Quantum Field Theory

Set 9: solutions

Exercise 1

- By dimensional analysis we have the equation:

$$4 = [\mathcal{L}] = 4[\partial] + 2[\eta] + 3[\phi] + [\lambda]$$

  Since $[\phi] = [\eta] = 1$ (due to the canonical kinetic terms) this implies that $[\lambda] = -5$

- The most obvious symmetry of the Lagrangian is the invariance under proper Poincaré transformations

$$\begin{aligned} x^\mu \to x'^\mu &= \Lambda^\mu{}_\nu x^\nu + a^\mu \\ \phi(x) \to \phi'(x') &= \phi(x)\,. \end{aligned}$$

  The symmetry is obvious since all the Lorentz indices in the Lagrangian are contracted, but let's check explicitly the Lorentz invariance for the last term, since the presence of the ϵ tensor makes things slightly less trivial. The derivative transforms as

$$\partial_\mu \to \partial'_\mu = \Lambda_\nu{}^\mu \partial_\nu\,.$$

  Thus we have

$$\begin{aligned} &(\eta_i(x)\partial_\mu\eta_j(x))(\partial_\nu\phi_I(x)\partial_\rho\phi_J(x)\partial_\sigma\phi_K(x))\epsilon^{ij}\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma} \\ \to&(\eta'_i(x')\partial'_\mu\eta'_j(x'))(\partial'_\nu\phi'_I(x')\partial'_\rho\phi'_J(x')\partial'_\sigma\phi'_K(x'))\epsilon^{ij}\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma} \\ =&\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta\Lambda_\rho{}^\gamma\Lambda_\sigma{}^\delta(\eta_i(x)\partial_\alpha\eta_j(x))(\partial_\beta\phi_I(x)\partial_\gamma\phi_J(x)\partial_\delta\phi_K(x))\epsilon^{ij}\epsilon^{IJK}\epsilon^{\mu\nu\rho\sigma} \\ =&(\eta_i(x)\partial_\alpha\eta_j(x))(\partial_\beta\phi_I(x)\partial_\gamma\phi_J(x)\partial_\delta\phi_K(x))\epsilon^{ij}\epsilon^{IJK}\epsilon^{\alpha\beta\gamma\delta} \end{aligned}$$

  where we have used the identity

$$\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta\Lambda_\rho{}^\gamma\Lambda_\sigma{}^\delta\epsilon^{\mu\nu\rho\sigma} = \det\Lambda\epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta}\,,$$

  where the last step is only true for the proper subgroup of the Lorentz group.

- The largest internal symmetry group of this Lagrangian is $SO(2)_\eta \times SO(3)_\phi$:

$$\begin{aligned} \eta_i &\to O^i_j\eta_j \\ \phi_I &\to R^I_J\phi_J \end{aligned}$$

  where O and R are two independent 2-dimensional and 3-dimensional rotation matrices respectively. Indeed, the kinetic terms are trivially invariant as being scalar products of two vectors, while the interaction term is invariant because the Levi-Civita tensors are invariant tensors of $SO(2)$ and $SO(3)$:

$$\begin{aligned} R^{il}R^{jm}\epsilon^{ij} &= \epsilon^{lm} \\ ThO^{IL}O^{JM}O^{KN}\epsilon^{IJK} &= \epsilon^{LMN} \end{aligned}$$

  These relations can be proved at the infinitesimal level. Expanding at first order the previous equations in the angles we would get respectively (in matrix notation):

$$\begin{aligned} \epsilon^{il}\epsilon^{im} + \epsilon^{jm}\epsilon^{lj} &= 0 \\ \epsilon^{PIL}\epsilon^{IMN} + \epsilon^{PJM}\epsilon^{LJN} + \epsilon^{PKN}\epsilon^{LMK} &= 0 \end{aligned}$$

  The first equation is trivially equivalent to the antisymmetry of the Levi-Civita tensor in 2 dimensions, while the second one is equivalent to the Jacobi identity for the algebra of $SO(3)$.

- The Noether currents will depend not only on the kinetic terms but also on the interaction coefficient λ, as the interaction term also contains derivatives. (We do not report the explicit result here, but it would be a good exercise for you to compute it)
- If the η are complex, with the Lagrangian given in the text, the symmetry group is enlarged to $U(2)_\eta \times SO(3)_\phi$. The $U(1)$ group (baryon number) arises because η only appears in complex-conjugate pairs. Notice that for the Lagrangian to be real we multiplied the interaction by a factor of i.
- In the last case the symmetry group is $SU(2)_\eta \times SO(3)_\phi$, because, as mentioned in the text, ϵ^{ij} is an invariant tensor of $SU(2)$:
$$\mathcal{U}^{il}\mathcal{U}^{jm}\epsilon^{ij} = \epsilon^{lm}$$
At the infinitesimal level the previous equation is equivalent to (in matrix notation):
$$\sigma_i^T \epsilon + \epsilon\sigma_i = 0 \tag{1}$$
which can be checked explicitly for every σ_i.

Exercise 2

Let us recall that the group $SO(N)$ is defined as:

$$SO(N) = \{O:\ OO^T = \mathbb{1},\ \det(O) = 1\}.$$

The case $N = 1$ corresponds to the trivial group and thus we get the most general Lagrangian with terms whose dimension is less or equal than four:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4.$$

We did not write a linear term $\mu^3\phi$ since this can always be eliminated shifting $F\phi \to \phi - \mu^3/m^2$. Also, we did not write total derivatives like $n\partial^2\phi$.
For $N \geq 2$, we can build invariants contracting the two invariant tensors of $O(N)$:

$$\delta_i^j, \quad \epsilon^{i_1 \ldots i_N}.$$

Contracting the first we get the invariants, with $d \leq 4$:

$$\partial_\mu\Phi^T\partial^\mu\Phi, \quad \Phi^T\Phi, \quad (\Phi^T\Phi)^2.$$

The epsilon tensor instead does not give non vanishing invariants. For instance $\epsilon^{i_1\cdots i_N}\phi_{i_1}\ldots\phi_{i_N} = 0$ by antisymmetry. Then we can write The $SO(N)$ model Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^T\partial^\mu\Phi - \frac{m^2}{2}\Phi^T\Phi - \frac{\lambda}{4}(\Phi^T\Phi)^2. \tag{2}$$

This Lagrangian is really invariant under $O \in O(N)$, i.e. also under transformations such that $\det(O) = -1$. Indeed the only requirement for $\Phi^T\Phi$ to be invariant is $O^TO = \mathbb{1}$.

Now we want to build the most general Lorentz invariant Lagrangian of two scalars with terms up to dimension 4, that is symmetric under the following three transformations separately:

1. $\phi_1 \to -\phi_1$
2. $\phi_2 \to -\phi_2$
3. $\phi_1 \leftrightarrow \phi_2$

The first two transformations imply that we can only write terms which are separately quadratic in the fields. Taking into account the last one, we conclude that the required Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \lambda_1\left(\phi_1^4 + \phi_2^4\right) - \lambda_2\phi_1^2\phi_2^2. \tag{3}$$

Each of the three transformations above taken alone forms a group which is isomorphic to $\mathbb{Z}_2$. However combined together they form a group which is different from $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, since they do not commute with each other. Consider for instance

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} \phi_2 \\ -\phi_1 \end{pmatrix}, \qquad \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} -\phi_2 \\ \phi_1 \end{pmatrix}.$$

This is called the *dihedral* group $\mathbb{D}_4$ and describes the symmetry of a square.
Let us call D_1, D_2, D_3 the action of the three transformations on the fields

$$D_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\phi_1 \\ \phi_2 \end{pmatrix}, \qquad D_2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix}, \qquad D_3 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}.$$

By combining the action of these, it is easy to see that the group is formed by eight elements. Indeed the most general transformation of the field doublet takes the form

$$D \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \pm\phi_{1/2} \\ \pm\phi_{2/1} \end{pmatrix}, \qquad D \in \mathbb{D}_4.$$

We can write all the elements as

$$\mathbb{D}_4 = \{\mathbb{1}, D_1, D_2, D_3, D_1D_2, D_1D_3, D_2D_3, D_1D_2D_3\}.$$

It is now easy to check that by taking different products one does not get new elements. For instance the following hold

$$D_1D_2 = D_2D_1, \quad D_3D_1 = D_2D_3, \quad D_3D_2 = D_1D_3,$$
$$D_3D_1D_2 = D_3D_2D_1 = D_1D_2D_3, \qquad D_1D_3D_2 = D_2D_3D_1 = D_3.$$

We can build a matrix representation of this group, by looking at its action on the field doublet $(\phi_1, \phi_2)^T$:

$$D_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the most general element belonging to $\mathbb{D}_4$ takes the form

$$D = \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix} \qquad \text{or} \qquad D = \begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}.$$

Finally the Lagrangian (3) reduces to (2) when $\lambda_2 = 2\lambda_1$, in which case the symmetry group is enhanced to $O(2) \supset \mathbb{D}_4$.

Exercise 3

Let's first introduce some notation for delta functions, used also in next exercise.

$$\int d^3x \; e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k}),$$
$$\int d^3k \; e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{x}),$$
$$\int d^3x \; \delta^3(\vec{x}) = \int d^3k \; \delta^3(\vec{k}) = 1.$$

When $\vec{k}$ can assume only discrete values $\vec{k}_n = 2\pi\vec{n}/L$, the Dirac delta becomes a Kronecker delta, since it has to give 1 when summed, *not* when integrated, and the integral on momenta becomes a sum over numbers. Basically the discrete case can be deduced from the continuous one making the following formal replacements.

$$\delta^3(\vec{k}) \longrightarrow \left(\frac{L}{2\pi}\right)^3 \delta^3_{\vec{n},\vec{0}},$$
$$\int d^3k \longrightarrow \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{n}\in\mathbb{Z}^3},$$

and in particular

$$\int d^3x\; e^{i\vec{k}\cdot\vec{x}} \quad\longrightarrow\quad \int d^3x\; e^{i\vec{k}_n\cdot\vec{x}} = (2\pi)^3\left(\frac{L}{2\pi}\right)^3 \delta^3_{\vec{n},\vec{0}}.$$

Now one can compute explicitly the required expression:

$$\int d^3x \sum_{\vec{n}\in\mathbb{Z}^3} \frac{1}{L^{3/2}}\phi_n(t)\partial_i\, e^{i\frac{2\pi}{L}\vec{n}\cdot\vec{x}} \sum_{\vec{m}\in\mathbb{Z}^3} \frac{1}{L^{3/2}}\phi_m(t)\partial_i\, e^{i\frac{2\pi}{L}\vec{m}\cdot\vec{x}} = \frac{1}{L^3}\left(\frac{i2\pi}{L}\right)^2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^3} \vec{n}\cdot\vec{m}\;\phi_n(t)\phi_m(t)\int d^3x\, e^{i\frac{2\pi}{L}(\vec{m}+\vec{n})\cdot\vec{x}}$$

$$= \frac{1}{L^3}\left(\frac{i2\pi}{L}\right)^2 \sum_{\vec{n},\vec{m}\in\mathbb{Z}^3} \vec{n}\cdot\vec{m}\;\phi_n(t)\phi_m(t) L^3 \delta^3_{\vec{n}+\vec{m},\vec{0}} = \left(\frac{2\pi}{L}\right)^2 \sum_{\vec{n}\in\mathbb{Z}^3} |\vec{n}|^2|\phi_n(t)|^2.$$

where we used the fact that $\phi_{-n}(t) = \phi_n^*(t)$ since $\phi(\vec{x},t)$ is real.

Exercise 4

Under a transformation parametrized by Lie parameter $\alpha_i(x)$, the coordinates and field change (to linear order in $\alpha_i(x)$) according to

$$x^\mu \longrightarrow x'^\mu = x^\mu - \alpha^i(x)\epsilon_i^\mu(x), \tag{4}$$
$$\phi_a(x) \longrightarrow \phi'_a(x) = \phi_a(x) + \alpha^i(x)\Delta_{ai}(x), \tag{5}$$

where ϵ_i^μ and Δ_{ai} depend on the specific transformation. The difference with respect to what was done in class is that $\alpha_i = \alpha_i(x)$ is local.

We now compute the variation of the action due to the local transformation. To linear order we have

$$S' = \int d^4x' \mathcal{L}\left(\phi'_a(x'), \partial'_\mu\phi'_a(x')\right) \tag{6}$$
$$= \int d^4x \left[1 - \partial_\mu(\alpha^i\epsilon_i^\mu)\right]\left[\mathcal{L}\left(\phi'_a(x), \partial_\mu\phi'_a(x)\right) - \alpha^i\epsilon_i^\mu\partial_\mu\mathcal{L}\left(\phi'_a(x), \partial_\mu\phi'_a(x)\right)\right] \tag{7}$$
$$= \int d^4x \left[1 - \partial_\mu(\alpha^i\epsilon_i^\mu)\right]\left[\mathcal{L}\left(\phi'_a(x), \partial_\mu\phi'_a(x)\right) - \alpha^i\epsilon_i^\mu\partial_\mu\mathcal{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)\right] \tag{8}$$

where in the last equality we made use of the fact that $\alpha^i\epsilon_i^\mu\partial_\mu\mathcal{L}\left(\phi'_a(x), \partial_\mu\phi'_a(x)\right) = \alpha^i\epsilon_i^\mu\partial_\mu\mathcal{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right)$ to leading order in α_i.

So for we only used the change of the coordinates, eq. (4). We now make use of the change of the field, eq. (5). We have (again to linear order)

$$\mathcal{L}\left(\phi'_a(x), \partial_\mu\phi'_a(x)\right) = \mathcal{L}\left(\phi_a(x), \partial_\mu\phi_a(x)\right) + \alpha^i(x)\Delta_{ai}(x)\frac{\partial\mathcal{L}}{\partial\phi_a} + \partial_\mu(\alpha^i(x)\Delta_{ai}(x))\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}. \tag{9}$$

Plugging into (8) we get (at linear order)

$$S' = \int d^4x \left[\left(1 - \partial_\mu(\alpha^i\epsilon_i^\mu)\right)\mathcal{L}\left(\phi_a, \partial_\mu\phi_a\right) - \alpha^i\epsilon_i^\mu\partial_\mu\mathcal{L}\left(\phi_a, \partial_\mu\phi_a\right) + \alpha^i\Delta_{ai}\frac{\partial\mathcal{L}}{\partial\phi_a} + \partial_\mu(\alpha^i\Delta_{ai})\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right] \tag{10}$$

where for notational purposes we did not write the argument over x. At this point it should be clear that every expression in the integrand is evaluated at x, in particular $\phi_a \equiv \phi_a(x)$ and $\alpha_i \equiv \alpha_i(x)$.

The variation of the action then reads

$$\delta S = S' - S = \int d^4x \left[-\partial_\mu(\alpha^i\epsilon_i^\mu)\mathcal{L}\left(\phi_a, \partial_\mu\phi_a\right) - \alpha^i\epsilon_i^\mu\partial_\mu\mathcal{L}\left(\phi_a, \partial_\mu\phi_a\right) + \alpha^i\Delta_{ai}\frac{\partial\mathcal{L}}{\partial\phi_a} + \partial_\mu(\alpha^i\Delta_{ai})\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right] \tag{11}$$
$$= \int d^4x \left[\alpha^i\left[\Delta_{ai}\frac{\partial\mathcal{L}}{\partial\phi_a} + \partial_\mu\Delta_{ai}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \epsilon_i^\mu\partial_\mu\mathcal{L}\left(\phi_a, \partial_\mu\phi_a\right) - \partial_\mu\epsilon_i^\mu\mathcal{L}\left(\phi_a, \partial_\mu\phi_a\right)\right]\right. \tag{12}$$
$$\left. +\ \partial_\mu\alpha^i\left[\Delta_{ai}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \epsilon_i^\mu\mathcal{L}\left(\phi_a, \partial_\mu\phi_a\right)\right]\right], \tag{13}$$

where we separated terms proportional to α^i from terms proportional to $\partial_\mu \alpha^i$. This is because we are told that the global transformation is a symmetry, i.e. $\delta S = 0$ for constant α^i. In this case the second term drops, because $\partial_\mu \alpha^i = 0$, and if $\delta S = 0$ we have to conclude that the integral over the first term is zero. As with the derivation of Noether's theorem (see lecture notes) the integration region is arbitrary, therefore, not only the integral but the integrand itself must vanish.

We conclude that even if α^i is taken to depend on space-time point, only the second term contributes and we have

$$\delta S = \int d^4x \, \partial_\mu \alpha^i J^\mu_i \tag{14}$$

where $J^\mu_i = \Delta_{ai} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \epsilon^\mu_i \mathcal{L}(\phi_a, \partial_\mu \phi_a)$ is the Noether current associated with the global symmetry for which $\alpha^i = \text{constant}$.

This is a good way to obtain the Noether current associated with the symmetry. Let us consider a complex scalar field charged under $U(1)$. The Lagrangian reads

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi. \tag{15}$$

Under the $U(1)$ we have

$$\phi \to e^{i\alpha(x)} \phi \approx \phi + i\alpha(x)\phi, \qquad \phi^\dagger \to e^{-i\alpha(x)} \phi^\dagger \approx \phi^\dagger - i\alpha(x)\phi^\dagger, \tag{16}$$

so that $\Delta_\phi = i\phi$ and $\Delta_{\phi^\dagger} = -i\phi^\dagger$ and $\epsilon^\mu = 0$.

The Lagrangian will change as

$$\begin{aligned}
\mathcal{L} \to \mathcal{L}' &= \partial_\mu \phi^\dagger \partial^\mu \phi + \partial_\mu(-i\alpha\phi^\dagger)\partial^\mu \phi + \partial_\mu \phi^\dagger \partial^\mu (i\alpha\phi) - m^2 \phi^\dagger \phi - m^2 (i\alpha\phi\phi^\dagger - i\alpha\phi\phi^\dagger) &(17)\\
&= \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + \partial_\mu \alpha \left[-i\phi^\dagger(\partial^\mu \phi) + i(\partial^\mu \phi^\dagger)\phi\right] = \mathcal{L} + \partial_\mu \alpha J^\mu &(18)
\end{aligned}$$

with

$$J^\mu = i\left[(\partial^\mu \phi^\dagger)\phi - \phi^\dagger(\partial^\mu \phi)\right]. \tag{19}$$

Exercise 5

- Let $\pi^a(x) \to \pi'^a(x) = \pi^a(x) + i\alpha^i \Theta^a_{ib} \pi^b(x)$. We can now fix Θ^a_{ib} using the Poisson bracket, $\{\pi_a(x), \phi_b(y)\} = \delta_{ab}\delta^3(x-y)$.

  $$\begin{aligned}
  \{\pi^a(x), \phi_b(y)\} \to \{\pi'^a(x), \phi'_b(y)\} &= \{\pi^a(x), \phi_b(y)\} + i\alpha^i \Theta^a_{ic}\{\pi^c(x), \phi_b(y)\} + i\alpha^i T^b_{ic}\{\pi^a(x), \phi_c(y)\} &(20)\\
  &= \{\pi^a(x), \phi_b(y)\} + i\alpha^i(\Theta^a_{ib} + T^b_{ia})\delta^3(x-y). &(21)
  \end{aligned}$$

  Since the Poisson bracket is invariant (it is a number) we must have $\Theta^a_{ib} = -T^b_{ia}$.
  Recall from exercise 3 of last set,

  $$Q_i = \int d^3x \; J^0_i = \int d^3x \left(\pi^a \, \Delta_{ai} - \epsilon^0_i \mathcal{L}\right) = iT^b_{ia} \int d^3x \, \pi^a \, \phi_b \tag{22}$$

  where now $\Delta_{ai} = iT^b_{ia}\phi_b$ and $\epsilon^0_i = 0$.
  Therefore, we have

  $$\{Q_i, \pi^a(x)\} = iT^c_{ib} \int d^3y \, \pi^b(y) \, \{\phi_c(y), \pi^a(x)\} = -iT^a_{ib}\pi^b(x) \tag{23}$$

  where we made use of $\{\pi^a(x), \phi_b(y)\} = \delta_{ab}\delta^3(x-y)$ in the last step.

- The Jacobi identity for ϕ_a, Q_i and Q_j reads

  $$\{\phi_a, \{Q_i, Q_j\}\} + \{Q_i, \{Q_j, \phi_a\}\} + \{Q_j, \{\phi_a, Q_i\}\} = 0 \tag{24}$$

  we now use $\{Q_i, \phi_a\} = iT^b_{ia}\phi_b$ on the second and third terms to get

  $$\{\phi_a, \{Q_i, Q_j\}\} = (T^c_{ib}T^b_{ja} - T^c_{jb}T^b_{ia})\phi_c = [T_i, T_j]^c_a \phi_c = if_{ijk}T^c_{ka}\phi_c = f_{ijk}\{Q_k, \phi_a\} = -f_{ijk}\{\phi_a, Q_k\} \tag{25}$$

  which for arbitrary ϕ_a implies

  $$\{Q_i, Q_j\} = -f_{ijk}Q_k. \tag{26}$$

- We can now confirm the above result by computing the Poisson bracket $\{Q_i, Q_j\}$ explicitly. Using (22) we find

$$\frac{\delta Q_i}{\delta \pi^a(z)} = iT^c_{ia}\phi_c(z), \qquad \frac{\delta Q_j}{\delta \phi_a(z)} = iT^a_{jb}\pi^b(z), \tag{27}$$

and, similarly, we find

$$\{Q_i, Q_j\} = \int d^3z \left[\frac{\delta Q_i}{\delta \pi^a(z)}\frac{\delta Q_j}{\delta \phi_a(z)} - \frac{\delta Q_j}{\delta \pi^a(z)}\frac{\delta Q_i}{\delta \phi_a(z)}\right] = -[T_i, T_j]^c_b \int d^3z\, \phi_c(z)\pi^b(z) = -f_{ijk}Q_k. \tag{28}$$