Quantum Field Theory

Set 2: solutions

Exercise 1

We can evaluate the propagator $K(\vec{x}, t, \vec{0}, 0)$ as

$$\langle \vec{x}|e^{-iHt}|\vec{0}\rangle = \int d^3k \, \langle \vec{x}|\vec{k}\rangle \, \langle \vec{k}|e^{-iHt}|\vec{0}\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}} t} \,, \tag{1}$$

where $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$. Passing to polar coordinates and defining $x \equiv |\vec{x}|$, $k \equiv |\vec{k}|$ we have

$$\begin{aligned} \langle \vec{x}|e^{-iHt}|\vec{0}\rangle = \int_0^{+\infty} \frac{k^2}{(2\pi)^3} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi e^{ikx\cos\theta - i\omega_k t} &= \frac{-i}{x(2\pi)^2} \int_0^{+\infty} dk k \left(e^{ikx} - e^{-ikx}\right) e^{-i\omega_k t} \\ &= \frac{-i}{x(2\pi)^2} \int_{-\infty}^{+\infty} dk k e^{ikx - i\omega_k t} \,. \end{aligned} \tag{2}$$

For $x > t$ we can estimate the last integral through a saddle point approximation. The exponent has a saddle in

$$\frac{d}{dk}\left(kx - \omega_k t\right) = 0 \Rightarrow k = \frac{imx}{\sqrt{x^2 - t^2}} \,, \tag{3}$$

and expanding the integral around this point we get

$$\langle \vec{x}|e^{-iHt}|\vec{0}\rangle \sim e^{-m\sqrt{x^2 - t^2}} \,. \tag{4}$$

[Further reading: https://arxiv.org/pdf/1110.5013.pdf (pg. 14)]

Exercise 2

Let's call $\tilde{p}$'s the momenta of the incoming particles in the laboratory frame, p's the momenta of the incoming particles in the center of mass frame, k's the momenta of the outgoing particles in the center of mass frame. One has generically

$$\tilde{p}^{(\gamma)}_\mu = (\tilde{E}_\gamma, 0, 0, \tilde{E}_\gamma), \qquad \tilde{p}^{(e)}_\mu = (m_e, 0, 0, 0),$$

$$\begin{aligned} &p^{(\gamma)}_\mu = (E_\gamma, 0, 0, E_\gamma), \qquad p^{(e)}_\mu = (\sqrt{E_\gamma^2 + m_e^2}, 0, 0, -E_\gamma), \\ &k^{(\gamma)}_\mu = (E'_\gamma, 0, E'_\gamma \sin\theta, E'_\gamma \cos\theta), \qquad k^{(e)}_\mu = (E'_e, 0, k'_e \sin\theta', k'_e \sin\theta'), \\ &k^{(+)}_\mu = (E^+, \vec{k}^+), \qquad k^{(-)}_\mu = (E^-, \vec{k}^-). \end{aligned}$$

Note that the total energy $E^{(tot)} = E_\gamma + \sqrt{E_\gamma^2 + m_e^2}$ is a monotonically increasing function of E_γ, so that the kinematical configuration which minimizes E_γ also minimizes $E^{(tot)}$.
The energies of the outgoing particles satisfy $E'_\gamma \geq 0$, $E'_e \geq m_e$, $E^+ \geq m_e$, and $E^- \geq m_e$, and the configuration with smallest total energy is the one in which the bounds are minimally satisfied, namely $E'_\gamma = 0$, $E'_e = E^+ = E^- = m_e$. This is the situation in which in the center of mass frame no final-state particle has kinetic energy,

i.e. the three massive final-state particles are produced at rest and the final-state photon is soft. Thus, in this threshold configuration

$$
\begin{aligned}
&p_\mu^{(\gamma)} = (E_\gamma, 0, 0, E_\gamma), \qquad p_\mu^{(e)} = (\sqrt{E_\gamma^2 + m_e^2}, 0, 0, -E_\gamma), \\
&k_\mu^{(\gamma)} = (0, 0, 0, 0), \qquad k_\mu^{(e)} = (m_e, 0, 0, 0), \\
&k_\mu^{(+)} = (m_e, 0, 0, 0), \qquad k_\mu^{(-)} = (m_e, 0, 0, 0).
\end{aligned}
$$

and

$$k_\mu^{(\gamma)} + k_\mu^{(e)} + k_\mu^{(+)} + k_\mu^{(-)} = (3\,m_e, 0, 0, 0) \equiv k_\mu^{(tot)}.$$

Now, exploiting the invariance under Lorentz transformations of the square of four-vectors, and using momentum conservation, one can deduce

$$9\,m_e^2 = (k^{(tot)})^2 = (p^{(\gamma)} + p^{(e)})^2 = (\tilde{p}^{(\gamma)} + \tilde{p}^{(e)})^2 = m_e^2 + 2\,m_e\,\tilde{E}_\gamma^{\min} \implies \tilde{E}_\gamma^{\min} = 4\,m_e.$$

Exercise 3

We call $y_i(t)$ the displacement of the i-th mass from the x axis in the $\hat{y}$ direction. The potential of the system is given by the sum of the elastic potentials of each spring. The length of the spring stretching between the i-th and the $i+1$-th mass is given by

$$L_{i,i+1} = \sqrt{(y_i(t) - y_{i+1}(t))^2 + a^2}$$

therefore the total potential is

$$V_{tot} = \sum_i \frac{1}{2} k L_{i,i+1}^2 = \frac{k}{2} \sum_i \left((y_i(t) - y_{i+1}(t))^2 + a^2\right)$$

Notice that a given y_i enters two times in the potential, since each mass is attached to two springs. In order to visualize this one can write some terms explicitly. For instance:

$$V_{tot} = ... + \frac{k}{2}\left((y_1(t) - y_2(t))^2 + a^2\right) + \frac{k}{2}\left((y_2(t) - y_3(t))^2 + a^2\right) + \frac{k}{2}\left((y_3(t) - y_4(t))^2 + a^2\right) + ...$$

and therefore all the times we derive the potential with respect to one coordinate two pieces will appear (indeed there are two strengths acting to each mass). The Lagrangian of the system is

$$L = T - V = \sum_i \frac{1}{2} m \dot{y}_i^2 - \frac{k}{2} \sum_i \left((y_i(t) - y_{i+1}(t))^2 + a^2\right)$$

The equation of motion are obtained as usual from

$$
\begin{aligned}
&\partial_t \left(\frac{\partial L}{\partial \dot{y}_i}\right) = \frac{\partial L}{\partial y_i} \\
\Rightarrow \quad &\partial_t (m \dot{y}_i(t)) = m \ddot{y}_i(t) = -k(y_i(t) - y_{i+1}(t)) + k(y_{i-1}(t) - y_i(t))
\end{aligned}
$$

We can also describe the system using the Hamiltonian formalism. Let us define the conjugate momentum of the i-th variable

$$p_i = \frac{\partial L}{\partial \dot{y}_i} = m \dot{y}_i$$

Then the Hamiltonian reads

$$
\begin{aligned}
H &= \sum p_i \dot{y}_i - L = \sum_i \left(\frac{\dot{y}_i}{m} \dot{y}_i - \frac{m}{2}\left(\frac{\dot{y}_i}{m}\right)^2\right) + V \\
&= \sum_i \frac{p_i^2}{2m} + \frac{k}{2} \sum_i \left((y_i(t) - y_{i+1}(t))^2 + a^2\right)
\end{aligned}
$$

In the Hamiltonian formalism the equations of motion are of the first order and read

$$\begin{aligned}
\dot{y}_i &= \frac{\partial H}{\partial p_i} = \frac{p_i}{m}\,, \\
\dot{p}_i &= -\frac{\partial H}{\partial y_i} = -k(y_i - y_{i+1}) + k(y_{i-1} - y_i)\,.
\end{aligned}$$

The equivalence of the two formalisms (Lagrangian and Hamiltonian) is manifest once we differentiate the first equation with respect to time and plug it into the second one: we recover exactly the Lagrangian equation of motion.

The continuum limit $a \to 0$ corresponds to considering the oscillators closer and closer together so that the position of the $i-$th oscillator (which would be $i \cdot a$) can be labeled by a continuous variable x. Instead of having a discrete set of functions of time one has a function of two variables, space and time:

$$y_i(t) \longrightarrow y(x,t)$$

The Lagrangian becomes

$$\lim_{a\to 0} L = \lim_{a\to 0} a \sum_i \frac{1}{2}\left(\frac{m}{a}\dot{y}(x,t)^2 - ka\left(\frac{y(x,t)-y(x+a)}{a}\right)^2\right)$$

In the $a \to 0$ limit, the term appearing in the parenthesis represents the derivative with respect to the x variable while the sum translates in an integral over x:

$$\begin{aligned}
&\lim_{a\to 0} \frac{y(x,t)-y(x+a,t)}{a} = -\frac{\partial y(x,t)}{\partial x} \\
&\lim_{a\to 0} a \sum_i = \int dx
\end{aligned}$$

Finally the Lagrangian reads

$$\int dx \underbrace{\frac{1}{2}\left(\mu\left(\frac{\partial y(x,t)}{\partial t}\right)^2 - Y\left(\frac{\partial y(x,t)}{\partial x}\right)^2\right)}_{\mathcal{L}(x,t)}$$

In the above expression the function $y(x,t)$ is called *field*. In Field Theory the quantity $\mathcal{L}(x,t)$ is called *Lagrangian density* and is a function of space and time. The integral on space of the Lagrangian density is the *Lagrangian* $L(t)$ while the integral on time as well defines the *Action*:

$$S = \int dt L(t) = \int dt\, dx \mathcal{L}(x,t)$$

The equation of motion that determines the dynamic of the field $y(x,t)$ is given by the Euler Lagrange equations:

$$\sum_I \partial_I\left(\frac{\partial \mathcal{L}}{\partial(\partial_I y)}\right) = \frac{\partial \mathcal{L}}{\partial y}$$

where ∂_I stands for derivative with respect to t or x. Therefore:

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial(\partial_t y)}\right) + \partial_x\left(\frac{\partial \mathcal{L}}{\partial(\partial_x y)}\right) = \frac{\partial \mathcal{L}}{\partial y}$$

In order to compute $\partial_I\left(\frac{\partial \mathcal{L}}{\partial(\partial_I y)}\right)$ one has to think of $\mathcal{L}$ as a function of the variable $\partial_I y$ and derive as usual. Hence.

$$\partial_t(\mu\partial_t y(x,t)) - \partial_x(Y\partial_x y(x,t)) = 0 \;\;\Rightarrow\;\; \partial_t^2 y(x,t) = \frac{Y}{\mu}\partial_x^2 y(x,t)$$

One can compare with the direct $a \to 0$ limit in the equation of motion derived previously:

$$\ddot{y}_i(t) \to \partial_t^2 y(x,t) = \lim_{a\to 0} -\frac{ka^2}{m}\frac{\frac{y(x,t)-y(x+a,t)}{a} - \frac{y(x-a,t)-y(x)}{a}}{a} = \lim_{a\to 0}(-)\frac{Y}{\mu}\frac{-\partial_x y(x,t) + \partial_x y(x-a)}{a} = \frac{Y}{\mu}\partial_x^2 y(x,t)$$

The result is the same as that obtained from the Euler Lagrange equation for the field $y(x,t)$. In order to find a solution of the above *wave equation* let us define two set of variables

$$\xi = vt + x\,, \qquad \eta = vt - x\,, \qquad v = \sqrt{\frac{Y}{\mu}}$$

$$x = \frac{\xi - \eta}{2}\,, \quad t = \frac{\eta + \xi}{2v}$$

Then the derivatives with respect to x and t become

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$$
$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = v\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta}$$

The equation of motion then becomes

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) y(x,t) = 4\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} y(\xi,\eta) = 0$$

The general solution of the above equation has the following form:

$$y(x,t) = f(\xi) + g(\eta) = f(x+vt) + g(vt-x) \tag{5}$$

where f and g are two arbitrary functions. The above solution corresponds to a superposition of two waves, one moving in the right direction and one moving in the left direction.

Regarding the last point, if we add the springs of frequency ω' the Lagrangian becomes:

$$L = \sum_i \frac{1}{2} m \dot{y}_i^2 - \frac{k}{2}\sum_i \left((y_i(t) - y_{i+1}(t))^2 + a^2\right) - \frac{m\,\omega'^2}{2}\sum_i y_i(t)^2$$

which in the continuum limit, repeating the steps outlined in the previous points, reads:

$$\int dx \frac{1}{2}\left(\mu\left(\frac{\partial y(x,t)}{\partial t}\right)^2 - Y\left(\frac{\partial y(x,t)}{\partial x}\right)^2 - \mu\omega'^2 y(x,t)^2\right)$$

The equation of motion can be easily calculated:

$$\partial_t^2 y(x,t) - v^2 \partial_x^2 y(x,t) = -\omega'^2 y(x,t)$$

After some simple rescaling of coordinates and working in natural units $c = 1$ we can bring the previous equation to a more familiar, relativistic form:

$$\partial_t^2 y(x,t) - \partial_x^2 y(x,t) = -M^2 y(x,t), \quad M^2 \equiv \omega'^2$$

This equation can be easily solved by taking its Fourier transforms:

$$k_0^2 \tilde{y}(k_0,k) - k^2 \tilde{y}(k_0,k) = M^2 \tilde{y}(k_0,k) \tag{6}$$

where $\tilde{y}(k_0,k)$ is the Fourier transform of $y(t,x)$ and (k_0,k) are the variables conjugated to (t,x). This admits the general solution:

$$\tilde{y}(k_0,k) = c(k_0,k)\delta(k_0^2 - k^2 - M^2) \tag{7}$$

where $c(k_0,k)$ is an arbitrary function of k_0 and k. By taking the anti-Fourier transform and making use of standard properties of the Dirac delta function one can cast the solution into the form:

$$y(x,t) = \int_{-\infty}^{\infty} dk e^{ikx}\left(A(k)e^{i\omega_k t} + B(k)e^{-i\omega_k t}\right) \tag{8}$$

where $A(k), B(k)$ are arbitrary functions. $y(x,t)$ is a general superposition of plane waves satisfying the dispersion relation $k_0 = \pm\omega_k$, $\omega_k = \sqrt{k^2 + M^2}$. Therefore, the difference with the solution (**??**) (which can be retrieved in the limit $M = 0$) is basically that we have in both cases a superposition of plane waves, but following a different dispersion relation. In the first (second) case in the continuum limit one retrieves the Lagrangian of a relativistic massless (massive) field.

Exercise 4

The Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

give

$$\partial_\mu \left(\frac{\partial}{\partial(\partial_\mu \phi)} \frac{1}{2} \partial_\rho \phi \partial_\sigma \phi \eta^{\rho\sigma} \right) = \partial_\mu (\eta^{\rho\sigma} \delta^\mu_\rho \partial_\sigma \phi) = \partial_\mu \partial^\mu \phi = -\frac{\partial V(\phi)}{\partial \phi}.$$

For the massive $\lambda\phi^4$ theory one gets:

$$\partial_\mu \partial^\mu \phi = -m^2 \phi - \frac{\lambda}{6} \phi^3.$$

Let us now consider the term

$$\alpha(\partial_\mu \phi \, \partial^\mu \phi)^2 \equiv \alpha(\partial_\mu \phi \, \partial^\mu \phi)(\partial_\nu \phi \, \partial^\nu \phi).$$

In order to compute the dimension in energy of the constant α let us compute first the dimension of the field ϕ. We recall that the dimension of the action is power of $\hbar$ (depending on the spacetime dimension), therefore in natural units the action is dimensionless. From this one can extract the dimension of the Lagrangian density $\mathcal{L}$:

$$S = \int dt \, d^{D-1} x \mathcal{L}(\vec{x}, t)$$

$$[S] = E^0 \,, \qquad [dx] = [dt] = E^{-1} \qquad \Rightarrow \qquad [\mathcal{L}] = E^D$$

Therefore in 3+1 dimensions all the terms appearing in the Lagrangian density must have dimension E^4. Let us consider the kinetic term: knowing the dimension $[\partial_\mu] = [\frac{\partial}{\partial x^\mu}] = E$, we can extract the dimension of ϕ:

$$[\partial_\mu \phi \, \partial^\mu \phi] = [\partial_\mu]^2 \times [\phi]^2 = E^2 \times [\phi]^2 \qquad \Rightarrow \qquad [\phi] = E$$

The field ϕ has dimension of a mass. One can repeat in arbitrary dimension and get $[\phi] = E^{\frac{D-2}{2}}$. Let us come back to the parameter α:

$$[\mathcal{L}] = E^4 \qquad \Rightarrow \qquad [\alpha] \times [\partial_\mu]^4 \times [\phi]^4 = E^4 \qquad \Rightarrow \qquad [\alpha] = E^{-4}$$

Finally let us compute the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu (\partial^\mu \phi + 4\alpha(\partial_\nu \phi \, \partial^\nu \phi) \partial^\mu \phi) = -\frac{\partial V}{\partial \phi}.$$

Exercise 5

The Lagrangian of the system is

$$L = \int d^3 x \mathcal{L}(x,t) \qquad \mathcal{L}(x,t) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.$$

The conjugate momentum is

$$\pi(x,t) = \frac{\delta \, L}{\delta(\partial_0 \phi)} = \frac{\partial \, \mathcal{L}(x,t)}{\partial(\partial_0 \phi(x,t))} = \partial_0 \phi(x,t).$$

The Hamiltonian reads

$$H = \int d^3 x \mathcal{H}(x,t) = \int d^3 x \left[\pi \partial_0 \phi - \mathcal{L} \right] = \int d^3 x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\partial_i \phi)(\partial_i \phi) + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right].$$

Given two functionals of ϕ, π

$$F[\phi, \pi](t) = \int f(\phi(x,t), \pi(x,t)) d^3 x,$$

One defines the equal time Poisson brackets between the two as

$$\{F(t), G(t)\} = \int \left(\frac{\delta \, F}{\delta \pi(z,t)} \frac{\delta \, G}{\delta \phi(z,t)} - \frac{\delta \, F}{\delta \phi(z,t)} \frac{\delta \, G}{\delta \pi(z,t)} \right) d^3 z,$$

where as usual $\frac{\delta F}{\delta\phi(z,t)} = \frac{\partial f(z,t)}{\partial\phi(z,t)}$. In particular

$$\begin{aligned}
\{\pi(x,t),\phi(y,t)\}_t = \int \Bigg(&\frac{\delta\left(\int d^3x_1\pi(x_1,t)\delta^3(x-x_1)\right)}{\delta\pi(z,t)}\frac{\delta\left(\int d^3x_2\phi(x_2,t)\delta^3(y-x_2)\right)}{\delta\phi(z,t)} \\
&- \frac{\delta\left(\int d^3x_1\pi(x_1,t)\delta^3(x-x_1)\right)}{\delta\phi(z,t)}\frac{\delta\left(\int d^3x_1\phi(x_2,t)\delta^3(y-x_2)\right)}{\delta\pi(z,t)}\Bigg) d^3z \\
&= \int \left(\delta^3(x-z)\delta^3(y-z)\right) d^3z = \delta^3(x-y).
\end{aligned}$$

The equations of motion become:

$$\begin{aligned}
\dot{\phi} &= \{H,\phi\}, \\
\dot{\pi} &= \{H,\pi\},
\end{aligned}$$

and therefore

$$\begin{aligned}
\dot{\phi}(y,t) &= \{H,\phi(y,t)\} = \int d^3x \left\{\frac{1}{2}\pi^2(x,t),\phi(y,t)\right\} = \int d^3x\ \pi(x,t)\{\pi(x,t),\phi(y,t)\} = \pi(y,t), \\
\dot{\pi}(y,t) &= \{H,\pi(y,t)\} = \int d^3x \left\{\frac{1}{2}(\partial_i\phi(x,t))^2 + \frac{1}{2}m^2\phi^2(x,t) + \frac{\lambda}{4!}\phi^4(x,t),\pi(y,t)\right\} = \\
&= \int d^3x \left(\partial_i\phi(x,t)\{\partial_i\phi(x,t),\pi(y,t)\} + m^2\phi(x,t)\{\phi(x,t),\pi(y,t)\} + \frac{\lambda}{3!}\phi^3(x,t)\{\phi(x,t),\pi(y,t)\}\right) \\
&= -\int d^3x \left(\partial_i\phi(x,t)\frac{\partial}{\partial x^i}\delta^3(x-y) + (m^2\phi(x,t) + \frac{\lambda}{3!}\phi^3(x,t))\delta^3(x-y)\right) \\
&= \partial_i\partial_i\phi(y,t) - m^2\phi(y,t) - \frac{\lambda}{3!}\phi^3(y,t).
\end{aligned}$$

Substituting the former in the latter on can show the equivalence with the Lagrangian formalism:

$$\partial_t^2\phi(y,t) - \partial_i\partial_i\phi(y,t) = \Box\phi(y,t) = -m^2\phi(y,t) - \frac{\lambda}{3!}\phi^3(y,t).$$