}\, a(-\vec{q},t) a(\vec{q},t) \frac{\partial}{\partial q^i} \left(\frac{q^j}{2q_0} \right) - (i \leftrightarrow j) \right) \\
&= - \left(\int \frac{d^3q}{(2\pi)^3 2q_0}\, a(-\vec{q},t) q^j \frac{\partial}{\partial q^i} a(\vec{q},t) - (i \leftrightarrow j) \right) - \underbrace{\left(\int \frac{d^3q}{(2\pi)^3}\, a(-\vec{q},t) a(\vec{q},t) \left(\frac{\delta^{ij}}{2q_0} - \frac{q^i q^j}{2q_0^3} \right) - (i \leftrightarrow j) \right)}_{\text{symmetric} \Longrightarrow =0} \\
&= - \left(\int \frac{d^3q}{(2\pi)^3 2q_0} \left(q^j \frac{\partial}{\partial q^i} a(-\vec{q},t) \right) a(\vec{q},t) - (i \leftrightarrow j) \right) = 0,
\end{aligned}$$

since we have shown that this term is equal to minus itself. In the last line we have commuted the operators and changed sign to $\vec{q}$. At the very end the generators J^{ij} read:

$$J^{ij} = i \int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q},t) \left(q^j \frac{\partial}{\partial q^i} - q^i \frac{\partial}{\partial q^j} \right) a(\vec{q},t).$$

Finally one can show that these charges don't depend on time, even if $a(\vec{q},t) = a(\vec{q}) e^{-iq_0 t}$ does. Clearly in the product $a^\dagger(\vec{q},t)a(\vec{q},t)$ the factors cancels. Therefore:

$$
\begin{aligned}
P^\mu &= \int \frac{d^3q}{(2\pi)^3 2q_0} q^\mu\, a^\dagger(\vec{q},t)a(\vec{q},t) = \int \frac{d^3q}{(2\pi)^3 2q_0} q^\mu\, a^\dagger(\vec{q})a(\vec{q}),\\
K^i &= \int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q},t)\left(tq^i - iq_0\frac{\partial}{\partial q^i}\right)a(\vec{q},t)\\
&= -\int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q})\left(iq_0\frac{\partial}{\partial q^i}\right)a(\vec{q}) + \int \frac{d^3q}{(2\pi)^3 2q_0}\, a^\dagger(\vec{q})a(\vec{q})\underbrace{\left(tq^i - iq_0(-it)\frac{\partial}{\partial q^i}q_0\right)}_{=0},\\
J^{ij} &= -i\int \frac{d^3q}{(2\pi)^3 2q_0} a^\dagger(\vec{q})\left(q^i\frac{\partial}{\partial q^j} - q^j\frac{\partial}{\partial q^i}\right)a(\vec{q}) - \int \frac{d^3q}{(2\pi)^3 2q_0}\, a^\dagger(\vec{q})a(\vec{q})\underbrace{\left(q^i\frac{\partial}{\partial q^j} - q^j\frac{\partial}{\partial q^i}\right)(iq_0t)}_{\propto\, q^iq^j - q^jq^i = 0}.
\end{aligned}
$$

Additional note

Instead of using the representation of the canonical fields $\phi(\vec{x},t)$, $\pi(\vec{x},t)$ in terms of the ladder operators $a(\vec{k})$, $a^\dagger(\vec{k})$, one could directly work with ϕ and π. Indeed, the following equation holds:

$$\frac{dQ_i}{dt} = \frac{\partial Q_i}{\partial t} + i[H,Q_i]$$

since H is the generator of time translations. Thus, just by using the commutation relations $[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$, one could check that the right-hand side of the previous equation vanishes. Notice that only the boost generators K_i have an explicit time-dependence, $\frac{\partial K_i}{\partial t} \neq 0$. For the other ones only the relation $[H, Q_i(t)] = 0$ must be checked (which means that the Hamiltonian is invariant under the transformations generated by Q_i, as expected).

Exercise 3

Given the canonical commutation relation at equal time:

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}),$$

we want to show that the Noether charges are the generators of the infinitesimal transformation in the following sense: if a transformation acts on coordinates and fields as $x'^\mu = x^\mu - \epsilon^\mu_i(x)\alpha^i$, $\phi'(x) = \phi(x) + \Delta_i(x)\alpha^i$ then:

$$[Q_i,\phi(x)] = i\Delta_i(x).$$

In Solution8 we have shown the analogous of this expression for classical field theory, where the commutators are replaced by Poisson brackets. One could start from the Noether's charge

$$Q_i = \int d^3x\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\Delta_i - \epsilon^0_i\mathcal{L}\right),$$

and derive the result following the same steps, since the canonical commutation relation have the same form as the Poisson brackets. Indeed, since the charges do not depend on time we can choose t in order to use equal time commutation rules. For example:

$$
\begin{aligned}
[J^{ij},\phi(\vec{x},t)] &= \int d^3y\ [y^i\pi(\vec{y},t)\partial^j\phi(\vec{y},t) - y^j\pi(\vec{y},t)\partial^i\phi(\vec{y},t),\phi(\vec{x},t)]\\
&= \int d^3y\left\{y^i[\pi(\vec{y},t),\phi(\vec{x},t)]\partial^j\phi(\vec{y},t) - (i\leftrightarrow j)\right\}\\
&= -i\int d^3y\ \delta^3(\vec{x}-\vec{y})\left\{y^i\partial^j\phi(\vec{y},t) - (i\leftrightarrow j)\right\}\\
&= -i\left(x^i\partial^j - x^j\partial^i\right)\phi(\vec{x},t).
\end{aligned}
$$

Similarly:

$$\begin{aligned}
[P^i, \phi(\vec{x},t)] &= \int d^3y\ [\pi(\vec{y},t)\partial^i\phi(\vec{y},t), \phi(\vec{x},t)] \\
&= \int d^3y [\pi(\vec{y},t), \phi(\vec{x},t)]\partial^i\phi(\vec{y},t) \\
&= -i\int d^3y\, \delta^3(\vec{x}-\vec{y})\partial^i\phi(\vec{y},t) = -i\partial^i\phi(\vec{x},t).
\end{aligned}$$

From the Poincaré algebra $[J^{ij}, P^k] = -i(\delta^{ik}\delta^{jn} - \delta^{jk}\delta^{in})P^n$, we deduce

$$\begin{aligned}
[[J^{ij}, P^k], \phi(\vec{x},t)] &= -i(\delta^{ik}\delta^{jn} - \delta^{jk}\delta^{in})[P^n, \phi(\vec{x},t)] \\
&= (\delta^{jk}\delta^{in} - \delta^{ik}\delta^{jn})\partial^n\phi(\vec{x},t) \\
&= \left(x^i\partial^j - x^j\partial^i\right)\partial^k\phi(\vec{x},t) - \partial^k\left(x^i\partial^j - x^j\partial^i\right)\phi(\vec{x},t) \\
&= -[P^k, [J^{ij}, \phi(\vec{x},t)]] + [J^{ij}, [P^k, \phi(\vec{x},t)]].
\end{aligned}$$

Whihc is precisely the Jacobi identity.

Exercise 4

The commutation relations for the ladder operators a_i, $a_i^\dagger$ are (in the normalization convention such that $d\Omega_k = \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}$):

$$[a_i(\vec{k}), a_j^\dagger(\vec{q})] = \delta_{ij}\delta^3(\vec{k}-\vec{q})$$

$$[a_i(\vec{k}), a_j(\vec{q})] = [a_i^\dagger(\vec{k}), a_j^\dagger(\vec{q})] = 0$$

- From the above commutation relations, after simple algebraic manipulations one finds:

  $$[a(\vec{k}), a^\dagger(\vec{q})] = [b(\vec{k}), b^\dagger(\vec{q})] = \delta^3(\vec{k}-\vec{q})$$

  $$[a(\vec{k}), b(\vec{q})] = [a^\dagger(\vec{k}), b(\vec{q})] = [a(\vec{k}), b^\dagger(\vec{q})] = [a^\dagger(\vec{k}), b^\dagger(\vec{q})] = 0$$

  This results can be simply interpreted as the fact that the creation operators $a^\dagger$ and $b^\dagger$ create two different kinds of particles.

- Since the Lagrangian is the sum of two pieces dependent separately on ϕ_1, ϕ_2, the same holds also for the Hamiltonian:

  $$H = H_{KG}[\phi_1] + H_{KG}[\phi_2]$$

  In previous problems we found:

  $$: H_{KG}[\phi_i] := \int d^3k\, \omega_k \mathcal{N}_i(\vec{k}), \qquad \mathcal{N}_i(\vec{k}) = a_i^\dagger(\vec{k})a_i(\vec{k})$$

  where $\mathcal{N}_i(\vec{k})$ is the number operator for particles of type i. After simple algebraic manipulations we can express H as:

  $$: H := \int d^3k\, \omega_k \left(\mathcal{N}_a(\vec{k}) + \mathcal{N}_b(\vec{k})\right), \qquad \mathcal{N}_a(\vec{k}) = a^\dagger(\vec{k})a(\vec{k}), \quad \mathcal{N}_b(\vec{k}) = b^\dagger(\vec{k})b(\vec{k})$$

- The total charge Q is:

  $$Q = i\int d^3x \left(\dot{\phi}^\dagger\phi - \phi^\dagger\dot{\phi}\right)$$

  After plugging in the expression for ϕ and some computational steps:

  $$Q = \int d^3k \left(\mathcal{N}_b(\vec{k}) - \mathcal{N}_a(\vec{k})\right)$$

where we used the representation for the δ function: $\int d^3x\, e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k})$. It should be clear from the last expression that the total charge is given by the number of particles of type b minus the number of particles of type a. The particles created by a (b) therefore have positive (negative) charge, and their are said to be the *antiparticle* to each other. In this example, the presence of particle and antiparticle arose naturally, but it is a very fundamental property for the consistency of a quantum field theory that to each particle there it should corresponding an antiparticle (look up CPT theorem if you are interested).

- It can be checked easily that $Q|\psi\rangle = (m-n)|\psi\rangle$

Exercise 5

Since the potential depends only on the modulus of the field $|\phi|$ we can minimize it with respect to this variable:

$$\frac{\partial V}{\partial|\phi|} = 2m^2|\phi| + \lambda|\phi|^3$$

- For $m^2 > 0$ the only real solution is $|\phi| = 0$, corresponding to $\phi = 0$. Therefore, we can expand ϕ around this minimum as $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$. In this parametrization the Lagrangian looks like:

$$\mathcal{L} = \partial_\mu\phi_1\partial^\mu\phi_1 + \frac{m^2}{2}\phi_1^2 + \partial_\mu\phi_2\partial^\mu\phi_2 + \frac{m^2}{2}\phi_2^2 + \frac{\lambda}{16}(\phi_1^2+\phi_2^2)^2$$

  representing two fields of mass m with an interaction term. The $U(1)$ symmetry is simply manifested as a rotation in the ϕ_1,ϕ_2 plane.

- Repeating the same steps above, but now setting $m^2 = -\mu^2$ such that $\mu^2 > 0$, we find extrema at the potential at $|\phi| = 0$ and $|\phi| = v \equiv \sqrt{\frac{2\mu^2}{\lambda}}$. Taking the second derivative of one can check that the former is a local maximum, while the latter is a minimum. This corresponds to a continuous line of values for ϕ in the complex plane, which are all connected to each other by the $U(1)$ symmetry. However the vacuum has to lie in a particular value, say $\phi = v$. It is said that the symmetry is *spontaneously broken* by the vacuum.

- With the parametrization $\phi = v + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ the kinetic term reads:

$$\partial_\mu\phi\partial^\mu\phi^* = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2\,.$$

  and the potential:

$$V = -\frac{\mu^4}{\lambda} + \frac{2\mu^2}{2}\phi_1^2 + \frac{\lambda}{16}\phi_1^4 + \frac{\lambda}{16}\phi_2^4 + \frac{\lambda}{2}\sqrt{\frac{\mu^2}{\lambda}}\phi_1^3 + \frac{\lambda}{2}\sqrt{\frac{\mu^2}{\lambda}}\phi_1\phi_2^2 + \frac{\lambda}{8}\phi_1^2\phi_2^2\,.$$

  Thus, we see that the field ϕ_1 has mass $\sqrt{2}\mu$, while the field ϕ_2 is massless (a quadratic term in ϕ_2 is absent). In this parametrization it is seems that the original symmetry is gone.

- By choosing the parametrization $\phi = \frac{v+h}{\sqrt{2}}e^{i\pi}$, the kinetic terms looks like:

$$\frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}(v+h)^2\partial_\mu\pi\partial^\mu\pi$$

  and contains interactions between the fields. On the contrary the potential depends only on h:

$$V = \frac{m^2}{2}v^2 + \frac{\lambda}{16}v^4 + \frac{m^2}{2}h^2 + \frac{\lambda}{4}vh^3 + \frac{\lambda}{16}h^4$$

  Notice how the field π appears only in combinations with derivatives (and of course it is massless). Therefore, the Lagrangian in invariant under the transformation $\pi(x) \to \pi(x) + c$ for any constant c, and the field π is said to be a Goldstone boson. This *shift* symmetry for π is the manifestation of the original $U(1)$ symmetry after it is spontaneously broken.