Quantum Field Theory

Set 4: solutions

Exercise 1

In this exercise we will study the finite group $\mathcal{S}_3$. This is the group of permutations of three objects. Each element of the group is a rearrangement of a given set of three objects. We can easily list all the possible rearrangements of the set $\{1,2,3\}$

$$
\begin{gathered}
e:\{1,2,3\}\to\{1,2,3\},\ a_1:\{1,2,3\}\to\{2,3,1\},\ a_2:\{1,2,3\}\to\{3,1,2\},\\
a_3:\{1,2,3\}\to\{2,1,3\},\ a_4:\{1,2,3\}\to\{1,3,2\}, a_5:\{1,2,3\}\to\{3,2,1\}.
\end{gathered}
$$

The number of these rearrangements is the *order* of the group and it is usually denoted as $|\mathcal{S}_3|$. In our case this is clearly

$$|\mathcal{S}_3| = 3! = 6.$$

It is easy to convince yourself that this is a group: any combinations of permutations is still a permutation, e is the identity matrix and since every permutation is a bijection it is invertible and associativity holds. From the basic definitions we just saw we can build the product table

$\times$	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	e	a_5	a_3	a_4
a_2	a_2	e	a_1	a_4	a_5	a_3
a_3	a_3	a_4	a_5	e	a_1	a_2
a_4	a_4	a_5	a_3	a_2	e	a_1
a_5	a_5	a_3	a_4	a_1	a_2	e

We now want to find a representation of this group, that is an explicit realization of these 6 transformations using matrices. The most obvious way to do it is to consider a basis of three vectors $\{1,2,3\}$ and building matrices that implement this rearranging. For example we can look for a matrices that implements the swap between the first two basis vector and this will be a representation of the group element a_3 on a vector space of dimension 3

$$D_3[a_3]1 = 2,\ D_3[a_3]2 = 1,\ D_3[a_3]3 = 3.$$

In matrix form this just corresponds to

$$D_3[a_3] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can do the same for the swap between 2 and 3 to find a representation of a_4

$$D_3[a_4] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The other matrices are then built by multiplying this two.

$$
\begin{gathered}
D_3[a_3]D_3[a_4] = D_3[a_1] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D_3[a_4]D_3[a_3] = D_3[a_2] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\\
D_3[a_1]D_3[a_3] = D_3[a_5] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{gathered}
$$

Finally the identity element is obviously represented by

$$D_3[e] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can represent groups with matrices with any dimension. In this case we chose 3×3 matrices for simplicity, but we might wonder if there are smaller dimension representations. In other words we want to see if this representation is *reducible*. To see this se need to find *invariant subspace*. There is an obvious invariant subspace given by the vectors of the form

$$U = \{u = \alpha(1 + 2 + 3), \forall \alpha \in \mathbb{R}\}$$

since permuting any of the basis vectors maps u into itself. This is a 1-dimensional vector space, so there must be another invariant subspace that is 2-dimensional and orthogonal to this. A basis vector for the U space is

$$u = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The orthogonal space to this is given by

$$V = \{v = \alpha_1 1 + \alpha_2 2 + \alpha_3 3), \forall (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 0\}.$$

This subspace is also invariant since the condition $\alpha_1 + \alpha_2 + \alpha_3 = 0$ is left invariant by permutations. We can pick the following two orthogonal basis vectors for this space

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

If we change the basis from our initial basis to this we will see that our 3×3 matrices will become block diagonal. The matrix that implements the change of basis is simply given by stacking these three basis vector as column of a matrix

$$S = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

and we can use this to go to the new basis as follows

$$D_3'[g] = S^T D_3[g] S.$$

D_3' is another 3-dimensional representation and it's equivalent to D_3, since they are related by a similarity transformation. This representation looks as follows

$$D_3'[e] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad D_3'[a_1] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{array}\right), \quad D_3'[a_2] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{array}\right),$$
$$D_3'[a_3] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad D_3'[a_4] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{array}\right), \quad D_3'[a_5] = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1/2 & -\sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{array}\right).$$

You can see that all the matrices are in block diagonal form, where the top-left block is always a 1 while the bottom-right is a 2×2 block. The first block is the trivial representation of $\mathcal{S}_3$ called D_0, where we represent each group element with the number 1, while the second block is a 2-dimensional representation of the group D_2.

D_0 is obviously irreducible. To show that D_2 is also irreducible requires showing that it does not possess any invariant subspace. Since it is a 2 dimensional representation the only non-trivial invariant subspace it can have is 1-dimensional. A line is spanned by a single basis vector v. It can only be invariant if acting on v with any matrix belonging to D_2 gives another vector which is proportional to v, say λv with $\lambda \neq 0$.

To be concrete, given

$$v = (\alpha_1, \alpha_2), \tag{1}$$

we want to show that for all $g \in S_3$,

$$D(g)v = \lambda v \tag{2}$$

does not admit a solution for some α_1 and α_2.

Taking for example

$$D_2(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

we find that it has the following eigenvectors, i.e. it solves eq. (2), for

$$v = (1, 0), \qquad \text{or} \qquad v = (0, 1). \tag{4}$$

If we now take, for example,

$$D_2(a_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \tag{5}$$

and test (2) on either of the v's given in (4) we find that there is no $\lambda \neq 0$ that satisfies it.

We conclude that there is no (non-trivial) invariant subspace. We therefore found the following decomposition of D_3 in irreducible representations

$$D_3 = D_0 \oplus D_2.$$

The 2×2 matrix representation looks like rotation matrices of $60°$ angles. This is because the group $\mathcal{S}_3$ is isomorphic to the group $\mathcal{D}_3$ that are the symmetries of the equilateral triangles (rotations + reflections).

Exercise 2

The following groups are the most common groups one can deal with in theoretical physics.

- $$U(N) \equiv \left\{ U \in GL(N, \mathbb{C}) \middle| UU^\dagger = U^\dagger U = 1_N \right\}$$

  This is the group of $N \times N$ complex unitary matrices. Clearly the inverse corresponds to the hermitian conjugate. One can consider the associated algebra $u(N)$ and take a complete basis T^a of this vector space. Here T^a represents a basis of generators and the label a runs from 1 to dim(algebra). In order to identify the structure of the algebra one can make use of the exponential map to write a generic element U of the group in terms of the generator T^a and some coordinate α^a:

  $$U_\alpha = e^{i\alpha^a T^a} \simeq 1_N + i\alpha^a T^a + O(\alpha^2).$$

  The unitarity of U implies that

  $$1_N = U_\alpha U_\alpha^\dagger \simeq (1_N + i\alpha^a T^a)\left(1_N - i\alpha^a (T^a)^\dagger\right) \simeq 1_N + i\alpha^a T^a - i\alpha^a (T^a)^\dagger.$$

  Therefore the generators are all the matrices that satisfy $T = T^\dagger$, that is to say the hermitian $N \times N$ matrices. One can easily compute the dimension of this vector space counting the number of independent parameters appearing in a generic hermitian matrix.

  $$T_{ij} = (T^\dagger)_{ij} = (T_{ji})^* \implies \begin{cases} \text{Elements on the diagonal are real: } N \text{ components.} \\ \text{Elements symmetric w.r.t the diagonal are} \\ \text{complex conjugate: } N(N-1) \text{ components.} \end{cases}$$

  In the end the dimension of the algebra (equal to the dimension of the vector space of complex hermitian matrices) is N^2. A complete set of generators for the group $U(N)$ is given by a complete basis of the complex hermitian $N \times N$ matrices.

- $$SU(N) \equiv \left\{ U \in GL(N, \mathbb{C}) \middle| UU^\dagger = U^\dagger U = 1_N, \ \det(U) = 1 \right\}.$$

  The latter group is similar to the previous one but with an additional constraint: if in $U(N)$ the determinant of a matrix satisfies $|\det(U)| = 1$, here we choose only $\det(U) = 1$. This corresponds to considering only the subgroup of $U(N)$ connected to the identity. The additional requirement can be translated to the algebra using the relation

$$\det(e^A) = e^{\text{Tr}[A]}.$$

Therefore the algebra is now composed by complex hermitian *traceless* $N \times N$ matrices. The tracelessness constraint consists in only one relation between the components of an hermitian matrix T since one already knows that all diagonal elements are real. The dimension of the algebra is therefore:

$$\dim(su(N)) = N^2 - 1.$$

- $$SO(N) \equiv \left\{ R \in GL(N, \mathbb{R}) \middle| RR^T = R^T R = 1_N,\, \det(R) = 1 \right\}.$$

  This is the group of orthogonal real $N \times N$ matrices. Still using the exponential map

  $$R_\alpha = e^{\alpha^a T^a} \simeq 1_N + \alpha^a T^a + O(\alpha^2).$$

  This time it's better to define the generator without the i in the exponent: in this way, since R is real also the T^a are real instead of purely imaginary. The orthogonality implies:

  $$1_N = R_\alpha R_\alpha^T \simeq \left(1_N + \alpha^a T^a\right)\left(1_N + \alpha^a (T^a)^T\right) \simeq 1_N + \alpha^a T^a + \alpha^a (T^a)^T,$$

  that is to say the algebra is formed by antisymmetric real matrices. The tracelessness is automatically satisfied since antisymmetric matrices have all zero components in the diagonal. The number of components of such a matrix are $N(N-1)/2$, which corresponds to the dimension of the algebra $so(N)$.

- $$O(N) \equiv \left\{ R \in GL(N, \mathbb{R}) \middle| RR^T = R^T R = 1_N \right\}.$$

  The structure of the algebra is the same as the previous one since the removed constraint has no implication at the algebra level. However the group is not the same: one can think about $O(N)$ as $SO(N)$ with additional parities that invert an odd number of coordinates. For example $O(3)$ can be thought as the rotation group $SO(3)$ together with the following matrices

  $$P_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

  The latter are discrete symmetries: composing a generic element of $SO(N)$ with one of these, one can generate the whole $O(N)$. Note that in this case the exponential map doesn't cover all the group since it's formed by several disconnected pieces: the one containing the identity is the subgroup $SO(N)$ and one can reach the others acting with the parities.

- $$SL(N, \mathbb{C}) \equiv \left\{ V \in GL(N, \mathbb{C}) \middle| \det(V) = 1 \right\}.$$

  This is the group of complex $N \times N$ matrices with unitary determinant. Using the exponential map one obtains the constraint for the algebra:

  $$\det(V) = 1 = e^{i\alpha^a \text{Tr}[T^a]} \;\Rightarrow\; Tr[T^a] = 0.$$

  Since the tracelessness this time is a complex statement, it contains two independent constraints and the dimension of the algebra is

  $$\dim(sl(N, \mathbb{C})) = 2N^2 - 2 = 2(N^2 - 1).$$

Exercise 3

Let us write with no loss of generality:

$$D = \left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right), \qquad A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right).$$

Then $[D, A] = 0$, gives the constraints:

$$\begin{cases} [D_1, A_{11}] = 0 \\ [D_2, A_{22}] = 0, \end{cases} \qquad \begin{cases} D_1 A_{12} - A_{12} D_2 = 0 \\ D_2 A_{21} - A_{21} D_1 = 0. \end{cases}$$

Then first Schur lemma implies $A_{12} = 0,\;\; A_{21} = 0$, while second Schur lemma gives $A_{11} = \lambda_1 \mathbb{1}$ and $A_{22} = \lambda_2 \mathbb{1}$.

Exercise 4

Given an algebra

$$\left[T^a, T^b\right] = i f^{abc} T^c \ ,$$

one can consider the following identity

$$\begin{aligned}
&\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] = \\
&T^a \left(T^b T^c - T^c T^b\right) - \left(T^b T^c - T^c T^b\right) T^a + T^b \left(T^c T^a - T^a T^c\right) - \left(T^c T^a - T^a T^c\right) T^b \\
&+T^c \left(T^a T^b - T^b T^a\right) - \left(T^a T^b - T^b T^a\right) T^c = 0.
\end{aligned}$$

Substituting in the first line the result of each commutator one gets

$$\begin{aligned}
&\left[T^a, \left[T^b, T^c\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] \\
&= \sum_d i f^{bcd} \left[T^a, T^d\right] + i f^{cad} \left[T^b, T^d\right] + i f^{abd} \left[T^c, T^d\right] \\
&= \sum_{d,f} -f^{bcd} f^{adf} T^f - f^{cad} f^{bdf} T^f - f^{abd} f^{cdf} T^f .
\end{aligned}$$

The latter is a vanishing linear combination of generators that are a basis of the algebra, therefore the whole coefficient has to be zero:

$$\sum_d \left(f^{adf} f^{bcd} + f^{bdf} f^{cad} + f^{cdf} f^{abd}\right) = 0.$$

This identity can also be used to show that the quantities f^{abc}, called *structure constants*, provide themselves a representation of the group. Let's define a set of matrices $\{A^a\}$ as

$$(A^a)_b{}^c \equiv -i f^{abc}.$$

Then the Jacobi identity can be rewritten as

$$\begin{aligned}
&f^{adf} f^{bcd} - f^{bdf} f^{acd} + f^{cdf} f^{abd} = 0, \\
&(A^b)_c{}^d (A^a)_d{}^f - (A^a)_c{}^d (A^b)_d{}^f = -i f^{abd} (A^d)_c{}^f , \\
&\left[A^b, A^a\right] = i f^{bad} A^d .
\end{aligned}$$

Thus the matrices satisfy the algebra and therefore provide a representation of the group. The vector space on which these matrices act is the algebra itself. This is called *adjoint representation.*