Quantum Field Theory

Set 3: solutions

Exercise 1

The Lagrangian of the system is

$$L = \int d^3x \mathcal{L}(x,t) \qquad \mathcal{L}(x,t) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The conjugate momentum is

$$\pi(x,t) = \frac{\delta\, L}{\delta(\partial_0\phi)} = \frac{\partial\, \mathcal{L}(x,t)}{\partial(\partial_0\phi(x,t))} = \partial_0\phi(x,t).$$

The Hamiltonian reads

$$H = \int d^3x \mathcal{H}(x,t) = \int d^3x\, [\pi\partial_0\phi - \mathcal{L}] = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\phi)(\partial_i\phi) + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right].$$

Given two functionals of ϕ, π

$$F[\phi,\pi](t) = \int f(\phi(x,t),\pi(x,t))d^3x,$$

One defines the equal time Poisson brackets between the two as

$$\{F(t),G(t)\} = \int \left(\frac{\delta\, F}{\delta\pi(z,t)}\frac{\delta\, G}{\delta\phi(z,t)} - \frac{\delta\, F}{\delta\phi(z,t)}\frac{\delta\, G}{\delta\pi(z,t)}\right) d^3z,$$

where as usual $\frac{\delta\, F}{\delta\phi(z,t)} = \frac{\partial\, f(z,t)}{\partial\phi(z,t)}$. In particular

$$\begin{aligned}
\{\pi(x,t),\phi(y,t)\}_t = \int \Bigg(& \frac{\delta\,\left(\int d^3x_1\pi(x_1,t)\delta^3(x-x_1)\right)}{\delta\pi(z,t)}\frac{\delta\,\left(\int d^3x_2\phi(x_2,t)\delta^3(y-x_2)\right)}{\delta\phi(z,t)} \\
& - \frac{\delta\,\left(\int d^3x_1\pi(x_1,t)\delta^3(x-x_1)\right)}{\delta\phi(z,t)}\frac{\delta\,\left(\int d^3x_1\phi(x_2,t)\delta^3(y-x_2)\right)}{\delta\pi(z,t)}\Bigg) d^3z \\
= \int & \left(\delta^3(x-z)\delta^3(y-z)\right) d^3z = \delta^3(x-y).
\end{aligned}$$

The equations of motion become:

$$\begin{aligned}
\dot{\phi} &= \{H,\phi\}, \\
\dot{\pi} &= \{H,\pi\},
\end{aligned}$$

and therefore

$$\begin{aligned}
\dot{\phi}(y,t) &= \{H,\phi(y,t)\} = \int d^3x \left\{\frac{1}{2}\pi^2(x,t),\phi(y,t)\right\} = \int d^3x\; \pi(x,t)\{\pi(x,t),\phi(y,t)\} = \pi(y,t), \\
\dot{\pi}(y,t) &= \{H,\pi(y,t)\} = \int d^3x \left\{\frac{1}{2}(\partial_i\phi(x,t))^2 + \frac{1}{2}m^2\phi^2(x,t) + \frac{\lambda}{4!}\phi^4(x,t),\pi(y,t)\right\} = \\
&= \int d^3x \left(\partial_i\phi(x,t)\{\partial_i\phi(x,t),\pi(y,t)\} + m^2\phi(x,t)\{\phi(x,t),\pi(y,t)\} + \frac{\lambda}{3!}\phi^3(x,t)\{\phi(x,t),\pi(y,t)\}\right) \\
&= -\int d^3x \left(\partial_i\phi(x,t)\frac{\partial}{\partial x^i}\delta^3(x-y) + (m^2\phi(x,t) + \frac{\lambda}{3!}\phi^3(x,t))\delta^3(x-y)\right) \\
&= \partial_i\partial_i\phi(y,t) - m^2\phi(y,t) - \frac{\lambda}{3!}\phi^3(y,t).
\end{aligned}$$

Substituting the former in the latter on can show the equivalence with the Lagrangian formalism:

$$\partial_t^2\phi(y,t) - \partial_i\partial_i\phi(y,t) = \Box\phi(y,t) = -m^2\phi(y,t) - \frac{\lambda}{3!}\phi^3(y,t).$$

Exercise 2

The Maxwell equations read

$$\vec{\nabla}\cdot\vec{E}=\rho, \qquad \vec{\nabla}\wedge\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}\vec{B}=0,$$
$$\vec{\nabla}\cdot\vec{B}=0, \qquad \vec{\nabla}\wedge\vec{B}-\frac{1}{c}\frac{\partial}{\partial t}\vec{E}=\frac{\vec{J}}{c}.$$

One can also rewrite the latter expression in components; recalling that:

$$\begin{aligned} &\vec{E}=(E^1,E^2,E^3), \qquad \vec{B}=(B^1,B^2,B^3), \qquad \vec{J}=(J^1,J^2,J^3), \qquad \rho=J^0, \\ &\vec{\nabla}=(\partial_1,\partial_2,\partial_3), \qquad \frac{\partial}{\partial t}=\partial_0, \end{aligned} \tag{1}$$

one obtains

$$\partial_i E^i=J^0, \qquad \epsilon_{ijk}\partial_j E^k+\frac{1}{c}\partial_0 B^i=0,$$
$$\partial_i B^i=0, \qquad \epsilon_{ijk}\partial_j B^k-\frac{1}{c}\partial_0 E^i=\frac{J^i}{c}.$$

In defining four components quantities one must pay attention to the position of spatial indices; since these are lowered and raised with a metric $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, with Minkosky signature the spacial indices acquire a minus sign in the transition. For example:

$$V^\mu=(V^0,V^i)=(V_0,-V_i), \qquad \partial_\mu=(\partial_0,\partial_i)=(\partial^0,-\partial^i).$$

The field strength F can be expressed in terms of the vector potential $A_\mu=(A_0,A_i)$ as follows:

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu.$$

One should notice the antisymmetric nature of the tensor F for the exchange of the indices $\mu\leftrightarrow\nu$. In order to verify that this expression reflects the definition of F in terms of the physical fields E, B on can compute

$$F^{0i}=\partial^0 A^i-\partial^i A^0=\partial_0 A^i+\partial_i A_0=(\partial_0\vec{A}+\nabla A_0)^i=-E^i,$$
$$F^{12}=\partial^1 A^2-\partial^2 A^1=-\partial_1 A^2+\partial_2 A^1=-(\vec{\nabla}\wedge\vec{A})^3=-B^3,$$

where we have lowered indices of derivatives to get the standard form as defined before.
In order to compute the equations of motion for the field A_μ one has to apply the Euler-Lagrange equation. This time the field with respect to which we differentiate carries an additional space-time index:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)=\frac{\partial\mathcal{L}}{\partial A_\nu}.$$

The previous equation contains a free index ν, which selects the component of the field A_ν with respect to which one derives (therefore one has 4 "independent" equations), and a summed index μ. The equations read

$$\begin{aligned} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)&=\partial_\mu\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[-\frac{1}{4}\left(\partial_\rho A_\sigma-\partial_\sigma A_\rho\right)\left(\partial^\rho A^\sigma-\partial^\sigma A^\rho\right)\right]=\partial_\mu\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[-\frac{1}{2}\left(\partial_\rho A_\sigma-\partial_\sigma A_\rho\right)\partial^\rho A^\sigma\right] \\ &=\partial_\mu\frac{\partial}{\partial(\partial_\mu A_\nu)}\left[-\frac{1}{2}\left(\partial_\rho A_\sigma-\partial_\sigma A_\rho\right)\partial_\alpha A_\beta\eta^{\rho\alpha}\eta^{\sigma\beta}\right] \\ &=\partial_\mu\left[-\frac{1}{2}\left(\delta^\mu_\rho\,\delta^\nu_\sigma-\delta^\mu_\sigma\,\delta^\nu_\rho\right)\partial_\alpha A_\beta\eta^{\rho\alpha}\eta^{\sigma\beta}-\frac{1}{2}\left(\partial_\rho A_\sigma-\partial_\sigma A_\rho\right)\delta^\mu_\alpha\,\delta^\nu_\beta\,\eta^{\rho\alpha}\eta^{\sigma\beta}\right]=-\partial_\mu F^{\mu\nu}, \\ \frac{\partial\mathcal{L}}{\partial A_\nu}&=\frac{\partial}{\partial A_\nu}\left(-J^\rho A_\rho\right)=-J^\rho\delta^\nu_\rho=-J^\nu, \end{aligned} \tag{2}$$

where we have derived using the relation

$$\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\rho A_\sigma)}=\delta^\rho_\mu\delta^\sigma_\nu,$$

that is to say that one gets non-vanishing contribution only if the indices of derivative and of the vector A match. Otherwise the derivative gives zero since it's like deriving a variable with respect to an independent one. Finally the equations of motion are given by

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

At this point it's straightforward to verify that one has obtained exactly the Maxwell equations: the component 0 reads

$$\partial_\mu F^{\mu 0} = \partial_i F^{i0} = J^0 = -\partial_i F^{0i} = \partial_i E^i = \rho,$$

while the component i is

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = -\partial_0 E^i - \partial_j \epsilon_{jik} B^k = J^i,$$

where we have used the property $F^{mn} = F_{mn} = -\epsilon_{mnp} B^p$. However we immediately see that the Euler-Lagrange equations reproduce only the inhomogeneous Maxwell equations, that is to say the ones with a source on their l.h.s.. The homogeneous equations derive from the so called *Bianchi identity*:

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 2\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^\rho A^\sigma = 0,$$

since the tensor $\partial^\nu\partial^\rho$ is contracted with the total antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. Expanding the identity one gets:

$$\epsilon_{0\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \epsilon_{0ijk}\partial^i F^{jk} = -\epsilon_{ijk}\partial^i \epsilon^{jkl} B^l = -2\partial_i B^i = 0$$

$$\begin{aligned}
&\epsilon_{i\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \epsilon_{i0jk}\partial^0 F^{jk} + 2\epsilon_{ij0k}\partial^j F^{0k} \\
&= \epsilon_{ijk}\partial^0 \epsilon^{jkl} B^l + 2\epsilon_{ijk}\partial^j(-E^k) = 2\partial_0 B^i + 2\epsilon_{ijk}\partial_j E^k = 0.
\end{aligned}$$

The Bianchi identities are relation encoded in the structure of the field strength and find their natural explication in the formalism of differential forms.
One can solve the latter equation for a simple external source. The simplest current one can think about is the one generated by a static pointlike charge. In general the current generated by a pointlike particle has the form

$$J^\mu = \left(e\delta^3(x - x(t)),\, e\vec{v}(t)\delta^3(x - x(t))\right),$$

however, since the particle doesn't move, its velocity $\vec{v}$ is null and the current has only the 0-component. Therefore $J^\mu = \left(e\delta^3(x),\, \vec{0}\right)$. For such a configuration one should expect to find the Coulomb potential generated by a charge e. Since the current is time independent we can look for a static solution. We start considering the equation for the scalar potential $A_0(\vec{x})$:

$$\partial_i E^i = \partial_i(-\partial_i A_0 - \partial_0 A^i) = -\nabla^2 A_0(x) = e\delta^3(x).$$

The solution of this Laplace equation can be easily obtained in Fourier transform; define

$$A_0(\vec{x}) = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\tilde{A}(\vec{p})e^{i\vec{p}\cdot\vec{x}}, \qquad \delta^3(\vec{x}) = \int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}.$$

therefore the equation becomes

$$\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\left(\tilde{A}(\vec{p})\partial_i\partial_i + e\right)e^{i\vec{p}\cdot\vec{x}} = \int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\left(-\tilde{A}(\vec{p})|\vec{p}|^2 + e\right)e^{i\vec{p}\cdot\vec{x}} = 0.$$

The latter expression states that the function $\left(-|\vec{p}|^2\tilde{A}(\vec{p}) + e\right)$, thought as an element of the Hilbert space where the Fourier transform is defined, has vanishing scalar product with all the functions $e^{i\vec{p}\cdot\vec{x}}$ which is a complete basis in that space. Therefore it must be

$$\tilde{A}(\vec{p}) = \frac{e}{|\vec{p}|^2}.$$

The solution in coordinate space is simply obtained by Fourier transforming the one in momentum space:

$$\begin{aligned}
A_0(\vec{x}) &= e\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{x}}}{|\vec{p}|^2} = e\int_0^\infty dp\frac{p^2}{(2\pi)^3}\int_{-1}^{1}d(\cos\theta)\int_0^{2\pi}d\phi\frac{e^{ipx\cos\theta}}{p^2} \\
&= \frac{e}{4\pi^2}\int_0^\infty dp\left(\frac{e^{-ipx} - e^{ipx}}{-ipx}\right) = \frac{e}{2\pi^2}\int_0^\infty dp\left(\frac{\sin(px)}{px}\right) \\
&= \frac{e}{2\pi^2 x}\underbrace{\int_0^\infty dy\frac{\sin y}{y}}_{\pi/2} = \frac{e}{4\pi}\frac{1}{|\vec{x}|},
\end{aligned}$$

which is the usual Coulomb scalar potential generated by a static charge and gives an electric field

$$\vec{E} = -\vec{\nabla} A_0(\vec{x}) = \frac{e}{4\pi^2} \frac{\vec{x}}{|\vec{x}|^3}.$$

The vector potential satisfies the homogeneous equation $\nabla^2 \vec{A}(\vec{x}) - \vec{\nabla}(\vec{\nabla} \cdot \vec{A}(\vec{x})) = 0$. Thus, we can choose $\vec{A} = 0$.

Exercise 3

In order to define a group $\mathcal{G}$, a set of transformations $\{g_i\}$ has to satisfy the following four properties:

- An operation $\circ$ must be defined on the set $\{g_i\}$ such that for each $g_1,\, g_2 \in \mathcal{G}$: $g_1 \circ g_2 \equiv g_3 \in \mathcal{G}$. For $\tilde{U}$ this operation is realized by the usual composition of functions. We can compute the group product law using the given realization:

$$\begin{aligned}
&\tilde{U}(\alpha_2, \beta_2)(\tilde{U}(\alpha_1, \beta_1)(x)) = \tilde{U}(\alpha_2, \beta_2)(e^{\alpha_1} x + \beta_1) \\
=&e^{\alpha_2}(e^{\alpha_1} x + \beta_1) + \beta_2 = (e^{\alpha_1+\alpha_2})x + (e^{\alpha_2}\beta_1 + \beta_2) \\
=&\tilde{U}(\alpha_1 + \alpha_2, e^{\alpha_2}\beta_1 + \beta_2)(x)
\end{aligned}$$

  So the group product law is

$$U(\alpha_2, \beta_2) \circ U(\alpha_1, \beta_1) = U(\alpha_1 + \alpha_2, e^{\alpha_2}\beta_1 + \beta_2)$$

- Associativity: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$. Function composition is associative so this requirement is automatically fulfilled.
- The set $\{g_i\}$ must contain the *identity element* e, such that, for each $g \in \mathcal{G}$, $e \circ g = g \circ e = g$. In our case the identity element is precisely at the origin of the parameter space

$$e = U(0, 0) : x \to x$$

- For each $g \in \mathcal{G}$ there must exist a $g^{-1} \in \mathcal{G}$, the *inverse element*, such that $g^{-1} \circ g = g \circ g^{-1} = e$. In order to find such an element one can denote $U^{-1}(\alpha, \beta) \equiv U(\bar{\alpha}, \bar{\beta})$ and require

$$U(\alpha, \beta)U(\bar{\alpha}, \bar{\beta}) = U(\alpha + \bar{\alpha}, e^{\alpha}\bar{\beta} + \beta) = U(0, 0) \implies \begin{cases} \bar{\alpha} = -\alpha \\ \bar{\beta} = -e^{-\alpha}\beta. \end{cases}$$

  Thus,

$$U^{-1}(\alpha, \beta) = U(-\alpha, -e^{-\alpha}\beta).$$

  We can check that it is also a left-inverse,

$$U(\bar{\alpha}, \bar{\beta})U(\alpha, \beta) = U(\bar{\alpha} + \alpha, e^{\bar{\alpha}}\beta + \bar{\beta}) = U(0, 0).$$

An abelian group satisfies $g_1 \circ g_2 \equiv g_2 \circ g_1$ for any $g_1,\, g_2 \in \mathcal{G}$. To check this we use the composition law derived above. We see that

$$U(\alpha_2, \beta_2) \circ U(\alpha_1, \beta_1) = U(\alpha_1 + \alpha_2, e^{\alpha_2}\beta_1 + \beta_2) \neq U(\alpha_1 + \alpha_2, e^{\alpha_1}\beta_2 + \beta_1) = U(\alpha_1, \beta_1) \circ U(\alpha_2, \beta_2).$$

Thus, the collinear group is not abelian.

Exercise 4

We consider a particle with position $q(t)$ and mass m obeying the differential equation

$$m\ddot{q}(t) + kq^2(t) = 0$$

with k a constant.

- The constant k has dimension

$$[k] = \frac{\text{mass}}{\text{length time}^2} = \frac{M}{LT^2}$$

- Under the transformation

$$q(t) \to q'(t') = \lambda^{-p} q(t), \quad t' = \lambda t$$

the equation transforms as

$$m\lambda^p \lambda^2 \frac{d^2}{d(\lambda t)^2} q'(\lambda t) + k\lambda^{2p} {q'}^2(\lambda t) = 0$$

The equation is invariant if

$$2 + p = 2p \quad \Longrightarrow \quad p = 2$$

- The value of p could have been easily guessed from the dimension of k. Here we explain why.

  The dimension of k was obtained by dimensional analysis: we are requiring that each term has the same units. This is so that when changing units, the equation stays invariant. Changing units can be seen as applying different dilatations :

$$M \to \lambda_M M, \quad T \to \lambda_T T, \quad L \to \lambda_L L$$

  where for example going from hours to minutes would be given by choosing $\lambda_T = 60$. We can thus think of dimensional analysis as asking for invariance under different dilatation transformations. This is however not necessarily a symmetry of the system as the parameters also transform, not only the variables and the coordinates.

  For example, considering time dilatation alone ($\lambda_M = \lambda_L = 0$), we see that the equation has the correct units and respect dimensional analysis. However, under this dilatation transformation, the constant k transforms. Thus, this is not a symmetry of the system. (Sometimes, when the parameters transform also, this is called a spurious transformation. This is useful for studying approximate symmetries.)

  In the previous point, we showed that the equation is actually invariant under a more generic transformation that combines several of the previously mentionned one:

$$M \to M, \quad T \to \lambda T, \quad L \to \lambda^{-2} L$$

  such that under this transformation, masses have dimension 0, times dimension 1 and lengths dimension -2. Under this transformation, both parameters m and k are dimensionless, and thus don't transform. Because the equation doesn't contain any dimensionful parameter under this symmetry, it is invariant. Thus the transformation is a symmetry for the value of p for which k is dimensionless.

Exercise 5

(Recall 1 eV $= 1.602 \cdot 10^{-19}$ J, $M_p = 0.938$ GeV$/c^2$)

Velocity of protons coming from SPS:

$$M_p \gamma c^2 = 450 \text{ GeV} \Rightarrow \frac{1}{1-\beta^2} = \frac{(450 \text{ GeV})^2}{M_p^2 c^4} \Rightarrow \beta = \sqrt{\frac{\frac{(450)^2}{(0.938)^2} - 1}{\frac{(450)^2}{(0.938)^2}}} = 0.999998$$

Protons energy:

$$E = M_p \gamma c^2 = 7 \text{ TeV}$$

Number of protons per beam:

$$N_P = 2.8 \cdot 10^{14}$$

Total energy of the beam:

$$E_{\text{tot}} = 7 \text{ TeV} \cdot N_P = 1.96 \times 10^{15} \text{ TeV} = 1.96 \times 1.6 \times 10^{27-19} \text{ J} = 314 \text{ MJ}$$

One can compare this quantity with the kinetic energy of a running TGV. The velocity of the train can be easily extracted:

$$\frac{1}{2}MV_T^2 = 314 \cdot 10^6 \text{ J} \Rightarrow V_{TGV} = \sqrt{2 \cdot \frac{3140}{4}} \cdot \text{ m s}^{-1} = 39.6 \text{ m s}^{-1} = 143 \text{ Km/h}$$

One can also compute the total current circulating inside the ring: in the interval of 1 s a number of bunches $N_B = \left(27 \text{ Km}/3 \cdot 10^5\right)^{-1}$ passes in a given point of the ring. The total charge passing through that point in a second is given by:

$$Q = N_B \cdot 2800 \cdot 1.6 \cdot 10^{11-19} \text{ C} = 0.498 \text{ C} \;\Rightarrow\; I = \frac{Q}{\text{s}} = 0.498 \text{ A}.$$

Suppose we want to strike a target b at rest with a particle a with energy in the laboratory $\tilde{E}_a$ in such a way that in the center of mass the energy is $\sqrt{s} = 14$ TeV. We need to compute $\tilde{E}_a$ as a function of $\sqrt{s}$. We can use the property that the value of the square of a four momentum is independent of the frame in which it is computed:

$$P^2 = \tilde{P}^2 = (\tilde{P}_a + \tilde{P}_b)^2 \;\Rightarrow\; s = m_a^2 + m_b^2 + 2m_b\tilde{E}_a$$

Substituting $\sqrt{s} = 14$ TeV and $m_a = m_b \simeq 1$ GeV (at LHC we collide protons) we get:

$$\tilde{E}_a = \frac{s - m_a^2 - m_b^2}{2m_b} \simeq 98 \times 10^3 \text{ TeV}$$

Colliding with fixed ^{208}Pb target, i.e. replacing $m_b \simeq 208$ GeV, we get

$$\tilde{E}_a \simeq 4.7 \times 10^2 \text{ TeV}$$

We see immediately the convenience of having a collider with two beams circulating in opposite direction and equal energy.

Suppose now you want to build a fixed target collider. At regime the power loss must be balanced by the energy given per lap by the electric field. Calling r the radius of the accellerator, a proton makes $2\pi r/\beta \simeq 2\pi R$ number of laps per second (in natural units). Calling E_{in} the energy gained per lap by a single particle, we require

$$\frac{E_{in}}{2\pi r} \geq P = \frac{2e^2}{3r^2}(\beta\gamma)^4 \simeq \frac{2e^2}{3r^2}\left(\frac{\tilde{E}_a}{m_a}\right)^4.$$

The minimum value of r for which this is possible, is thus given by

$$r_{min} = \frac{4\pi e^2}{3E_{in}}\left(\frac{\tilde{E}_a}{m_a}\right)^4 \simeq \frac{4\pi e^2}{3E_{in}}\left(\frac{s}{2m_b m_a}\right)^4, \tag{3}$$

where we used the relation between the energy in the center of mass frame and in the lab frame and that $s \gg m_b, m_a$. Setting $m_a \simeq m_b \simeq 1\,\text{GeV}$, we have

$$r_{min} \simeq 2.4 \times 10^{16}\,\text{eV}^{-1}\left(\frac{\text{MeV}}{E_{in}}\right)\left(\frac{\sqrt{s}}{\text{TeV}}\right)^8 \simeq 4.7 \times 10^9\,\text{m}\left(\frac{1\text{ MeV}}{E_{in}}\right)\left(\frac{\sqrt{s}}{1\text{ TeV}}\right)^8. \tag{4}$$

Setting for instance $\sqrt{s} \simeq 1.5$ TeV and $E_{in} \simeq 10^3\,\text{MeV}$, we get that r should be of the order of the earth radius:

$$r_{min} \sim 10^8\,\text{m}. \tag{5}$$

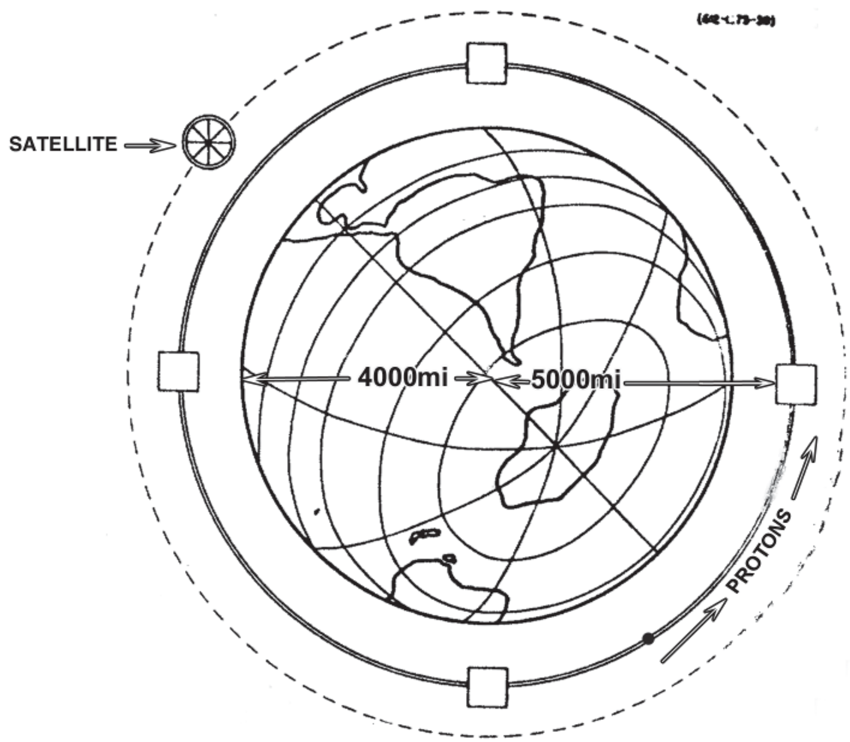

Figure 1: Fermi proposal for a fixed target high energy collider in 1954.