

Quantum Field Theory

Homework : solutions

Exercise 1

Given the Lorentz transformation:

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \\ \psi'_L(x') = \Lambda_L \psi_L(\Lambda^{-1}x') = e^{-\frac{1}{2}(i\theta^i + \eta^i)\sigma^i} \psi_L(\Lambda^{-1}x'),$$

where θ, η are the parameters associated respectively to rotations and boosts, one can consider the bilinear $\psi_L^\dagger \bar{\sigma}^\mu \psi_L$, where as usual $\bar{\sigma}^\mu = (1, -\sigma^i)$. We recall the notation for spinorial indices:

$$\psi_{L\alpha}, \quad \psi_{L\dot{\beta}}^\dagger, \quad (\bar{\sigma}^\mu)^{\dot{\beta}\alpha},$$

since the $\dagger$ transforms undotted indexes in dotted ones and vice versa. Thus the transformation properties of the bilinear are:

$$\psi_{L\dot{\beta}}'^\dagger (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \psi_{L\alpha}' = \psi_{L\dot{\gamma}}^\dagger (\Lambda_L^\dagger)^{\dot{\gamma}\dot{\beta}} (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} (\Lambda_L)^\delta_\alpha \psi_{L\delta}.$$

We now recall that $\Lambda_L^\dagger \bar{\sigma}^\mu \Lambda_L = \Lambda^\mu_\nu \bar{\sigma}^\nu$ to conclude that this bilinear transforms in the representation $(1/2, 1/2)$. This is the expected results since all the spinorial indices are contracted while one vector index is free (in practice the $\bar{\sigma}^\mu$ represent the Clebsch-Gordan coefficient needed to pass from the $(1/2, 0) \otimes (0, 1/2)$ to the $(1/2, 1/2)$). Let us consider now the left doublet Ψ_L and the right singlet ψ_R with transformation properties under $SU(2)$ and Lorentz given by:

$$\begin{aligned} \text{Lorentz} & \begin{cases} x'^\mu = \Lambda^\mu_\nu x^\nu \\ (\Psi'_L)_\alpha(x') = (\Lambda_L)_\alpha^\beta (\Psi_L)_\beta(\Lambda^{-1}x') \\ (\psi'_R)^{\dot{\beta}}(x') = (\Lambda_R)^{\dot{\beta}}_{\dot{\delta}} (\psi_R)^{\dot{\delta}}(\Lambda^{-1}x') \end{cases} \\ \text{Isospin} & \begin{cases} x'^\mu = x^\mu \\ (\Psi'_L)_\alpha(x') = U^a_b (\Psi_L)_\alpha^b(x) \\ (\psi'_R)^{\dot{\beta}}(x') = (\psi_R)^{\dot{\beta}}(x) \end{cases} \end{aligned}$$

Hence the bilinear $\psi_R^\dagger \Psi_L = \psi_R^{\dagger\alpha} \Psi_{L\alpha}$ transforms as:

$$\text{Lorentz:} \quad \psi_R^{\dagger\alpha} \Psi'_{L\alpha} = (\Lambda_R \psi_R)^{\dagger\alpha} (\Lambda_L \Psi_L)_\alpha = \psi_R^{\dagger\gamma} (\Lambda_R^\dagger)_\gamma^\alpha (\Lambda_L)_\alpha^\beta \Psi_{L\beta} = \psi_R^{\dagger\gamma} \Psi_{L\gamma},$$

since $\Lambda_R^\dagger = \Lambda_L^{-1}$. Hence the latter is a scalar under Lorentz transformations. Note that we have omitted the x dependence but clearly the complete relation would be:

$$\psi_R^{\dagger\alpha}(x') \Psi'_{L\alpha}(x') = \psi_R^{\dagger\alpha}(\Lambda^{-1}x') \Psi_{L\alpha}(\Lambda^{-1}x'),$$

which is the usual one for scalar quantities. Under $SU(2)$ transformation one gets:

$$\text{Isospin:} \quad \psi_R^{\dagger\alpha} \Psi_L'^a = \psi_R^{\dagger\alpha} U^a_b \Psi_L^b = U^a_b \psi_R^{\dagger\alpha} \Psi_L^b.$$

therefore the bilinear is an Isospin doublet. Still, this was expected since the spinor indices are all contracted while, concerning $SU(2)$, we are considering the product: $0 \otimes 1/2 = 1/2$.

Let's consider finally the term $\Psi_L^\dagger \sigma^i \not{\partial} \Psi_L$:

$$(\Psi_L^\dagger)_{\dot{\alpha}b} (\sigma^i)^b_a (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \partial_\mu (\Psi_L)_\beta^a,$$

where by convention the $\dagger$ exchanges dotted with undotted indices and lowers the Isospin index. Then the two transformations give:

$$\begin{aligned} \text{Lorentz:} \quad & (\Psi_L^\dagger)_{\dot{\alpha}b}(\sigma^i)^b_a(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\partial'_\mu(\Psi'_L)^a_\beta = (\Psi_L^\dagger)_{\dot{\gamma}}(\Lambda_L^\dagger)^{\dot{\gamma}}_{\dot{\alpha}}\sigma^i(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\Lambda^\nu_\mu\partial_\nu(\Lambda_L)^\delta_\beta(\Psi_L)_\delta = \Lambda^\mu_\nu\Lambda^\rho_\mu\Psi_L^\dagger\sigma^i\bar{\sigma}^\nu\partial_\rho\Psi_L \\ & = (\Lambda^{-1})^\mu_\nu\Lambda^\rho_\mu\Psi_L^\dagger\sigma^i\bar{\sigma}^\nu\partial_\rho\Psi_L = \Psi_L^\dagger\sigma^i\bar{\sigma}^\nu\partial_\nu\Psi_L, \\ \text{Isospin:} \quad & (\Psi'_L)^\dagger_b(\sigma^i)^b_a\partial(\Psi'_L)^a = (\Psi_L^\dagger)_c(U^\dagger)^c_b(\sigma^i)^b_a\partial U^a_d(\Psi_L)^d = R^{(j=1)}[U]^i_j(\Psi_L^\dagger)_b(\sigma^j)^b_a\partial(\Psi_L)^a, \end{aligned}$$

where we have made use of the relations:

$$\begin{aligned} (\Lambda_L^\dagger)^{\dot{\gamma}}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\Lambda_L)_\beta^\delta &= \Lambda^\mu_\nu(\bar{\sigma}^\mu)^{\dot{\gamma}\delta}, \\ (U^\dagger)^c_b(\sigma^i)^b_aU^a_d &= R^{(j=1)}[U]^i_j(\sigma^j)^c_d. \end{aligned}$$

In the end the latter bilinear is a Lorentz scalar while is an Isospin vector (that is to say it transforms in the $j = 1$ representation of $SU(2)$).

Exercise 2

SU(2)

- $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. Using the Pauli matrices σ^i we can explicitly decompose

$$u_a v_a^* \sim 0, \quad \sigma^i_{ab} u_b v_a^* \sim 1.$$

If U is an $SU(2)$ matrix then $U^\dagger U = 1$, or $U_{ba}^* U_{bc} = \delta_{ac}$. The doublets transform as $u_a \rightarrow U_{ac} u_c$ and $v_b^* \rightarrow U_{bd}^* v_d^*$. We can then check that the first term is indeed a singlet, while the second makes use of the identity $U^\dagger \sigma^i U = R_j^i \sigma^j$ (derive it), where R is in the 3 dimensional representation of $SU(2)$, the vector irrep.

- $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. Noting that $\epsilon_{ab} v_b \sim v_b^* \sim \frac{1}{2}$ we have

$$\epsilon_{ab} u_a v_b \sim u_{[a} v_{b]} \sim 0, \quad \sigma^i_{ab} \epsilon_{ac} u_b v_c \sim u_{(a} v_{b)} \sim 1.$$

To derive the above we can simply replace $v_a^* \rightarrow \epsilon_{ab} v_b$ in the previous solution for $\frac{1}{2} \otimes \frac{1}{2}$, or directly transform the above terms and see how they transform, as was done for $\frac{1}{2} \otimes \frac{1}{2}$. Alternatively, one can get the singlet as the antisymmetric product of u_a and v_b since an antisymmetric matrix in 2 dimensions only has 1 independent component, and since symmetry (anti-symmetry) of a matrix is preserved under $SU(2)$ transformations, that is symmetric (anti-symmetric) tensors form an invariant subspace, the only possibility is for $u_{[a} v_{b]}$, which has dimension 1, to be the singlet. The triplet comes from orthogonalizing with respect to the singlet, which is nothing but the symmetric combination of the product of u_a and v_b . Symmetric matrices in 2 dimensions have 3 independent components, matching the dimension of the triplet or vector irreducible representation, and thus the only possibility is for the triplet to be the symmetrized product $u_{(a} v_{b)}$.

Lorentz

- $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0, 0) \oplus (1, 0)$. Use that $\epsilon \Lambda_L = \Lambda_R^* \epsilon$ (derive it) and $\Lambda_R^\dagger = \Lambda_L^{-1}$ to get

$$\epsilon_{\alpha\beta} \psi_\alpha \phi_\beta \sim \psi_{[\alpha} \phi_{\beta]} \sim (0, 0), \quad \sigma^i_{\alpha\beta} \epsilon_{\alpha\gamma} \psi_\beta \phi_\gamma \sim \psi_{(\alpha} \phi_{\beta)} \sim (1, 0);$$

- $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, 0) = (0, \frac{1}{2}) \oplus (1, \frac{1}{2})$. Noting that $A_{\beta\delta} \equiv \sigma_{\beta\delta}^\mu A_\mu \sim (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ we find

$$\epsilon_{\alpha\beta} \psi_\alpha A_{\beta\delta} \sim \psi_{[\alpha} A_{\beta]\delta} \sim \left(0, \frac{1}{2}\right), \quad \sigma^i_{\alpha\beta} \epsilon_{\alpha\gamma} \psi_\beta A_{\gamma\delta} \sim \psi_{(\alpha} A_{\beta)\gamma} \sim \left(1, \frac{1}{2}\right);$$

- $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (0, 0) \oplus (0, 1) \oplus (1, 0) \oplus (1, 1)$. Again write $B_{\alpha\sigma} \equiv \sigma_{\alpha\sigma}^\nu B_\nu \sim (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$. Noting the identities $\bar{\sigma}^\mu \sigma^\nu + \sigma^\mu \bar{\sigma}^\nu = \eta^{\mu\nu}$ and $\epsilon^{-1} \sigma^\mu \epsilon = \bar{\sigma}^{*\mu}$ we find

$$\epsilon_{\alpha\beta} \epsilon_{\sigma\delta} B_{\alpha\sigma} A_{\beta\delta} = \eta^{\mu\nu} A_\mu B_\nu = A \cdot B \sim (0, 0).$$

The remainder are found by following this procedure sistematically. However, note that Lorentz transformations preserve symmetry and anti-symmetry, meaning that the space of (anti-)symmetric matrices is an *invariant subspace*. The same goes with the trace, which is invariant under Lorentz transformations. If we take out the trace of symmetric matrices we end up with a 9 independent component vector space, the same as $\dim(1,1) = 3 \times 3 = 9$. This is no coincidence. You can show that the $(1,1)$ irrep only depends on the projection of $A_\mu B_\nu$ into this invariant subspace. Similarly, an antisymmetric tensor only has 6 independent components, which we can identify with $(1,0) \oplus (0,1)$ which has dimension $3+3=6$. Therefore, one guesses

$$A_{(\mu} B_{\nu)} - \frac{1}{4} \eta_{\mu\nu} A \cdot B \sim (1,1), \quad A_{[\mu} B_{\nu]} \sim (1,0) \oplus (0,1).$$

More details on Lorentz

A generic tensor transforming in an irreducible representation $(j/2, \bar{j}/2)$ of the Lorentz group (or, more precisely, of $SL(2, \mathbb{C})$), is an object with j and $\bar{j}$ unprimed and primed indices which transforms as follows:

$$t_{A_1 \dots A_j}^{A'_1 \dots A'_j} \longrightarrow (\Lambda_L)_{A_1}^{B_1} \dots (\Lambda_L)_{A_j}^{B_j} (\Lambda_R)_{B'_1}^{A'_1} \dots (\Lambda_R)_{B'_j}^{A'_j} t_{B_1 \dots B_j}^{B'_1 \dots B'_j}.$$

Notice that for this tensor to belong to the specified irreducible representations all indices must be symmetrized. Indeed any pair anitisymmetric of indices must be proportional to the tensors ϵ_{AB} or $\epsilon^{A'B'}$, which are invariant under $SL(2, \mathbb{C})$ transformations.

A generic tensor with latin indices can be decomposed into irreducible representations explicitly via contraction with the Pauli Matrices $\sigma_{AA'}^\mu$ and $\bar{\sigma}^{\mu A'A}$. For instance a vector V_μ belongs to a $(1/2, 1/2)$ irreducible representation, since it can be written as

$$V_{AA'} \equiv \sigma_{AA'}^\mu V_\mu$$

which transform as (recall that indices can be raised and lowered using ϵ_{AB} , ϵ^{AB} , $\epsilon_{A'B'}$, $\epsilon^{A'B'}$)

$$V_{AA'} \longrightarrow (\Lambda_L)_A^B (\Lambda_R)_{A'}^{B'} V_{BB'}.$$

Finally, since the algebra of $SL(2, \mathbb{C})$ is isomorphic to the complexified sum of two independent $SU(2)$ subalgebras,

$$\mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{su}(2) \oplus_{\mathbb{C}} \mathfrak{su}(2),$$

tensor prducts can be formally decomposed using the usual $SU(2)$ rule both for j and $\bar{j}$:

$$\begin{aligned} (j_1, \bar{j}_1) \otimes (j_2, \bar{j}_2) &= (j_1 \otimes j_2, \bar{j}_1 \otimes \bar{j}_2) \\ &= (|j_1 - j_2| \oplus |j_1 - j_2| + 1 \oplus \dots \oplus j_1 + j_2, |\bar{j}_1 - \bar{j}_2| \oplus |\bar{j}_1 - \bar{j}_2| + 1 \oplus \dots \oplus \bar{j}_1 + \bar{j}_2) \\ &= (|j_1 - j_2|, |\bar{j}_1 - \bar{j}_2|) \oplus (|j_1 - j_2| + 1, |\bar{j}_1 - \bar{j}_2|) \oplus \dots \oplus (j_1 + j_2, \bar{j}_1 + \bar{j}_2). \end{aligned}$$

Here we formally wrote, abusing of notation, $(j_a \oplus j_b, \bar{j}_a \oplus \bar{j}_b)$ to really mean $(j_a, \bar{j}_a) \oplus (j_b, \bar{j}_a) \oplus (j_a, \bar{j}_b) \oplus (j_b, \bar{j}_b)$.

Let us now discuss two applications.

- Consider the product of a vector and a left handed spinor:

$$A_\mu \psi_A.$$

The tensor product decomposition can be computed formally as:

$$(1/2, 1/2) \otimes (1/2, 0) = (1/2 \otimes 1/2, 1/2) = (0 \oplus 1, 1/2) = (0, 1/2) \oplus (1, 1/2).$$

The two representations can be obtained explicitly considering

$$\sigma_{BB'}^\mu V_\mu \psi_A = V_{BB'} \psi_A \equiv \Phi_{AB B'}.$$

Symmetrizing and antisymmetrizing indices the representations are obtained:

$$\Phi_{(AB) B'} \equiv \frac{1}{2} (\Phi_{AB B'} + \Phi_{BA B'}) \sim (1, 1/2),$$

$$\Phi_{[AB] B'} \equiv \frac{1}{2} (\Phi_{AB B'} - \Phi_{BA B'}) = \frac{1}{2} \epsilon_{AB} \Phi_{B'} \sim (0, 1/2).$$

where we defined $\Phi_{B'} \equiv \epsilon^{AB} \Phi_{AB B'}$ and we used the fact that any $2d$ antisymmetric tensor must be proportional to ϵ_{AB} ; the $1/2$ in the last equation is fixed contracting with ϵ^{AB} . Finally we use parenthesis $()$ for symmetrization of indices, square parenthesis $[]$ for antisymmetrization.

- Consider now the product

$$A_\mu B_\nu.$$

Recalling that a vector belongs to $(1/2, 1/2)$, the tensor product gives:

$$(1/2, 1/2) \otimes (1/2, 1/2) = (0, 0) \oplus (1, 0) \oplus (0, 1) \oplus (1, 1).$$

To write explicitly the tensor product decomposition one could proceed systematically, but there is a simpler (and more instructive) way. Recall that Lorentz transformations preserve the symmetricity or antisymmetricity of indices. Also, the trace of a tensor is trivially left unchanged by Lorentz transformations. Hence we can divide the product $A_\mu B_\nu$ into three tensors which transform independently:

$$\begin{aligned} S_{\mu\nu} &\equiv A_{(\mu} B_{\nu)} - \frac{1}{4} \eta_{\mu\nu} A \cdot B \\ A_{\mu\nu} &\equiv A_{[\mu} B_{\nu]} \\ T &\equiv \eta^{\mu\nu} A_\mu B_\nu = A \cdot B. \end{aligned}$$

These are, respectively, the traceless symmetric part, the antisymmetric part and the trace. Now we can match with the decomposition we found before simply counting the components of these tensors. Indeed we know from the theory of $SU(2)$ that a $(0, 0)$ representation (scalar) has just one component, the representations $(1, 0)$ and $(0, 1)$ have 3 each, while the representation $(1, 1)$ has $3 \times 3 = 9$ components. Then, after counting the components of the tensors, the only possible identifications are:

$$S_{\mu\nu} \sim (1, 1), \quad A_{\mu\nu} \sim (1, 0) \oplus (0, 1), \quad T \sim (0, 0).$$

This means that $S_{\mu\nu}$ can be written as a tensor $S_{AB}^{A'B'}$, $A_{\mu\nu}$ as $A_{AB} \epsilon^{A'B'} + A^{A'B'} \epsilon_{AB}$ and T is a scalar. To be more explicit, we can define $S_{AB}^{A'B'}$ as:

$$\sigma_{AA'}^\mu \sigma_{BB'}^\nu S_{\mu\nu} \equiv S_{AB A'B'} \sim (1, 1).$$

To write the antisymmetric part, we notice that the only two objects at our disposal to convert an antisymmetric tensor in latin indices to $(0, 1)$ and $(1, 0)$ representations are

$$(\sigma^{\mu\nu})_A^B = \frac{1}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_A^B, \quad (\bar{\sigma}^{\mu\nu})_{B'}^{A'} = \frac{1}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{B'}^{A'}.$$

Then we can define

$$(\sigma^{\mu\nu})_A^B A_{\mu\nu} \equiv A_A^B \sim (1, 0), \quad (\bar{\sigma}^{\mu\nu})_{B'}^{A'} A_{\mu\nu} \equiv A_{B'}^{A'} \sim (0, 1).$$

Exercise 3

The action for a free massless scalar field in d dimension is

$$\mathcal{S} = \frac{1}{2} \int dt d^{d-1}x \partial_\mu \phi(x) \partial^\mu \phi(x).$$

We consider the scale transformation labelled by the parameter $\lambda \in \mathbb{R}$ and defined as

$$\begin{aligned} x'^\mu &= e^\lambda x^\mu, \\ \phi'(x') &= e^{k\lambda} \phi(x) = e^{k\lambda} \phi(e^{-\lambda} x'). \end{aligned}$$

Expanding for infinitesimal parameter one gets

$$\begin{aligned} x'^\mu &\simeq x^\mu + \lambda x^\mu + O(\lambda^2) \implies \epsilon^\mu = -x^\mu, \\ \phi'(x) &= (1 + k\lambda + O(\lambda^2)) \phi(x - \lambda x + \dots) \simeq (1 + k\lambda + O(\lambda^2)) (\phi(x) - \lambda x^\mu \partial_\mu \phi(x) + O(\lambda^2)) \\ &\simeq \phi(x) + k\lambda \phi(x) - \lambda x^\mu \partial_\mu \phi(x) + O(\lambda^2) \implies \Delta(x) = k\phi(x) - x^\mu \partial_\mu \phi(x). \end{aligned}$$

Here the usual indices a, i labeling different fields and parameters have disappeared since they assume only the value $a = i = 1$. In order to define a symmetry of the theory these transformation must leave invariant the action:

$$\begin{aligned} x' &= e^\lambda x & d^d x' &= e^{d\lambda} d^d x & \partial'_\mu &= e^{-\lambda} \partial_\mu, \\ \partial'_\mu \phi'(x') &= e^{k\lambda} \partial'_\mu \phi(x) = e^{(k-1)\lambda} \partial_\mu \phi(x), \\ \frac{1}{2} \int d^d x \partial_\mu \phi(x) \partial^\mu \phi(x) &\longrightarrow \frac{1}{2} \int d^d x \partial_\mu \phi(x) \partial^\mu \phi(x) e^{(2k-2+d)\lambda} \\ \implies (2k-2+d)\lambda &= 0 \implies k = 1 - \frac{d}{2}. \end{aligned}$$

In last equation we have discarded the solution $\lambda = 0$, which corresponds to the identical transformation, which is always an uninteresting symmetry.

In four dimension, $k = -1$ and the Noether's current reads

$$\begin{aligned} S^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta - \epsilon^\mu \mathcal{L} = -(\phi + x^\nu \partial_\nu \phi) \partial^\mu \phi + \frac{1}{2} x^\mu (\partial_\nu \phi(x) \partial^\nu \phi(x)) \\ &= -\phi \partial^\mu \phi - x^\nu \partial_\nu \phi \partial^\mu \phi + \frac{1}{2} x^\mu \partial_\nu \phi \partial^\nu \phi. \end{aligned}$$

Recalling the definition of the energy momentum tensor associated to this Lagrangian

$$T^\mu_\rho = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L} = \partial_\rho \phi \partial^\mu \phi - \frac{1}{2} \delta^\mu_\rho (\partial_\nu \phi \partial^\nu \phi),$$

one has

$$T^\mu_\mu = \partial_\mu \phi \partial^\mu \phi - \frac{4}{2} \partial_\nu \phi \partial^\nu \phi = -\partial_\nu \phi \partial^\nu \phi.$$

One can consider an *improved energy momentum tensor* K^μ_ρ adding the terms

$$K^\mu_\rho = T^\mu_\rho + A \delta^\mu_\rho \square \phi^2 + B \partial_\rho \partial^\mu \phi^2.$$

The choice of the constant A, B is fixed by the requirement that the above expression be conserved (as the original energy-momentum tensor) and in addition traceless:

$$\begin{aligned} \partial_\mu K^\mu_\rho &= \partial_\mu T^\mu_\rho + (A+B) \partial_\rho \square \phi^2 = 0 \implies A+B=0, \\ K^\mu_\mu &= T^\mu_\mu + 4A \square \phi^2 - A \square \phi^2 = 0. \end{aligned}$$

Using the identity

$$\square \phi^2 = 2 \partial_\mu \phi \partial^\mu \phi + 2 \phi \square \phi,$$

and making use of the equation of motion $\square \phi = 0$, we can write the trace of the improved energy momentum tensor as

$$K^\mu_\mu = T^\mu_\mu + 6A \partial_\mu \phi \partial^\mu \phi = (-1+6A) \partial_\mu \phi \partial^\mu \phi = 0 \implies A = \frac{1}{6}.$$

At the end the improved energy momentum tensor reads

$$K^\mu_\rho = \partial_\rho \phi \partial^\mu \phi - \frac{1}{2} \delta^\mu_\rho (\partial_\nu \phi \partial^\nu \phi) + \frac{1}{6} (\delta^\mu_\rho \square \phi^2 - \partial_\rho \partial^\mu \phi^2).$$

We can write the dilatations current S^μ in terms of the above improved energy momentum tensor

$$S^\mu = -x^\nu K^\mu_\nu - \phi \partial^\mu \phi + x^\rho \frac{1}{6} (\delta^\mu_\rho \square \phi^2 - \partial_\rho \partial^\mu \phi^2).$$

The invariance of the theory under scale transformations implies the vanishing of $\partial_\mu S^\mu$ and therefore

$$\begin{aligned} 0 &= \partial_\mu S^\mu = -K^\mu_\mu - x^\nu \partial_\mu K^\mu_\nu - \partial_\mu \phi \partial^\mu \phi + \frac{1}{6} (4 \square \phi^2 - \square \phi^2) \\ \implies K^\mu_\mu &= 0 \end{aligned}$$

where we have again expanded $\square\phi^2$ and used the equation of motion $\square\phi = 0$ and the conservation of K^ν_μ . The invariance of the theory under dilatations forces the improved energy momentum tensor to be traceless. For free theories we already know that this is the case since K^μ_ν has been constructed in such a way as to have this property. However one could extend the definition of K for a more general theory with a potential

$$K^\mu_\rho = \partial_\rho\phi\partial^\mu\phi - \delta^\mu_\rho\left(\frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - V\right) + \frac{1}{6}(\delta^\mu_\rho\square\phi^2 - \partial_\rho\partial^\mu\phi^2),$$

and it is possible to check that the tracelessness of K^μ_ν represents a non trivial constraint on the potential V .

The addition of a potential of the form $c_n\phi^n$ brings an additional constraint between k and d which can fix definitively the dimension. In order to have an invariant theory one needs:

$$\int d^d x' \phi'^n(x') = e^{d\lambda+nk\lambda} \int d^d x \phi^n(x) = \int d^d x \phi^n(x) \implies \begin{cases} d+nk=0 \\ k=1-\frac{d}{2}. \end{cases}$$

The solution for the above system of equation doesn't exist for $n=2$. Instead:

$$\text{For } n=3 \implies d=6,$$

$$\text{For } n=4 \implies d=4.$$

The dimensions in energy of the parameters appearing in the potential are then:

$$\begin{aligned} [\text{Action}] &= E^0, & [d^d x] &= E^{-d}, & [\mathcal{L}] &= E^d, \\ [\partial] &= E, & [\phi] &= E^{\frac{d}{2}-1}, \\ [m] &= E, & [\beta] &= E^{3-\frac{d}{2}}, & [\alpha] &= E^{4-d}. \end{aligned}$$

Therefore the couplings α, β are both adimensional in the dimension in which the Lagrangian is invariant under scale transformation. This is not unexpected because the scale transformation deforms lengths and energies as well. The invariance of the theory under such transformation means that the dynamics is the same at all energy scales. In order for this to be true there mustn't be any reference scale in the theory. Therefore in a scale invariant theory only dimensionless parameters are allowed in the potential. This also explains why there is no solution for the term $m^2\phi^2$: the dimension of m doesn't depend on the dimension d , hence it always introduces a reference scale which is the indeed the mass of the field.

Exercise 4

Consider a symmetry defined by the transformation acting on fields:

$$\begin{aligned} x' &= x, \\ \phi'_a(x') &= \mathcal{R}_a^b \phi_b(x) \simeq \phi_a(x) + i\alpha^A (T^A)_a^b \phi_b(x), \end{aligned}$$

where $(T^A)_a^b$ are the generators of the symmetry in the appropriate representation and satisfy the Lie algebra with the ordinary commutator: $[T^A, T^B] = i f^{ABC} T^C$. One can easily compute the conserved Noether's charge:

$$Q^A = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \Delta_a(x) \right) = i \int d^3x \pi^a (T^A)_a^b \phi_b(x).$$

Therefore the Poisson brackets between two charges give:

$$\{Q^A, Q^B\} = \int d^3z \left(\frac{\delta Q^A}{\delta \pi^c(z)} \frac{\delta Q^B}{\delta \phi_c(z)} - \frac{\delta Q^A}{\delta \phi_c(z)} \frac{\delta Q^B}{\delta \pi^c(z)} \right).$$

Since

$$\begin{aligned} \frac{\delta Q^A}{\delta \pi(z)^c} &= i \frac{\partial (\pi^a (T^A)_a^b \phi_b)}{\partial \pi^c}(z) = i (T^A)_c^b \phi_b(z), \\ \frac{\delta Q^B}{\delta \phi(z)_c} &= i \frac{\partial (\pi^a (T^B)_a^b \phi_b)}{\partial \phi_c}(z) = i \pi^a(z) (T^B)_a^c, \end{aligned}$$

hence:

$$\{Q^A, Q^B\} = \int d^3z \pi^a [T^A, T^B]_a^b \phi_b = i f^{ABC} \int d^3z \pi^a (T^C)_a^b \phi_b = f^{ABC} Q^C.$$

One can finally define $Q^A = -i\tilde{Q}^A$ so that

$$\{\tilde{Q}^A, \tilde{Q}^B\} = i f^{ABC} \tilde{Q}^C.$$

There is however a shorter way to obtain the commutation rules for the charges and it involves the Jacobi identity; recall indeed that the Poisson brackets, as all the Lie products, satisfy the Jacobi relation:

$$\{\{Q^A, Q^B\}, \phi_a\} + \{\{Q^B, \phi_a\}, Q^A\} + \{\{\phi_a, Q^A\}, Q^B\} = 0.$$

Since the charges are the generators of the transformation:

$$\{Q^A, \phi_a\} = \Delta_a^A = i(T^A)_a^b \phi_b,$$

then, applying two times this definition one gets

$$\{\{Q^A, Q^B\}, \phi_a\} = -(T^B)_a^c (T^A)_c^b \phi_b + (T^A)_a^c (T^B)_c^b \phi_b = i f^{ABC} (T^C)_a^b \phi_b = f^{ABC} \{Q^C, \phi_a\} = \{f^{ABC} Q^C, \phi_a\}.$$